Localized solutions are obtained for the system of fermions and scalar bosons which interact via Yukawa type coupling. Such solutions are stable for every value of Yukawa coupling constant. In this model we present a picture that a fermion is always clothed with the cloud of scalar bosons and a naked fermion does not appear.

§ 1. Introduction

There have been many works to explore an attractive idea that the hadron can be understood as an extended object. Two approaches may be distinguished: (a) solitons which have topological quantum numbers, (b) nontopological solitons and bags.

An example of the type (a) is the monopole solution in the non-Abelian gauge theory, which is given by 't Hooft and Polyakov. It is pointed out that the bound system of a monopole and an isospin doublet scalar boson has half-odd angular momentum and behaves as a fermion. This implies that bosons are elementary particles and there is a possibility that we can regard the internal quantum number as half-integer spin. On the other hand, a fair success of the quark model may suggest us that hadrons are made of spin 1/2 quarks (and anti-quarks). From this point of view, it will be natural to study a theory of extended objects in the framework of a quantum field theory containing fermions (type (b)). Many authors have devoted their efforts to this subject.

Recently T. D. Lee et al. examined a model of a charged scalar field coupled with a Higgs scalar field, and showed that the localized solutions do exist, which have no topological quantum number. In their model, it is pointed out that the conservation of charge played an essential role for the existence of localized solutions. If we take a system which contain fermions in addition to bosons, the conservation law of fermion number guarantees the stability of localized solutions so that we have such a stable solution that a fermion is trapped in a bag.

In this paper, we are going to study a simple model, which consists of a massive fermion and a massive neutral scalar field (not a Higgs scalar field).
interacting with each other via Yukawa type coupling and we are looking for stable localized solutions in this model. As is shown by T. D. Lee and G. C. Wick, the expectation value of the scalar field is different from the vacuum expectation value in the domain where the fermion is distributed and the "mass" of the fermion there is lower than the normal value.

The Lagrangian that we take is

\[ L = \int d^3x \left\{ \bar{\psi} \left[ i \gamma^\mu \partial_\mu - (M + g\phi) \right] \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right\}, \quad (1.1) \]

where \((M + g\phi)\) will hereafter be referred to as the mass term of the fermion and other notations will be obvious. We perform the quantization in the same way as the SLAC bag. The c-number equations for the fields are obtained by the variational method. We discuss these equations and seek for static S-wave solutions for which the energy density vanishes exponentially far away from the origin, which we call localized solutions. In the case of the SLAC bag, the fermion has mass \((M)\) through the Higgs mechanism. Inside the bag the mass has a negative value \((-M)\) and the wave function of the fermion is localized around the surface of the bag. In our model the mass term in Eq. (1.1) becomes zero or at least takes a small negative value inside the bag in the strong coupling limit and the shape of the wave function is similar in form but broader than that of the SLAC bag.

If we consider a vector field instead of the scalar field in Eq. (1.1), we get localized solutions in the case of the weak coupling. When the coupling constant becomes stronger, the energy level of the fermion becomes lower and lower, and it will drop into negative energy continuum (Klein paradox). This means that Dirac's hole theory must be employed. In other words, we must take account of the effect of pair creations. But this is out of our scope because we restrict ourselves to discussing the c-number equations of the fields.

It is very difficult to solve the c-number equations analytically. We perform numerical calculations to obtain the solutions. The results of calculations suggest that the localized solutions exist for every non-vanishing value of the coupling constant \(g\), and these solutions are stable in the meaning that the total energies of these solutions are less than the mass of the fermion. So the fermion is always clothed with the cloud of the scalar field and the naked fermion does not appear.

Finally, what we must comment is that a scalar field does not distinguish a fermion from an antifermion, at least in this approximation. Therefore, the total energy for the bound system of two fermions has the same value as that for the system of one fermion and one antifermion, and the same statement holds for the systems with much more fermions and antifermions.

In the next section we derive the c-number equations. In § 3 we show the existence of localized solutions by using trial functions. In § 4 we introduce external scalar and vector potentials whose shapes are of square well and obtain
eigenvalues of the Dirac equations in these potentials. In § 5 some properties of the localized solutions are discussed analytically. The results of numerical calculations are shown in the last section.

§ 2. C-number equations

We adopt the same quantization as the SLAC bag. The boson field operator is represented as a superposition of plane waves,

\[ \phi(x) = \sum_{p} \frac{1}{\sqrt{2p}} [a_{p}e^{-ipx} + a_{p}^{\dagger}e^{ipx}] \tag{2.1} \]

As the orthonomal set of the fermion wave function, we choose the eigenfunctions of the Dirac equation which involves the c-number static scalar potential \( \phi_{0}(x) \),

\[ (-i\gamma\nabla + M + g\phi_{0}) u_{n} = E_{n}^{\phi} u_{n}, \tag{2.2a} \]
\[ (-i\gamma\nabla + M + g\phi_{0}) v_{n} = E_{n}^{\phi} v_{n}, \tag{2.2b} \]

where \( u_{n}(x) \) and \( v_{n}(x) \) are positive and negative energy solutions, respectively, and \( E_{n} = -E_{n} > 0 \). Then \( \phi(x) \) is expanded by \( u_{n}(x) \)'s and \( v_{n}(x) \)'s,

\[ \phi(x) = \sum_{n} [b_{n}u_{n}(x) + d_{n}^{\dagger}v_{n}(x)]. \tag{2.3} \]

We will regard \( a_{p}, b_{n}, d_{n} \) \((a_{p}^{\dagger}, b_{n}^{\dagger}, d_{n}^{\dagger})\) as annihilation (creation) operators, and the vacuum is defined by

\[ a_{p}|0\rangle = b_{n}|0\rangle = d_{n}|0\rangle = 0. \tag{2.4} \]

After such preparations we give the trial state with \( I \) fermions and \( J \) antifermions as

\[ |\text{trial}\rangle = \prod_{i=1}^{I} b_{i}^{\dagger} \prod_{j=1}^{J} d_{j}^{\dagger} \exp \left[ -i \int d^{3}x \phi_{0}(x) \pi(x) \right] |0\rangle, \tag{2.5} \]

where \( \pi(x) \) is the canonical momenta of the boson field. Taking the expectation value of the normal-ordered Hamiltonian with respect to the state (2.5), we get the total energy \( W \) of the state,

\[ W = \langle \text{trial}|H|\text{trial}\rangle = \int d^{3}x \left\{ \sum_{i=1}^{I} \bar{u}_{i}(-i\gamma\nabla + M + g\phi_{0}) u_{i} - \sum_{j=1}^{J} \bar{v}_{j}(-i\gamma\nabla + M + g\phi_{0}) v_{j} \right. \]

\[ + \frac{1}{2} \left( \phi_{0}^{2} + \frac{m^{2}}{2}\phi_{0}^{2} \right) \tag{2.6} \]

with the notation \( \gamma\nabla = \gamma^{i}(\partial/\partial x^{i}) + \gamma^{4}(\partial/\partial x^{4}) + \gamma^{5}(\partial/\partial x^{5}) \). A minus sign of the second term of the integrand is due to normal-ordering of the Hamiltonian. Hereafter we will write \( \phi_{0} \) as \( \phi \) without confusing.
We require the functional derivative of $W$ with respect to $\delta \phi(y)$ to vanish,
\[
0 = \frac{\delta W}{\delta \phi(y)} = \int d^3x \sum_i \frac{\delta u_i(x)}{\delta \phi(y)} (-i\gamma^\nu + M + \gamma^\nu \phi) u_i + \bar{u}_i (-i\gamma^\nu + M + \gamma^\nu \phi) \frac{\delta u_i(x)}{\delta \phi(y)}
\]
\[
- \sum_j \frac{\delta \bar{v}_j(x)}{\delta \phi(y)} (-i\gamma^\nu + M + \gamma^\nu \phi) v_j + \bar{v}_j (-i\gamma^\nu + M + \gamma^\nu \phi) \frac{\delta v_j(x)}{\delta \phi(y)}
\]
\[
+ g \left[ \sum_i u_i(y) u_i(y) - \sum_j \bar{v}_j(y) v_j(y) \right] - \Phi \phi(y) + m^2 \phi(y).
\] (2.7)

Using Eqs. (2.2a) and (2.2b) and normalization conditions for $u_i$’s and $v_j$’s,
\[
\int d^3x [u_i^+ \cdot \delta u_i + \bar{u}_i^+ \cdot u_i] = \int d^3x [v_j^+ \cdot \delta v_j + \bar{v}_j^+ \cdot v_j] = 0,
\] (2.8)

we can show that the first term in Eq. (2.7) vanishes. Then we have a c-number equation,
\[
(-\Phi^2 + m^2) \phi(x) = -g (\sum u - \sum \bar{v}).
\] (2.9)

At this stage we have obtained a set of c-number equations, (2.2a), (2.2b) and (2.9). To justify this formalism, we must show that for $E_i < 0$ and $E_j > 0$ there is no solution of Eqs. (2.2a), (2.2b) and (2.9). In § 5 we will perform such justification partially. By making use of Eqs. (2.2a), (2.2b) and (2.9), we can rewrite the total energy in a compact form,
\[
W = \sum_i E_i - \sum_j E_j + \int d^3x \left[ \frac{1}{2} (\Phi^2 + m^2)^2 \phi^2 \right].
\] (2.10)

From this equation we can see that $W$ is positive definite. It is not necessary to consider the negative energy solution $v_j(x)$, because $v_j(x)$ will be obtained from $u_i(x)$ by taking a new spinor $i\gamma^\nu \gamma^\mu \gamma^\mu u_i(x)$ with $\bar{E}_j = -E_j$.

Hereafter we only consider such a case that $\phi(x)$ is spherically symmetric and all fermions and antifermions occupy the S-wave state, that is, $u(x)$ is decomposed as
\[
u(x) = \begin{pmatrix} F(r) s \\ -iG(r) \frac{1}{r} \sigma \cdot x s \end{pmatrix},
\] (2.11)

where $F(r)$ and $G(r)$ are real scalar functions of $r = |x|$ and $s$ denotes the spin state of the fermion. We assume that the lowest of positive energy states is the S-wave state, and the corresponding wave functions $F(r)$ and $G(r)$ are nodeless. Hereafter we call such a state the ground state. Substituting (2.11) into Eqs. (2.2a) and (2.9), we obtain
\[
\frac{dF}{dr} = (M + g\phi + E) G,
\] (2.12a)
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\[ \frac{dG}{dr} = (M + g\phi - E) F - \frac{2}{r} G, \quad (2.12b) \]

\[ \left( -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + m^2 \right) \phi = Ng(F^2 - G^2), \quad (2.13) \]

where \( N \) is the total number of fermions and antifermions. As will be shown later (§ 5), if we put \( u(x) \) in the following form

\[ u(x) = \begin{pmatrix} iG' \frac{1}{r} & \sigma x \end{pmatrix} = i\gamma^1 \gamma_r \begin{pmatrix} F' \cr -iG' \frac{1}{r} \sigma x \end{pmatrix}, \quad (2.14) \]

instead of (2·11), Eqs. (2·2a) and (2·9) have no localized solution.

§ 3. Existence of localized solutions

In order to see the existence of localized solutions for the system of \( N \) fermions and antifermions, we show that \( W \) can be made less than \( NM \) by choosing suitable localized trial functions \( u_i \) and \( \phi_i \). If such trial functions can be obtained, there exist solutions whose total energy is less than \( W(u_i, \phi_i) \). With respect to such solutions, fermions cannot be plane wave because the total energy is less than \( NM \). And the number density of fermions have non-vanishing value only in a compact domain of space. Therefore \( \phi(x) \) vanishes exponentially out of this domain unless \( m = 0 \). The wave functions \( u(x) \) also vanish exponentially because \( E < M \), that is, the fermions are trapped in the potential \( g\phi(x) \). Then the energy density vanishes exponentially far away from the domain.

For the trial function \( u_i \), we choose the eigenfunction of the Dirac equation with a scalar square well potential \( \phi_w(r) \), which is defined by

\[ M + g\phi_w(r) = \begin{cases} 0 & \text{for } r \leq a \\ M & \text{for } r > a. \end{cases} \quad (3.1) \]

The trial function \( \phi_i(r) \) is given to take the form,

\[ M + g\phi_i(r) = \begin{cases} 0 & \text{for } r \leq a \\ M[1 - e^{-m(r-a)}] & \text{for } r > a. \end{cases} \quad (3.2) \]

We substitute \( u_i \) and \( \phi_i \) into Eq. (2·6),

\[ W(u_i, \phi_i) = Ng \int d^3 x \overline{u}_i (-i\gamma r + M + g\phi_w) u_i \]

\[ + \int d^3 x \left[ \frac{1}{2} (\mathbf{r}\phi_i)^2 + \frac{m^2}{2} \phi_i^2 \right] + Ng \int d^3 x \overline{u}_i u_i (\phi_i - \phi_w). \quad (3.3) \]

The first term in Eq. (3·3) reduces to the eigenvalue \( \varepsilon_i \) corresponding to the

*) In practice we will consider the cases in which \( N=1, 2, \) or 3.
ground state multiplied by $N$. We can exactly solve the Dirac equation in the square well potential and obtain the inequality $\varepsilon \cdot a < 4.5$. The second term can be easily calculated from Eq. (3.2). In § 5 it will be shown that $\bar{u}u$ is positive so that the last term in Eq. (3.3) is negative. Then we can obtain the following inequality,

$$W(u, \phi) \cdot a < 4.5N + \pi \frac{(Ma)^3}{g^2} \left[ 2 \frac{(ma)^2}{3} + 2(ma) + 2 + \frac{1}{ma} \right].$$  (3.4)

We will show that the right hand side of Eq. (3.4) is less than $NMa$

$$4.5N + \pi \frac{(Ma)^3}{g^2} \left[ 2 \frac{(ma)^2}{3} + 2(ma) + 2 + \frac{1}{ma} \right] < NMa.$$  (3.5)

This inequality holds if $Ng^2/4\pi > 23$, $Ma \sim 0.4Ng^2/4\pi > 9$, and $ma \sim 0.6$. What we obtained through this rough estimation is only the sufficient condition for the existence of localized solutions and not the necessary condition. In practice, however, localized solutions exist even if $g^2/4\pi$ is small.

§ 4. The Dirac equation with a scalar or a vector potential

Since we restrict ourselves to an $S$-wave state, the source term $\sum \bar{u}u - \sum \bar{v}v$ in Eq. (2.9) should be spherically symmetric. For the vector meson exchange Eq. (2.9) is replaced by the equation for the time component of the vector field $V_\nu$ with the source term $\sum u^\nu u - \sum v^\nu v$, which is to be spherically symmetric for the ground state. Localized solutions exist for the vector coupling as well as the scalar coupling.

An interesting question is whether localized solutions exist or not for other kinds of couplings (pseudoscalar, axial vector, tensor). So far as one fermion sector is considered, an $S$-wave solution does not exist for such couplings because the source term is not spherically symmetric. However, we cannot discard the possibility that non-spherically symmetric solutions exist.

In the last half of this section we discuss discrete eigenvalues of the Dirac equations with an external square well potential in cases of the scalar coupling and the vector coupling. We have two parameters for the square well potential, a depth and a radius. We will fix the radius and vary the depth. The eigenfunctions for the square well potential are written by the Bessel functions or the modified Bessel functions inside the potential and by the modified Bessel functions outside the potential. We can obtain the eigenvalues by imposing continuity of the wave functions at the surface of the potential. The changes of eigenvalues with respect to the depth are shown in Figs. 1 and 2 for the vector coupling and the scalar coupling, respectively.

In the vector coupling case the eigenvalues can take negative values. If the depth of the potential becomes very large, the discrete eigenvalues drop into negative energy continuum (Klein paradox). In the scalar coupling case the Klein
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paradox does not occur (see the next section). Positive and negative eigenvalues are symmetric with each other because the equation is symmetric by the replacement $u(x) \rightarrow i\gamma^3 u(x)$ and $E \rightarrow -E$. A characteristic feature of the scalar coupling is that when the potential becomes deep eigenstates except for the lowest one disappear one after another, and only the ground state survives when $M + g_\varphi = -M$. The fermion number density $\psi^\dagger \psi$ of this state increases inside the potential and then decreases outside the potential both exponentially.\(^\star\)

\[ \begin{align*}
\text{Fig. 1.} & \quad \text{The change of the energy eigenvalues} \\
& \text{with respect to the depth } V^2 \text{ of the square well vector potential. There is positive (negative) energy continuum for } E/M > 1 (E/M < -1).
\end{align*} \]

\[ \begin{align*}
\text{Fig. 2.} & \quad \text{The change of the eigenvalues with respect to the depth } \varphi \text{ of the square well scalar potential.}
\end{align*} \]

§ 5. Properties of the localized solutions

In this section we will examine the properties of the solutions analytically and we give them in the form of six statements.

(A) There is no localized solution with the eigenvalue $E = 0$ if the scalar potential is static and spherically symmetric.

Proof

We will prove the statement (A) for the $S$-wave state. The proof for the cases of higher angular momentum waves is straightforward.

We multiply Eqs. (2.12a) and (2.12b) by $F(r)$ and $G(r)$, respectively, and then subtract the latter from the former,

\[ \frac{1}{2} \frac{d}{dr} (F^2 - G^2) = \frac{2}{r} G^2 + 2EGG. \quad (5.1) \]

If we assume $E = 0$, $F^2 - G^2$ should be monotonically increasing. At $r \rightarrow 0$, $F^2 - G^2$ is positive because $F$ and $G$ behave like constant and $r$, respectively. As $r \rightarrow \infty$, $F^2 - G^2$ must vanish because of locality. So we cannot have localized solutions\(^\star\)

\[ \text{The solution discussed in Ref. 5) is this solution.} \]
if \( E = 0 \).

In the course of the proof the essential point is that the dimension of space is three. If we consider one space dimension, the term \( (2/r)G^2 \) in Eq. (5·1) does not appear so that the eigenvalue \( E = 0 \) is allowed. From the statement (A) it is easily seen that positive energy solutions do not mix with negative energy solutions. This is the generalization of what is shown in § 4 for the square well potential.

(B) There is no localized solution for Eqs. (2·12a) and (2·12b) with \( E < 0 \) which has no node.

\[ \text{Proof} \]

We define \( P(r) \) by

\[ P(r) = \frac{G(r)}{F(r)}. \quad (5·2) \]

From Eqs. (2·12a) and (2·12b), we obtain the equation for \( P(r) \),

\[ \frac{dP}{dr} = -E + M + g\phi - \frac{2}{r} P - (E + M + g\phi)P^2. \quad (5·3) \]

At \( r \sim 0 \), \( P(r) \) has a small negative value \( (P(0) = 0) \), that can be shown by the Taylor expansions around \( r = 0 \). Since we consider a nodeless solution only, \( P(r) \) is a continuous and negative definite function from \( r = 0 \) to \( r = \infty \). When \( r \sim \infty \), we can neglect \( \phi \) in Eq. (5·3) and \( P(r) \) behaves like

\[ P(r) \sim -\sqrt{\frac{M - E}{M + E}} - \frac{1}{M + E} \frac{1}{r}. \quad (5·4) \]

If we assume \( E < 0 \), \( P(r = \infty) \) is less than \(-1\). So there must exist a point \( r = r_0 \) where \( P(r_0) = -1 \) and \( dP/dr \bigg|_{r=r_0} \leq 0 \). However, from Eq. (5·2) we can show that

\[ \frac{dP}{dr} \bigg|_{r=r_0} = -2E + \frac{2}{r} > 0 \quad \text{for} \quad E < 0. \quad (5·5) \]

This is a contradiction and \( E \) must be positive.

(C) \( uu = F^2 - G^2 > 0 \) for a nodeless solution.

\[ \text{Proof} \]

We have just obtained the properties of \( P(r) \), that is, \( P(0) = 0, P(r = \infty) = -\sqrt{(M - E)/(M + E)} - 1 \) and \( P(r) < 0 \) for \( 0 < r < \infty \). If we assume that \( P(r) < -1 \) for \( r_1 < r < r_2 \) and \( P(r_1) = P(r_2) = -1 \), we obtain the following inequalities,

\[ \frac{dP}{dr} \bigg|_{r=r_1} = -2E + \frac{2}{r_1} \leq 0 \leq \frac{dP}{dr} \bigg|_{r=r_2} = -2E + \frac{2}{r_2}. \quad (5·6) \]

Equation (5·6) contradicts \( r_1 < r_2 \). Then we have proved \( 0 > P(r) > -1 \), that is, \( F^2 - G^2 > 0 \).

Next, with the help of (B) and (C) we can obtain statement (D).
(D) There is no localized solution which has the form of Eq. (2.14).

Proof

In this case $F'(r)$ and $G'(r)$ satisfy Eqs. (2.12a) and (2.12b) with the replacement $E \rightarrow -E$. The statements (B) and (C) tell us that $-E > 0$ and $(F')^2 - (G')^2 > 0$. The equation for $\phi(r)$ is given by

$$(-F^2 + m^2) \phi(r) = -g^2[F'' - G''] + \mathcal{R}.$$  \hspace{1cm} (5.7)

Since the source term is negative, $g\phi(r)$ is always positive. This can be easily seen if Eq. (5.7) is rewritten in the form,

$$g\phi(x) = \frac{g^2}{4\pi} \int d^3y \left\{ [F'(y)]^2 - [G'(y)]^2 \right\} \frac{e^{-m|x-y|}}{|x-y|}.$$ \hspace{1cm} (5.8)

When the potential $g\phi(x)$ is always positive, there is no localized solutions.

(Q.E.D.)

From the statements (B) and (D) we see that a localized solution $u(x)$ belongs to the positive eigenvalue and has the form of Eq. (2.11). By the replacement $v(x) \rightarrow i \gamma^i \tau^i v(x)$ and $E \rightarrow -E$, $v(x)$ belongs to the negative eigenvalue and has the form of Eq. (2.14). Thus our quantization method is justified partially since the total energy $W$ is shown to be positive definite, at least for the ground state.

(E) When the coupling constant $g^2$ increases, the total energy $W$ decreases.

Proof

We take the variations of $u_i$, $v_j$ and $g\phi$, $\delta u_i$, $\delta v_j$ and $\delta(g\phi)$ with respect to $\delta g^2$. With the help of Eqs. (2.2a), (2.2b) and (2.9), the variation of the total energy is obtained,

$$\delta W = \int d^3x \left[ \sum (\bar{u} \delta u + \delta \bar{u} \cdot u) - \sum (\bar{v} \delta v - \delta \bar{v} \cdot v) \right]$$

$$- \frac{\delta g^2}{g^2} \int d^3x \left[ \frac{1}{2} (\mathcal{R} g\phi)^2 + \frac{m^2}{2} (g\phi)^2 \right].$$ \hspace{1cm} (5.9)

The first term of the right hand side vanishes because of the normalization conditions for $u_i$ and $v_j$. The second term is always negative. Then we have $dW/dg^2 < 0$.

(Q.E.D.)

(F) In the strong coupling limit $g^2 \rightarrow \infty$, $W$ vanishes.

Proof

We use the same trial functions as used in § 3. The total energy is bounded by

$$\frac{W}{M} < \frac{E_a}{Ma} + \pi M \frac{1}{m} \left[ \frac{2}{3} \left( \frac{m}{M} \right)^2 (Ma)^2 + 2 \left( \frac{m}{M} \right)^2 (Ma)^2 + 2 \left( \frac{m}{M} \right) Ma + 1 \right].$$ \hspace{1cm} (5.10)

We assume $Ma \rightarrow \infty$ for $g^2 \rightarrow \infty$. In this limit $Ma \rightarrow \infty$, we can easily show that the eigenvalue $E_a$ has a nonzero finite value. Next in this limit we determine $Ma$ in such a way that the right hand side of Eq. (5.10) takes the minimum value...
with $g^2$ and $m/M$ fixed. For large $Ma$ the minimum is realized when

$$Ma \sim \left[ \frac{Ea}{2\pi (M/g)} \right]^{1/4} \sqrt{|g|}.$$  \hspace{1cm} (5.11)

From this equation the assumption $Ma \to \infty$ for $g^2 \to \infty$ is justified self-consistently. Substituting Eq. (5.11) into Eq. (5.10) we can obtain the result that $W/M$ vanishes like $O(1/\sqrt{|g|})$ or faster as $g^2$ tends to infinity. \hspace{1cm} (Q.E.D.)

The statement (F) is proved by the use of the trial functions for which the mass term $M + g \phi_i$ is set to be zero inside the bag. It is natural to expect that in the strong coupling case we can regard these trial functions as a good approximation to the true localized solutions. If we take such an expectation seriously, we can say that the mass term approaches zero inside the bag in the strong coupling limit.

§ 6. Numerical calculations

We introduce dimensionless variables,

$$\rho = m r, \quad \kappa = \frac{M}{m}, \quad \nu = \frac{E}{M},$$

$$A(\rho) = \frac{g}{m} \phi(r), \quad \tilde{F}(\rho) = m^{-3/2} \sqrt{\frac{Ng^2}{4\pi}} F(r), \quad \tilde{G}(\rho) = m^{-3/2} \sqrt{\frac{Ng^2}{4\pi}} G(r).$$  \hspace{1cm} (6.1)

The c-number Eqs. (2.12a), (2.12b) and (2.13) are written as

$$\frac{d\tilde{F}}{d\rho} = (\kappa + A + \kappa \nu) \tilde{G},$$  \hspace{1cm} (6.2)

$$\frac{d\tilde{G}}{d\rho} = (\kappa + A - \kappa \nu) \tilde{F} - \frac{2}{\rho} \tilde{G},$$  \hspace{1cm} (6.3)

$$\left( \frac{d^2}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} - 1 \right) A = \tilde{F}^2 - \tilde{G}^2.$$  \hspace{1cm} (6.4)

And we have the normalization condition

$$\int_0^\infty \rho^2 d\rho [\tilde{F}^2 + \tilde{G}^2] = N \frac{g^2}{4\pi}.$$  \hspace{1cm} (6.5)

In these variables the energy density is given by

$$\sigma = \nu \left[ \frac{1}{N(g^2/4\pi)} (\tilde{F}^2 + \tilde{G}^2) + \frac{1}{2} \frac{1}{4\pi \kappa} \frac{1}{N(g^2/4\pi)} \left[ \frac{dA}{d\rho} \right]^2 + A^2 \right]$$  \hspace{1cm} (6.6)

and the total energy is

$$\frac{W}{NM} = \int d^3 \rho \cdot \sigma = \nu + \frac{1}{2\kappa} \frac{1}{N(g^2/4\pi)} \int_0^\infty \rho^2 d\rho \left[ \frac{dA}{d\rho} \right]^2 + A^2.$$  \hspace{1cm} (6.7)

We performed the calculations when $\kappa = 10$ bearing in mind the results of § 3,
that is, $ma \sim 0.6$ and $Ma \geq 9$. It is not necessary to take $\kappa = 10$, we may choose $\kappa = 1$ or 100 or any other. In the numerical calculations we first give a value of $\nu$ and then search a solution which satisfies the boundary conditions at $\rho = 0$ and $\rho = \infty$. After we obtain a solution for a certain value of $\nu$, we can calculate $Ng^2/4\pi$ by Eq. (6·5) and $W/NM$ by Eq. (6·6).

We show in Figs. 3 and 4 $\nu (= E/M)$ as a function of $Ng^2/4\pi$ and $W/M$ as a function of $g^2/4\pi$, respectively. For one fermion sector the energies of the localized solutions are less than $M$ with any value of $g^2/4\pi$ (0.5 to 5000). This fact means that the fermion is accompanied by the cloud of scalar mesons (dressed fermion) and the naked fermion cannot exist. In the weak coupling we see from Eq. (2·9) that $\psi(x)$ is small with the order $g$. If the Hamiltonian includes the term $J\psi^4$, the correction from this term is of the order $g^4$ and we can neglect this term in comparison with the other terms, which are of the order $g^2$. So we may expect that the localized solutions are also stable in the weak coupling even if $\phi^4$ coupling is included.

The wave functions and the energy density are shown in Fig. 5(a) to Fig. 5(e). From these, we can see that the mass term approaches zero inside the bag as $g^2/4\pi \to \infty$, this is the fact expected from the statement (E) in § 5. In the strong coupling the number density of the fermion has a maximum at the surface of the bag, this is characteristic for the ground state.
Fig. 5. The wave functions $F(p)$, $G(p)$, $(M+g\phi)/m$, and the energy density $\sigma$ (dashed line). In these figures $F(p)$ and $G(p)$ are normalized as $\int p^2 dp (F^2 + G^2) = 1$. Five figures correspond to the case (a) $\nu = 0.995$, (b) $\nu = 0.7$, (c) $\nu = 0.15$, (d) $\nu = 0.08$, (e) $\nu = 0.04$, respectively.
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