most of them are well above three would indicate that magnetomechanical effects are not the predominant cause of damping at these stress levels.

A direct comparison between the "12 per cent chrome steel" used by the author and the similar steels diagrammed in Figs. 1 and 2 cannot be made since more details are required on the chemical composition and processing procedure used for the author's steel. However, if the damping of the type 403 stainless steel diagramed in Fig. 1, which contains 12.6 per cent chromium, or that of the other steels in this figure which contain from 12 to 20 per cent chromium, are compared in the same units with the author's steel, it is found that the damping at the fatigue limit is considerably larger than that at the critical stress indicated by the author.

Although considerably more data are required before definite conclusions can be formulated the following tentative statements appear justifiable:

1 Since the high stress damping exponent n averages approximately 7 for the temperature-resistant materials studied to date, magnetomechanical effects probably are not the cause of the damping observed near the fatigue limit.

2 Damping near the fatigue limit (probably caused by plastic-flow effect) appears to be much larger than that caused by magnetomechanical effects at the critical stress (approximately 10 per cent of the fatigue limit for the author's steel) for the materials investigated to date.

3 Thus, although magnetomechanical effects may be the most important cause of damping of ferromagnetic materials near their critical stress, the damping caused by plastic strain appears to predominate near the fatigue limit.

4 The relative importance of low-stress damping, produced, for example, by magnetomechanical effects, and high-stress damping as produced by plastic flow depends on the application and the magnitude of damping exponent n. In general, the more concentrated the stress (that is, the smaller the per cent of volume at near peak stress) and the lower the damping exponent, the more important does low-stress damping become as a limiter of near-resonance vibration. In cases, however, where the per cent of material at near peak stress is significant and the damping exponent n is large, the damping at high stress levels becomes the primary consideration. Thus additional work on "stress-distribution function" discussed by the author and the "cross-sectional shape" and "longitudinal stress-distribution" factors[4] is highly desirable to further understanding on this relationship.

It would be appreciated if the author would give more details on the 12 per cent chrome material and type of specimen used in his work so that more direct comparisons may be made with other work.

Author's Closure

The author is grateful to Professor Lazan for his interesting remarks. Most of the questions raised have been answered in a second paper.[5] The conclusions reached in that paper agree with Professor Lazan's statement 4, but not with statements 1 and 3. The question of whether the high-stress damping (or plastic-flow damping) is more significant than the damping due to magnetomechanical hysteresis depends on the material used and the stress system to which it is subjected. The magnetomechanical damping is of primary importance for ferromagnetic materials when they vibrate similar to cantilever or simple-supported beams. This is explainable since most of the volume of such members is subjected to only a fraction of the peak stress. The high-stress damping therefore does not contribute much to the total damping even if the peak stress is near the fatigue limit and regardless of whether the damping energy increases with the fifth or tenth power of stress at larger stresses.

There are, of course, cases where plastic flow is the primary source of the material damping. This has been explained in detail in the paper mentioned.

The composition of the "12 per cent chrome steel" is Fe—12.5 per cent Cr, 0.5 per cent Ni, and 0.1 per cent C. The alloy was forged at 1950 F and annealed for two hours at 1750 F. Finally, the specimens used were conventional Föppl-Pertz torsion specimens[6] having a diameter of $^1/_2$ inch, an effective length of $6^3/_4$ in., and ends $^5/_8 \times {}^5/_8$ in.

[4] See "Effect of Damping Constants and Stress Distribution on the Resonance Response of Members," by B. J. Lazan, Journal of Applied Mechanics, Trans. ASME, vol. 75, 1953, pp. 201–209.

[5] "A Method for Determining the Internal Damping of Machine Members," by A. W. Cochardt, to be presented at the Annual Meeting of the ASME, Nov. 30–Dec. 4, 1953. Paper No. 53—A-44.

[6] "Föppl-Pertz Damping Machine," *Metals and Alloys*, New Products Section, February, 1931, p. 28.

Axisymmetric Flexural Temperature Stresses in Circular Plates[1]

H. L. Langhaar.[2] The author has obtained a remarkably simple solution of a problem that is becoming increasingly important. The method of "cuts" that he employs provides us with an insight into the nature of the phenomena that accompany temperature gradients in plates. Much of the value of this method is lost, however, if we abandon the classic assumption that normals to the middle surface remain straight and normal. This assumption, which is equivalent to neglecting the shearing stress τ_{rz}, accounts for the simplicity and the generality of the results that have been obtained. Since the shearing stresses may be expected to depend on the dimensionless products $(h/T)\ (dT/dr)$ and $(h^2/T)\ (d^2T/dr^2)$, the conclusions that have been derived are perhaps valid only if the temperature varies gradually in the radial direction.

The case that has been treated belongs to the class of thermal-stress problems in which the temperature distributions are axially symmetrical. This whole class of problems is a fertile field for research. Presumably, some exact solutions can be obtained that represent thermal stresses in plates with special types of axially symmetrical temperature distributions. Such solutions are potentially valuable, not only because they apply for thick disks and for disks with large radial temperature gradients, but also because they supply a check on the assumption, $\tau_{rz} = 0$. However, an exact general solution of the plate problem for all axially symmetrical temperature distributions that vary linearly with the axial co-ordinate z can scarcely be expected.

Possibly approximate solutions for moderately thick plates, and for plates with rather large radial temperature gradients, may be obtained in the form of truncated power series in z. This method has been applied to shell problems by P. S. Epstein,[3] though not with consideration of thermal strains.

Author's Closure

The author is indebted to Professor Langhaar for his interesting and constructive comment. The solution which was presented, since it implied the usual assumptions of classical plate

[1] By J. E. Goldberg, published in the June, 1953, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 75, pp. 257–260.

[2] Department of Theoretical and Applied Mechanics, University of Illinois, Urbana, Ill. Mem. ASME.

[3] "On the Theory of Elastic Vibrations in Plates and Shells," by P. S. Epstein, *Journal of Mathematics and Physics*, vol. 21, 1942, pp. 198–208.

theory, is valid for thin plates in which the radial temperature gradients are "well-behaved." Indeed, the problem came to the author's attention in connection with the design of certain items of refinery equipment wherein these restrictions were presupposed. However, the solution may give results of sufficient accuracy for engineering purposes in many cases where the restrictions of the parameters mentioned by Professor Langhaar are overlooked and should, at least, serve as a first approximation in such cases.

One notes that the shearing stress τ_{rz} is identically zero within the "central disks" used in the synthesis of the solution. Also, in the case of unloaded plates, the average value of τ_{rz} through the thickness at any point must be zero to satisfy statical equilibrium. These facts lend some support to the intuitive feeling that the range of validity of the solution may be stretched a little for engineering purposes.

The present closure affords an opportunity to present a more compact derivation of Equations [8]. In terms of the radial and tangential moments, M_1 and M_2, respectively, the equilibrium condition for the axisymmetric problem of circular plates is

$$M_1 + r\frac{dM_1}{dr} - M_2 + Vr = 0 \qquad [1]$$

in which V is the unit shear acting on a circumferential arc. The slope-moment-temperature relations are

$$\left.\begin{aligned} M_1 &= D\left(\frac{d\phi}{dr} + \mu\frac{\phi}{r}\right) - \frac{\alpha(1+\mu)DT}{h} \\ M_2 &= D\left(\frac{\phi}{r} + \mu\frac{d\phi}{dr}\right) - \frac{\alpha(1+\mu)DT}{h} \end{aligned}\right\} \qquad [2]$$

where T is the temperature difference between the faces of the plate. Noting that $V = 0$ if the plate is not loaded, and substituting Equation [2] into Equation [1] yields

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = \frac{\alpha(1+\mu)}{h}\frac{dT}{dr} = k_s\frac{dT}{dr}$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi)\right] = k_s\frac{dT}{dr} \qquad [3]$$

Integrating twice, one obtains

$$\rho\phi = k_s\int_0^\rho Tr\,dr + \frac{C_1\rho^2}{2} + C_2$$

Regularity at the origin requires that $C_2 = 0$. Hence

$$\phi = \frac{k_s}{\rho}\int_0^\rho Tr\,dr + C_3\rho \qquad [4]$$

Substituting Equation [4] into Equation [2]

$$\left.\begin{aligned} M_1 &= -\frac{k_m}{\rho^2}\int_0^\rho Tr\,dr + C \\ M_2 &= \frac{k_m}{\rho^2}\int_0^\rho Tr\,dr - k_m T + C \end{aligned}\right\} \qquad [5]$$

Since the integrals in Equation [5] represent the statical moment about the origin of the area under the T curve bounded by the ordinate at ρ, they may be replaced by the quantity $A_1\bar{b}_1 + T_\rho\frac{\rho^2}{2}$, thus reducing Equation [5] to the form of Equation [8] in the paper.

One may also arrive at Equation [3] by a slightly more physical reasoning. We imagine that while plate is being brought up to its final temperature distribution, it is restrained against displacement by a set of surface loads. If the displacements are zero throughout then, initially, $M_1 = M_2 = -\alpha(1+\mu)DT/h = -k_sDT$. Substitution of these moments into Equation [1] defines, by means of shears, the surface loading necessary to prevent displacements, and it is found that $V = k_sD\left(\frac{dT}{dr}\right)$. To remove these constraints, we merely superimpose a set of equal and opposite surface loads, i.e. a set of loads characterized by $V = -k_sD(dT/dr)$, for which

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\phi}{r^2} = -\frac{V}{D} = k_s\frac{dT}{dr}$$

This is identical with the predecessor of Equation [3].

The End Problem of Rectangular Strips[1]

M. Z. v. KRZYWOBLOCKI.[2] The author should be congratulated for not following the bad habit of using the doubtful "strain-energy method" and for seeking some new methods of solution of elasticity problems. But since the author mentioned the writer's name and since he mispresented a few facts, it is believed that some remarks are in order to correct the situation.

It should be mentioned that the only rigorous solution of the problem of a minimum of an integral functional like that of minimum of potential or strain energy is connected inseparably with the calculus of variation and with its principal equation, i.e., Euler-Lagrange differential equation. This fact should be known to everybody who attacks any such problem and it is explained thoroughly in such properly written books on elasticity as that of Sokolnikoff.

However, it is also known that the contrary is true and that in this Journal there have appeared in the past many articles whose authors showed a complete lack of knowledge of the facts. Owing to the great difficulty of solving Euler-Lagrange differential equations some approximate solutions must be used. In 1908 a Swiss physicist, Walther Ritz (who died prematurely), proposed a new approximation method of solving the problems of a variational character. One may find its description in the original Ritz's paper, or in many papers on this method written in the last half century.

A few sections from R. Courant's address on this method follow:

"Suppose we seek the minimum d of an integral expression $I(\phi)$. We then start with a minimizing sequence $\phi_1, \phi_2, \ldots, \phi_n \ldots\ldots$, that is, a sequence of functions, admissible in our variational problem, for which $\lim_{n\to\infty} I(\phi_n) = d$, d, being the lower bound of the functional $I(\phi)$.

"Ritz's construction proceeds as follows: We start with an arbitrarily chosen system of co-ordinate functions $\omega_1, \omega_2, \ldots, \omega_n, \ldots$, which should satisfy two conditions: (a) any linear combination $\phi_n = c_1\omega_1 + c_2\omega_2 + \ldots c_n\omega_n$, of them is admissible in the variational problem; (b) they should form a complete system of functions in the sense that any admissible function ϕ and its relevant derivatives may be approximated with any degree of accuracy by a linear combination of co-ordinate functions and of their corresponding derivatives, respectively. If we begin with such a system of co-ordinate functions, it is clear that for n sufficiently large and for a suitable choice of the coefficients $c_1, c_2, \ldots\ldots, c_n$, we can find admissible functions ϕ_n for which $I(\phi_n)$ differs arbitrarily

[1] By G. Horvay, published in the March, 1953, issue of the JOURNAL OF APPLIED MECHANICS, Trans. ASME, vol. 75, pp. 87–94.

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