RESEARCH ARTICLE | SEPTEMBER 01 1994

Scalar invariants of the radiating Kerr–Newman metric: A simple application of GRTensor FREE

Peter Musgrave; Kayll Lake

Comput. Phys. 8, 589–599 (1994)
https://doi.org/10.1063/1.168482

Scalar invariants of the radiating Kerr–Newman metric: A simple application of GRTensor

Peter Musgrave and Kayll Lake
Department of Physics, Queen's University, Kingston, Ontario K7L 3N6, Canada

(Received 1 February 1994; accepted 17 March 1994)

GRTensor is an interactive PC-based program for tensor analysis primarily of interest for teaching and research in general relativity. It uses either MAPLEV or MATHEMATICA as its algebraic engine. In this paper we use GRTensor to evaluate the Ricci and Weyl invariants for the radiating Kerr–Newman metric. This includes, as a special case, all nonvanishing invariants of the Kerr metric—the archetypical black hole solution in general relativity.

INTRODUCTION

The purpose of this paper is to give a simple demonstration of the program GRTensor by way of an examination of some fundamental properties of the radiating Kerr–Newman metric. GRTensor, which is available free of charge from the authors,[1] is an interactive PC-based program for tensor analysis which uses either MAPLEV or MATHEMATICA as its algebraic engine. The Kerr metric[2] is arguably the single most important exact solution known in general relativity[3]—Einstein's theory of the gravitational field. It represents the archetypical black hole that can arise in the complete gravitational collapse of a star. This metric is a special case of the one examined here.

Despite the central role of the Kerr metric, even a direct check that it does represent a vacuum solution of the Einstein field equations involves cumbersome calculations.[4] It represents an ideal candidate for computer-aided study. Moreover, in view of the history of the misunderstanding of the Schwarzschild "singularity,"[5] an explicit demonstration of the true singularity of the Kerr metric is central to its study. Whereas the singular structure of the Kerr metric is known, justification of this structure is not explicitly demonstrated in standard texts. Most often, the statement given is to the effect that evaluation of scalars like the Kretschmann scalar $R^{abcd}R_{abcd}$ (where R_{abcd} is the Riemann tensor) exhibits the singularity.[6] In only one recent text is this scalar actually exhibited explicitly.[7] For vacuum it is known that there are in fact four scalar invariants which do not vanish identically.[8] In this paper we examine the somewhat more general situation where the field has both mass and charge and these are both functions of an "advanced" time.

The paper is organized as a commentary on an actual GRTensor session which is reproduced in the Appendix directly as printed from Microsoft Windows.[9] The algebraic platform for this session was MAPLEV Release 2. The session itself has been augmented by a commentary, embedded after the session, for the convenience of the reader.

I. COMMENTARY

The `readlib(grii):` command reads the GRTensor program, `readlib(griihelp):` the help files, and the statement `grtensor();` displays the header as shown. The command `groptions();` displays the adjustable environment set for this session. The metric is loaded with `qload(metricname);` where here the metric name is oknn which resides in the DOS directory c:\makeg\oknn. The file oknn.mpl is reproduced in Fig. 1. As with all but the most trivial metrics, it is most efficient to first calculate the inverse metric, and then simplify both the metric and the inverse metric before more involved routines are considered. The first step is accomplished with `grcalc(invmetric);` and the second with `gralter(metric, invmetric);`. In view of the nature of the metric, it is clear that trigonometric simplification is a reasonable first choice. Note that the size of the object is displayed after simplification. In order to save space in the session output shown here, the environmental settings are changed so as to suppress the object size display. Simplification routines can also be entered directly. This is shown by factoring the metric and its inverse. Einstein's equations for vacuum require that the Ricci tensor vanish identically. For the more general metric considered here, Ricci does not vanish. Ricci is calculated with `grcalc(Ricci);` and simplified with `gralter(Ricci,2,7);` where again 2 and 7 are appropriate in view of the form of the metric. The Ricci tensor components are not of direct interest here and so are not displayed.

In addition to the ability to define new objects, GRTensor contains a set of predefined objects. These include all independent algebraic invariants of the Riemann tensor.[10] The Ricci invariants are defined in Fig. 2. These are calculated by `grcalc(CMR);` and simplified. Since algebraic computing can give rise to very large objects, objects are displayed only by request with the display size governed by the options. Note that the Ricci invariants vanish for $q(\nu)=0$.

GRTensor allows the definition of new objects. We demonstrate this here by evaluation of the scalar $R4$ defined to be $R_a^bR_b^cR_c^dR_d^a$ (where R_{ab} is the Ricci tensor). As expected, this scalar is not independent.

The invariants which do not vanish identically in vacuum (zero Ricci tensor) are defined in Fig. 3. In the evaluation of these invariants an efficient first step is to evaluate and simplify the Riemann tensor. This is accom-

```
Ndim := 4:
x1 := v:
x2 := r:
x3 := theta:
x4 := phi:
rho:=r^2+a^2*cos(theta)^2:
delta:=r^2-2*m(v)*r+a^2+q(v)^2:
g11 := -(1-(2*m(v)*r-q(v)^2)/rho):
g12 := 1:
g13 := 0:
g14 := -a*(2*m(v)*r-q(v)^2)*sin(theta)^2/rho:
g22 := 0:
g23 := 0:
g24 := -a*sin(theta)^2:
g33 := rho:
g34 := 0:
g44 := ((r^2+a^2)^2-delta*a^2*sin(theta)^2)*sin(theta)^2/rho:
grDalias(m(v),q(v),v,'');
```

Figure 1. The input file oknn.mpl. This file was created with the GRTensor utility makeg, but could be produced simply with an editor. The grDalias line inputs an alias for the derivative. This does not affect the sample session.

plished here by `grcalc(Riemann);` and `gralter(Riemann,2,7);`. Since we are not interested here in the structure of the components of Riemann, we go directly to the evaluation of the Weyl invariants. This is accomplished by `grcalc(CMW);` where CMW signifies all four Weyl invariants. Simplification of the invariants is again accomplished by `gralter(CMW,2,7);`. It is clear that the radiating Kerr–Newman metric is singular at $r=0$ [for $m(v)$ and $q(v)\neq 0$] but only in the equatorial plane ($\theta=\pi/2$) for $a\neq 0$. The invariants simplify for zero charge. The charge is set to zero by the `grmap` function as shown using the MAPLEV substitution routine. Simply setting $m(v)=m$ gives the complete set of nonvanishing invariants for the Kerr metric.

A fundamental property of the coordinates is that trajectories with constant v, θ, and ϕ are null geodesics.

GRTensor Name	Definition
R	$R_a{}^a$
R1	$(1/4)S_a{}^bS_b{}^a$
R2	$(-1/8)S_a{}^bS_b{}^cS_c{}^a$
R3	$(1/16)S_a{}^bS_b{}^cS_c{}^dS_d{}^a$

Figure 2. The Ricci invariants. S_a^b is the trace-free Ricci tensor defined by $R_a^b-\delta^b{}_aR/4$, where R_a^b is the Ricci tensor.

GRTensor Name	Definition
W1R	$(1/8)C_{ab}{}^{cd}C_{cd}{}^{ab}$
W1I	$(1/8)C^*{}_{ab}{}^{cd}C_{cd}{}^{ab}$
W2R	$(-1/16)C_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab}$
W2I	$(-1/16)C^*{}_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab}$

Figure 3. The Weyl invariants. C_{abcd} is the Weyl (conformal) tensor, and C^*_{abcd} its tensor dual.

Whereas the nullity is obvious from the metric, we verify the geodesic character here by defining the associated vector field [$\nu^a=(0,c,0,0)$ where c is a constant] and evaluating the acceleration $\nu^b\nu^a{}_{;b}$. We also define and calculate the associated expansion $\nu^a_{;a}$. It is clear that the geodesics are affinely parametrized by r and that the expansion diverges at the singularity as expected.

II. DISCUSSION

Tensor analysis of metrics with explicit functions requires rather more robust routines than those which can handle metrics of a general form. For example, with GRTensor on the same system,[9] calculation and simplification of the Ricci tensor for the general Bondi metric[11] (four functions of three variables) requires only 2 CPU seconds. As regards a tool for teaching and research in general relativity, a usable graphical interface, interactivity, and speed are essential elements. We believe that GRTensor, in its current state of development, makes many prohibitively cumbersome calculations elementary. Whereas the Kerr metric itself is amenable to some hand calculation by way of modern techniques,[12] it is worth pointing out that the metric examined in this paper is of no special significance to the program GRTensor. Moreover, in view of the relatively modest computer used for this demonstration, we believe that GRTensor, which is not limited to classical techniques, will be of interest for research as well as interactive teaching.

*Note added in proof.*GRTensor is available for MAPLEV release 3 and will run on any hardware platform on which maple can run

ACKNOWLEDGMENTS

This work was supported in part by grants from the Advisory Research Committee of Queen's University, and the Natural Sciences and Engineering Research Council of Canada.

In this sample session we calculate the Ricci and Weyl invariants for the radiating Kerr-Newman metric. Comments were inserted after the session and appear in this font only. The first three commands read the GRTensor library, the help files and display the header as shown. Note that all commands entered here are preceeded by > and terminated by ; or :. Anything in this session which is not an inserted comment or command is output.

```
> readlib(grii):

> readlib(griihelp):

> grtensor();
```

GRTensorII Development Version 0.47

January 19, 1994

Copyright 1993-1994 by K.W. Lake and P.J. Musgrave

To initiate help do ?grtensor

The session default (but adjustable) parameters are shown below.

```
> groptions();
grOptionDisplayLimit   = 5000
grOptionAlterSize      = true
grOptionTrace          = true
grOptionTimeStamp      = true
grOptionDefaultSimp    = 8
grOptionMetricPath     = c:/makeg

grOptionDefaultSimp values: 0=None, 1=simplify, 2=simplify[trig],
  3=simplify[power] 4=simplify[hypergeom], 5=simplify[radical],
  6=expand, 7=factor, 8=normal, 9=sort
```

We now load the metric which is in c:\makeg\oknn.

```
> qload(oknn);
Aliases created
```

Default metric = oknn

```
For the oknn metric
```

Coordinates

$$x^{1} = v, x^{2} = r, x^{3} = \theta, x^{4} = \phi$$

Line element

$$ds^2 = \left(-1 + \frac{2\,m(v)\,r - q(v)^2}{\%1}\right) dv^2 + 2\,dv\,dr - 2\,\frac{a\left(2\,m(v)\,r - q(v)^2\right)\sin(\theta)^2\,dv\,d\phi}{\%1} - 2\,a\sin(\theta)^2\,dr\,d\phi + \%1\,d\theta^2 + \frac{\left(\left(r^2+a^2\right)^2 - \left(r^2 - 2\,m(v)\,r + a^2 + q(v)^2\right)a^2\sin(\theta)^2\right)\sin(\theta)^2\,d\phi^2}{\%1}$$

$$\%1 := r^2 + a^2 \cos(\theta)^2$$

We now start the session by calculating the inverse metric.

```
> grcalc(invmetric);
Calculated detg  (0.000000 sec.)
Calculated g(up,up)  (1.000000 sec.)
```

$$CPU\ Time = 1.000$$

An efficient first step is the simplification of the metric and its inverse.

```
> gralter(metric,invmetric);

Component Alteration of a grtensor object:

(use ?name for help on a particular simplification routine)

Choose which routine to apply:
 0)  none
 1)  simplify()           try all simplification techniques
 2)  simplify[trig]       apply trig simplification
 3)  simplify[power]      simplify powers, exp and ln
 4)  simplify[hypergeom]  simplify hypergeometric functions
 5)  simplify[radical]    convert radicals,log,exp to canonical form
 6)  expand()
 7)  factor()
 8)  normal()
 9)  sort()
10)  grCosToSin()         try and change 1-cos^2 -> sin^2
11)  other                user specified routine
Number of routine to apply (followed by ;) >
> 2;
Applying routine simplify[trig] to object g(dn,dn)
```

$$g_{\ v\,v}\ ,\ \textit{changed to 109 (words)}$$

$$g_{\ v\,r}\ ,\ \textit{changed to 1 (words)}$$

$$g_{\ v\,\phi}\ ,\ \textit{changed to 141 (words)}$$

$$g_{\ r\,\phi}\ ,\ \textit{changed to 29 (words)}$$

$$g_{\ \theta\,\theta}\ ,\ \textit{changed to 33 (words)}$$

$$g_{\ \phi\,\phi}\ ,\ \textit{changed to 349 (words)}$$

```
Applying routine simplify[trig] to object g(up,up)
```

$$g^{\ v\,v}\ ,\ \textit{changed to 73 (words)}$$

$$g^{\ v\,r}\ ,\ \textit{changed to 57 (words)}$$

$$g^{\ v\,\phi}\ ,\ \textit{changed to 41 (words)}$$

$$g^{\ r\,r}\ ,\ \textit{changed to 89 (words)}$$

23 April 2025 23:48:49

$g^{r\phi}$, *changed to 41 (words)*

$g^{\theta\theta}$, *changed to 37 (words)*

$g^{\phi\phi}$, *changed to 89 (words)*

$CPU\ Time = 3.000$

To save space here we supress the object size display.

```
> grOptionAlterSize:=false;
```

$grOptionAlterSize := false$

Simplification routines can also be entered directly as shown below.

```
> gralter(metric,invmetric,7);

Component Alteration of a grtensor object:

Applying routine factor to object g(dn,dn)
Applying routine factor to object g(up,up)
```

$CPU\ Time = 1.000$

We now calculate the Ricci tensor.

```
> grcalc(Ricci);
Calculated g(dn,dn,dn)  (1.000000 sec.)
Calculated Chr(dn,dn,dn)  (0.000000 sec.)
Calculated Chr(dn,dn,up)  (2.000000 sec.)
Calculated R(dn,dn)  (8.000000 sec.)
```

$CPU\ Time = 11.000$

In view of the form of the metric, the following are reasonable choices for simplification.

```
> gralter(Ricci,2,7);

Component Alteration of a grtensor object:

Applying routine simplify[trig] to object R(dn,dn)
Applying routine factor to object R(dn,dn)
```

$CPU\ Time = 13.000$

We are now in a position to evaluate the Ricci invariants.

```
> grcalc(CMR);
Calculated Ricciscalar  (1.000000 sec.)
Created definition for S(up,dn)
Calculated S(dn,dn)  (0.000000 sec.)
Calculated S(up,dn)  (1.000000 sec.)
Calculated R1  (0.000000 sec.)
Calculated R2  (0.000000 sec.)
```

```
Calculated R3  (0.000000 sec.)
```

$$CPU\ Time\ = 2.000$$

In this case the invariants are almost completely simplified.

```
> gralter(CMR,2,7);
```

```
Component Alteration of a grtensor object:

Applying routine simplify[trig] to object Ricciscalar
Applying routine simplify[trig] to object R1
Applying routine simplify[trig] to object R2
Applying routine simplify[trig] to object R3
Applying routine factor to object Ricciscalar
Applying routine factor to object R1
Applying routine factor to object R2
Applying routine factor to object R3
```

$$CPU\ Time\ = 0$$

Because of the potential size of objects, they must be requested. Objects are displayed if they satisfy the (adjustable) grOptionDisplayLimit. Otherwise only their size is given.

```
> grdisplay(CMR);
For the oknn metric
```

Ricci scalar

$$R\ = 0$$

CM invariant R1

$$R1\ = \frac{\mathrm{q}(v)^4}{\left(r^2 + a^2 \cos(\theta)^2\right)^4}$$

CM invariant R2

$$R2\ = 0$$

CM invariant R3

$$R3\ = \frac{1}{4} \frac{\mathrm{q}(v)^8}{\left(r^2 + a^2 \cos(\theta)^2\right)^8}$$

Note that the Ricci invariants vanish for q(v)=0. GRTensor allows the definition of new objects. Below we define and calculate a new scalar R4 which is four contractions of the Ricci tensor. (The second entry in grdefine predefines symmetries which we do not use in this session.)

```
> grdefine(`R4`,{},`R{a^b}R{b^c}R{c^d}R{d^a}`);
Created definition for R(dn,up)
Created definition for R4
```

```
> grcalc(R4);
  Calculated R(dn,up)  (1.000000 sec.)
  Calculated R4  (1.000000 sec.)
```

$$CPU\ Time\ = 2.000$$

```
> grdisplay(R4);
  For the oknn metric
```

$$R4$$

$$R4 = 4\,\frac{q(\nu)^8}{\left(r^2 + a^2\cos(\theta)^2\right)^8}$$

It is clear that this scalar is not independent!
We now calculate and simplify the Riemann tensor.

```
> grcalc(Riemann);
  Calculated R(dn,dn,dn,dn)  (8.000000 sec.)
```

$$CPU\ Time\ = 8.000$$

```
> gralter(Riemann,2,7);

  Component Alteration of a grtensor object:

  Applying routine simplify[trig] to object R(dn,dn,dn,dn)
  Applying routine factor to object R(dn,dn,dn,dn)
```

$$CPU\ Time\ = 19.000$$

We now evaluate and simplify the Weyl invariants.

```
> grcalc(CMW);
  Calculated C(dn,dn,dn,dn)  (3.000000 sec.)
  Calculated C(dn,dn,up,up)  (7.000000 sec.)
  Calculated WeylSq  (1.000000 sec.)
  Calculated W1R  (0.000000 sec.)
  Calculated LevC(dn,dn,dn,dn)  (1.000000 sec.)
  Created definition for C(up,up,up,up)
  Calculated C(up,up,up,up)  (5.000000 sec.)
  Calculated Cstar(dn,dn,up,up)  (2.000000 sec.)
  Calculated W1I  (2.000000 sec.)
  Calculated nkA(dn,dn,up,up)  (24.000000 sec.)
  Calculated W2R  (4.000000 sec.)
  Calculated W2I  (7.000000 sec.)
```

$$CPU\ Time\ = 56.000$$

```
> gralter(CMW,2,7);

  Component Alteration of a grtensor object:

  Applying routine simplify[trig] to object W1R
  Applying routine simplify[trig] to object W1I
  Applying routine simplify[trig] to object W2R
  Applying routine simplify[trig] to object W2I
  Applying routine factor to object W1R
  Applying routine factor to object W1I
  Applying routine factor to object W2R
```

```
Applying routine factor to object W2I
```

$$CPU\ Time\ = 30.000$$

```
> grdisplay(CMW);
For the oknn metric
```

CM invariant Re(W1)

$$W1R\ = -6\left(\mathrm{m}(v)\,r^3 - 3\,\mathrm{m}(v)\,r^2\,a\cos(\theta) - 3\,a^2\,\mathrm{m}(v)\,r\cos(\theta)^2 + \mathrm{m}(v)\,a^3\cos(\theta)^3 - r^2\,\mathrm{q}(v)^2 + 2\cos(\theta)\,\mathrm{q}(v)^2\,r\,a + a^2\,\mathrm{q}(v)^2\cos(\theta)^2\right)\left(-\mathrm{m}(v)\,r^3 - 3\,\mathrm{m}(v)\,r^2\,a\cos(\theta) + 3\,a^2\,\mathrm{m}(v)\,r\cos(\theta)^2 + \mathrm{m}(v)\,a^3\cos(\theta)^3 + r^2\,\mathrm{q}(v)^2 + 2\cos(\theta)\,\mathrm{q}(v)^2\,r\,a - a^2\,\mathrm{q}(v)^2\cos(\theta)^2\right)\Big/\left(r^2 + a^2\cos(\theta)^2\right)^6$$

CM invariant Im(W1)

$$W1I\ = -12\left(3\,a^2\,\mathrm{m}(v)\,r\cos(\theta)^2 - a^2\,\mathrm{q}(v)^2\cos(\theta)^2 + r^2\,\mathrm{q}(v)^2 - \mathrm{m}(v)\,r^3\right)\left(\mathrm{m}(v)\,a^2\cos(\theta)^2 + 2\,r\,\mathrm{q}(v)^2 - 3\,\mathrm{m}(v)\,r^2\right)a\cos(\theta)\Big/\left(r^2 + a^2\cos(\theta)^2\right)^6$$

CM invariant Re(W2)

$$W2R\ = -6\left(3\,a^2\,\mathrm{m}(v)\,r\cos(\theta)^2 - a^2\,\mathrm{q}(v)^2\cos(\theta)^2 + r^2\,\mathrm{q}(v)^2 - \mathrm{m}(v)\,r^3\right)\left(3\,\mathrm{m}(v)^2\cos(\theta)^6\,a^6 - 27\cos(\theta)^4\,\mathrm{m}(v)^2\,r^2\,a^4 + 18\cos(\theta)^4\,\mathrm{m}(v)\,a^4\,r\,\mathrm{q}(v)^2 - a^4\cos(\theta)^4\,\mathrm{q}(v)^4 + 33\cos(\theta)^2\,\mathrm{m}(v)^2\,r^4\,a^2 - 44\cos(\theta)^2\,\mathrm{m}(v)\,r^3\,a^2\,\mathrm{q}(v)^2 + 14\cos(\theta)^2\,r^2\,\mathrm{q}(v)^4\,a^2 - r^6\,\mathrm{m}(v)^2 + 2\,r^5\,\mathrm{m}(v)\,\mathrm{q}(v)^2 - r^4\,\mathrm{q}(v)^4\right)\Big/\left(r^2 + a^2\cos(\theta)^2\right)^9$$

CM invariant Im(W2)

$$W2I\ = 6\left(\mathrm{m}(v)\,a^2\cos(\theta)^2 + 2\,r\,\mathrm{q}(v)^2 - 3\,\mathrm{m}(v)\,r^2\right)\left(\mathrm{m}(v)^2\cos(\theta)^6\,a^6 - 33\cos(\theta)^4\,\mathrm{m}(v)^2\,r^2\,a^4 + 22\cos(\theta)^4\,\mathrm{m}(v)\,a^4\,r\,\mathrm{q}(v)^2 - 3\,a^4\cos(\theta)^4\,\mathrm{q}(v)^4 + 27\cos(\theta)^2\,\mathrm{m}(v)^2\,r^4\,a^2 - 36\cos(\theta)^2\,\mathrm{m}(v)\,r^3\,a^2\,\mathrm{q}(v)^2 + 10\cos(\theta)^2\,r^2\,\mathrm{q}(v)^4\,a^2 - 3\,r^6\,\mathrm{m}(v)^2 + 6\,r^5\,\mathrm{m}(v)\,\mathrm{q}(v)^2 - 3\,r^4\,\mathrm{q}(v)^4\right)a\cos(\theta)\Big/\left(r^2 + a^2\cos(\theta)^2\right)^9$$

Note that for this metric the derivatives of m(v) and q(v) do not appear in the Ricci or Weyl invariants. It is clear that the metric is singular at r=0 (unless m(v)=q(v)=0) but only in the equatorial plane (theta=Pi/2) for the parameter a not zero. We now set the charge to zero by mapping the Maple substitution command onto the CMW invariants, and resimplify them.

```
> grmap(CMW,subs,q(v)=0,`x`);
Applying routine subs to W1R
```

W1R, changed to 305 (words)

```
Applying routine subs to W1I
```

$W1I$, *changed to 178 (words)*

```
Applying routine subs to W2R
```

$W2R$, *changed to 235 (words)*

```
Applying routine subs to W2I
```

$W2I$, *changed to 251 (words)*

```
> gralter(CMW,2,7);

Component Alteration of a grtensor object:

Applying routine simplify[trig] to object W1R
Applying routine simplify[trig] to object W1I
Applying routine simplify[trig] to object W2R
Applying routine simplify[trig] to object W2I
Applying routine factor to object W1R
Applying routine factor to object W1I
Applying routine factor to object W2R
Applying routine factor to object W2I
```

$CPU\ Time = 7.000$

```
> grdisplay(CMW);
For the oknn metric
```

CM invariant Re(W1)

$$W1R = -6\,\mathrm{m}(v)^2\,(r+\cos(\theta)\,a)\,(-r+\cos(\theta)\,a)\left(r^2+4\,r\cos(\theta)\,a+a^2\cos(\theta)^2\right)\left(r^2-4\,r\cos(\theta)\,a+a^2\cos(\theta)^2\right)\Big/\left(r^2+a^2\cos(\theta)^2\right)^6$$

CM invariant Im(W1)

$$W1I = -12\,\frac{\mathrm{m}(v)^2\,r\,a\cos(\theta)\left(-3\,r^2+a^2\cos(\theta)^2\right)\left(-r^2+3\,a^2\cos(\theta)^2\right)}{\left(r^2+a^2\cos(\theta)^2\right)^6}$$

CM invariant Re(W2)

$$W2R = -6\,\mathrm{m}(v)^3\,r\left(-r^2+3\,a^2\cos(\theta)^2\right)\left(3\,a^6\cos(\theta)^6-27\,a^4\cos(\theta)^4\,r^2+33\,a^2\cos(\theta)^2\,r^4-r^6\right)\Big/\left(r^2+a^2\cos(\theta)^2\right)^9$$

CM invariant Im(W2)

$$W2I = 6\,\mathrm{m}(v)^3\,a\cos(\theta)\left(-3\,r^2+a^2\cos(\theta)^2\right)\left(a^6\cos(\theta)^6-33\,a^4\cos(\theta)^4\,r^2+27\,a^2\cos(\theta)^2\,r^4-3\,r^6\right)\Big/\left(r^2+a^2\cos(\theta)^2\right)^9$$

We now introduce a vector field in order to evaluate the associated acceleration and expansion. The following defines a contravariant vector field.

```
> grdefine(`v{^a}`, {});
Created definition for v(up)
```

The acceleration is defined as usual:

```
> grdefine(`A{^a}`,{},`v{^b}*v{^a;b}`);
Created definition for A(up)
```

as is the expansion.

```
> grdefine(`Expansion`,{},`v{^a;a}`);
Created definition for Expansion
```

The vector field must be defined before evaluation.

```
> grcalc(A(up));

Enter components for object v(up)

  If you wish to quit at any point and leave this object
  uninitialized, enter the string oops.
  REMEMBER to complete each entry with a semicolon.
```

$$v^{v}$$

```
  >
> 0;
```

$$v^{r}$$

```
  >
> c;
```

$$v^{\theta}$$

```
  >
> 0;
```

$$v^{\phi}$$

```
  >
> 0;
Calculated v(up)   (0.000000 sec.)
Calculated A(up)   (0.000000 sec.)
```

$$CPU\ Time\ = 0$$

```
> grdisplay(A(up));
For the oknn metric
```

A(up)

$$\mathrm{A}(up) = All\ components\ are\ zero$$

We see that the trajectories of constant v, theta and phi are null geodesics affinely parametrized by r.

```
> grcalc(Expansion);
Calculated Expansion  (0.000000 sec.)
```

$$CPU\ Time\ = 0$$

```
> grdisplay(Expansion);
```

```
For the oknn metric
```

Expansion

$$Expansion = 2\frac{r\,c}{r^2 + a^2\cos(\theta)^2}$$

The expansion diverges at the singularity as expected.

We conclude here by verifying that with m(v)=m and q(v)=0 the resultant Kerr metric is indeed vacuum.

```
> grmap(Ricci,subs,m(v)=m,q(v)=0,`x`);
Applying routine subs to R(dn,dn)
```

```
> grdisplay(Ricci);
For the oknn metric
```

Covariant Ricci

$$\mathrm{R}(dn, dn) = \text{All components are zero}$$

The entire session has a CPU execution time of 153 seconds. For a direct calculation from the Kerr metric it takes 11 seconds to show Ricci=0, and 21 seconds to evaluate and simplify the Kretschmann scalar.

REFERENCES

1. GRTensor is distributed free of charge. Send requests to GRTENSOR@ASTRO.QUEENSU.CA. The version of GRTensor used in this paper is an updated and expanded version of that previously distributed (also without charge) by the authors (GRTensor 0.26). It is distributed in the form of Maple "*.m*" files and as such is portable. The capabilities of GRTensor extend considerably beyond those demonstrated here. MAPLEV is a registered trademark of Waterloo Maple Software. MATHEMATICA is a registered trademark of Wolfram Research, Inc. At the time of writing, the MATHEMATICA version of GRTensor does not allow the definition of new objects. Further general information on computer algebra systems can be found in the database maintained by Paulo Ney de Souza at ca@math.berkeley.edu.
2. R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
3. We follow the sign conventions of C. W. Misner, K. S. Thorne, and J. A Wheeler, *Gravitation* (Freeman, San Francisco, 1973), and refer the reader to this text for the definitions of the Riemann, Ricci, and Weyl (conformal) tensors. The Riemann tensor reduces to the Weyl tensor in vacuum.
4. See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), p. 323.
5. See, for example, W. Israel, "Dark stars: the evolution of an idea" in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
6. Statements to this effect can, for example, be found in S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973); R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984); and R. D'Inverno, *Introducing Einstein's Relativity* (Clarendon, Oxford, 1992).
7. See F. de Felice and C. J. S. Clarke, *Relativity on Curved Manifolds* (Cambridge University Press, Cambridge, 1990), p. 401. This follows the work of F. de Felice and M. Bradley, Class. Quantum Gravit. **5**, 1577 (1988).
8. The invariants for vacuum are given by S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 146. These differ from those given in Fig. 3 by only a numerical coefficient.
9. The system used was an Intel 486 DX2 66 with a Norton Utilities(7) speed index of 132 and an overall performance index of 91.5. The operating system used was Microsoft MS-DOS 6.2 and the session was run under Microsoft Windows 3.11. Microsoft and MS-DOS are registered trademarks, and Windows is a trademark of Microsoft Corporation. Norton Utilities is a trademark of Symantec Corporation.
10. We have used the invariants given by J. Carminati and R. G. McLenaghan, J. Math. Phys. **32**, 3135 (1991), and have retained their definitions. This accounts for the leading numerical coefficients in Fig. 2, and Fig. 3.
11. H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London Ser. A **269**, 21 (1962).
12. See, for example, S. Chandrasekhar, "An introduction to the theory of the Kerr metric and its perturbations" in *General relativity*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).