material the extent of damage is independent of the energy, whether infinite or small, that is available in the pressure fluid and tank as a source for continuance of damage. After yielding, the extent of damage is dependent upon the plastic stress-strain characteristics of the shell material and the compressibility of the fluid inside the shell as well as upon the potential energy in any system that is exterior to the shell.

Before leaving the questions raised by discussing energy, there is one that should be clarified further. It is certain that there is theoretically a nondestructive test which requires no interior fluid in the specimen. Such a test is based on a suitably close matching of the volume changes of the pressure medium and tank with the volume change of the test specimen at incipient buckling. Further thinking along this line indicates how impractical it is to try to set up a test of this kind. We see that to set it up would call for so much pretest data on the displacements of the shell specimen that, after calculating these data, if this could be done without too much inaccuracy for the given shell, there would be no need to conduct the test. Even with careful metering of the pressure fluid, it appears unlikely that a reliable nondestructive pressure test could be established. To maintain the nondestructive feature, the metering response must be induced immediately and continuously by the condition of the shell; and, as quickly as it is required by the shell, the metering system must supply relief from or support of part of the increase of external pressure. These conditions prevail in a water-filled shell, but they have not, as yet, been obtained for an empty shell. Moreover, it seems doubtful that they can be readily obtained in the near future.

In regard to instrumentation, it has been found that radial displacements, elastic strains, and lobe shapes can be measured in the differential-pressure test. It is necessary that the measuring instruments be waterproofed. The writer has successfully measured deflections up to one and two inches with errors below two per cent under pressures up to 1100 psi using linear differential transformers which were enclosed in brass housings to isolate them from water-pressure effects. The leads were brought out of the pressure vessel through copper tubing. Likewise, waterproofing techniques for wire-resistance strain gages are well established. For measuring lobe shapes, it is felt that the differential-pressure test offers a distinct advantage in that not only can the lobe be measured but the progress of its formation can be studied due to slowing its rate by means of the control inherent in the test method. Measurements can be made by the use of a traversing system while the test is being conducted.

In using the stepwise procedure outlined in the writer's paper, one must exercise caution when reducing the internal pressure. If the shell is near the point of buckling, the buckling may occur during the internal-pressure reduction process attended by inadequate control of the test at this point. This can be avoided by slowly reducing the pressure accompanied by the same care in taking readings as that employed when the external pressure is being increased. If we accept the foregoing procedure, the capacity of the pressure vessel need be only slightly greater (of the order of 50 psi) than the capacity of a vessel used in applying external pressure to an empty test shell.

It is hoped that the foregoing discussion will help to clarify questions about the original paper, especially where there may be a tendency to believe that the test is applicable only under certain special conditions. Numerous tests made at the Naval Ordnance Laboratory have successfully demonstrated the general applicability of the method. For the sake of brevity some discussion of this test method has been omitted in this closure and in the original paper, but it may be found in an earlier report, which is available at the Naval Ordnance Laboratory.

The writer is grateful to Dr. Wenk for his stimulating discussion. He would also like to express appreciation to his colleague, Dr. Manford B. Tate, who not only has critically reviewed this discussion but also kindly offered much helpful advice and encouragement with respect to the original work.

# Response of a Cylindrical Shell to a Shock Wave[1]

M. C. Junger.[2] It is believed that certain simplifying assumptions in this analysis may give rise to appreciable errors. The approximation to be discussed here is connected with the differential equations of motion of the shell. In the paper, these equations do not include the coupling effect between flexural and extensional modes for $n > 0$. The importance of this effect increases very rapidly with frequency;[3] this is to be expected, by intuition, since high frequencies excite particularly the high-frequency (i.e., extensional) modes. Since a step-function type of disturbance necessarily has extremely high-frequency components, inevitably it will excite the extensional modes neglected in this analysis. The damping ratios connected with these modes are larger than those connected with the flexural modes.[3] Hence the extensional modes will be damped out at an early stage; the neglect of the coupling effect therefore does not give rise to a significant error in the later stages of the transient response of the shell. However, in the time interval immediately following the incidence of the shock wave, the error may be of the order of 50 per cent for modes $n > 0$. Fortunately, inclusion of the coupling effect introduces only minor complications in the analysis.

### Authors' Closure

The authors agree essentially with Mr. Junger's statement concerning the effect of the extensional modes for $n > 0$, particularly with the conclusion that neglecting them introduces no significant error for the later stages of the response. As a matter of fact, the authors made the approximation for this very reason.

A refined analysis, using the same general approach as the paper but including all extensional effects has been completed recently.[4] While no fundamental difficulty arises, it was found, contrary to the discusser's statement, that the numerical integrations required become much more cumbersome and time-consuming.

# Topics in Gyroscopic Motion[1]

F. B. Jennings.[2] If one wishes to analyze and understand the motion of gyroscopes, the most useful equations are Equations [72] and [74] of the paper. These equations are so im-

[1] By R. D. Mindlin and H. H. Bleich, published in the June, 1953, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 75, pp. 189–195.

[2] Research Fellow in Acoustics, Acoustics Research Laboratory, Harvard University, Cambridge, Mass. Jun. ASME.

[3] "Vibrations of Elastic Shells in a Fluid Medium and the Associated Radiation of Sound," by M. C. Junger, Journal of Applied Mechanics, Trans. ASME, vol. 74, 1952, pp. 439–445.

[4] "A Further Study of the Response of an Elastic Cylindrical Shell to a Transverse Shock Wave," by M. L. Baron, PhD dissertation, Columbia University, New York, N. Y., June, 1953.

[1] By H. Poritsky, published in the March, 1953, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 75, pp. 1–8.

[2] Supervisor of Aircraft Instrument, Development Engineering Unit, General Electric Company, Meter and Instrument Department, West Lynn, Mass.

portant that it will be worth while to investigate their physical significance.

They include the effect of gimbal inertias and apply when the gyroscope's case $R_4$ is stationary. The inner gimbal is assumed to be dynamically balanced, which is not always a valid assumption. Equation [72] gives the total angular momentum of the moving parts

$$\mathbf{M} = \mathbf{i}_2 I \dot{\alpha}_1 + \mathbf{\Phi}\cdot\boldsymbol{\omega}$$

The inner gimbal "effective" inertia dyadic and angular velocity are

$$\mathbf{\Phi} = \mathbf{i}_2\mathbf{i}_2(I + I_2 + K_3) + \mathbf{j}_2\mathbf{j}_2(K + J_2) + \mathbf{k}_2\mathbf{k}_2(K + K_2 + K_3)$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_3 + \boldsymbol{\omega}_2 = -\mathbf{i}_2\dot{\alpha}_3 \sin \alpha_2 + \mathbf{j}_2\dot{\alpha}_2 + \mathbf{k}_2\dot{\alpha}_3 \cos \alpha_2$$

It is interesting to note that the outer gimbal can be considered weightless, if its inertia $K_3$ is added to both the $\mathbf{i}_2$ and $\mathbf{k}_2$ principal inertias of the inner gimbal and locked rotor.

The torques applied to the gyro will be those needed to accelerate and precess the relative spin momentum $\mathbf{i}_2 I\dot{\alpha}_1$ and those needed to move the rigid body whose angular momentum is $\mathbf{\Phi}\cdot\boldsymbol{\omega}$

$$\mathbf{Q} = \dot{\mathbf{M}} = \mathbf{i}_2 I\ddot{\alpha}_1 + \boldsymbol{\omega} \times \mathbf{i}_2 I\dot{\alpha}_1 + \mathbf{\Phi}\cdot\partial\boldsymbol{\omega}/\partial t + \boldsymbol{\omega} \times \mathbf{\Phi}\cdot\boldsymbol{\omega}$$

The partial time derivative gives rate relative to the inner gimbal, $\mathbf{i}_2$, $\mathbf{j}_2$, and $\mathbf{k}_2$ being held constant. It is a useful approximation to ignore the quadratic term $\boldsymbol{\omega} \times \mathbf{\Phi}\cdot\boldsymbol{\omega}$ whenever the spin $\dot{\alpha}_1$ is large compared with $\boldsymbol{\omega}$.

Equations [74] give the torques applied by the rotor-driving motor $Q_{\alpha 1}$, the inner-gimbal torque motor $Q_{\alpha 2}$, and the outer-gimbal torque motor $Q_{\alpha 3}$. Bearing and windage torques are neglected, although they could be included in expressions for the torques. We shall not repeat the author's Equations [74] but instead give the approximate linear equations obtained by ignoring small quadratic terms. An expression for rotor driving torque from a synchronous motor also is included to show that $D\dot{\alpha}_1$ is the main source of damping in a gyro

$$Q_{\alpha 1} = kst - k\alpha_1 - D\dot{\alpha}_1 \approx I\ddot{\alpha}_1 - (I \sin \alpha_2)\ddot{\alpha}_3$$

$$Q_{\alpha 2} = \mathbf{j}_2\cdot\dot{\mathbf{M}} \approx (K + J_2)\ddot{\alpha}_2 + (I\dot{\alpha}_1 \cos \alpha_2)\dot{\alpha}_3$$

$$Q_{\alpha 3} = \mathbf{k}_3\cdot\dot{\mathbf{M}} \approx -(I \sin \alpha_2)\ddot{\alpha}_1 - (I\dot{\alpha}_1 \cos \alpha_2)\dot{\alpha}_2 + \{K_3 + (K + K_2)\cos^2 \alpha_2 + (I + I_2)\sin^2 \alpha_2\}\ \ddot{\alpha}_3$$

Often the motor stiffness $k$ will be great enough to permit us to set $\dot{\alpha}_1 = s =$ const. The bracketed quantities are considered to be constants in the linear equations. With no applied torques, these equations become the same as Equations [94] used by the author. We have found these linearized equations very useful in analyzing such problems as transient response, stability of servosystems, and effect of rotor acceleration.

### Author's Closure

The author wishes to thank Mr. Jennings for his interesting comments regarding the usefulness of the Equations [72] and [74] of the paper.

# On an Iterative Method for Nonlinear Vibrations[1]

K. Klotter.[2] In the writer's opinion, the significance of the paper, which discusses a previous one by J. E. Brock, rests on the following points:

1 The author frees Brock's procedure from some incidental accessories (e.g., the use of a particular method for numerical integration) and he appraises the role of that procedure from an analytical point of view.

2 The author shows that the same treatment may be applied to systems of more than one degree of freedom.

3 He shows, furthermore, that dissipative systems need not be excluded from the treatment.

The method, however, is reviewed exclusively for its own merits, and no comparison is attempted to assess its advantages or disadvantages in regard to other methods available.

### Author's Closure

Professor Klotter has given an admirable summation of the principal goals of the paper. With regard to the relative advantages of this method over others, the author feels that there probably are none when the method is considered to be a one-term analytical approximation. It may, however, have advantages when used as a purely numerical method, similar to Brock's original treatment. The most promising facet of the method is relative to higher approximations. However, as stated in the paper, the general questions of convergence and relation to classical existence theorems have not been examined thoroughly.

# Stress Singularities Resulting From Various Boundary Conditions in Angular Corners of Plates in Extension[1]

Prof. A. Erdélyi has called to the author's attention a paper by Lelia Ricci[2] which tabulates the minimum roots of the complex equation

$$\sin z = \pm kz$$

which equation is of the type occurring for several of the boundary-condition combinations given in the author's paper.

By using Ricci's values to check Fig. 1, a slight error was found in Case 2 between 180 and 360 deg. The following values should be used:

| Degrees | | | | |
|---|---|---|---|---|
| 180 | 225 | 270 | 300 | 360 |
| 1.000 | 0.736 | 0.604 | 0.555 | 0.500 |

The author wishes to thank Professor Erdélyi for this reference as the solution of this eigen equation is quite time-consuming.

[1] By R. E. Roberson, published in the June, 1953, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 75, pp. 237–240.

[2] Professor of Engineering Mechanics, Stanford University, Stanford, Calif.

[1] By M. L. Williams, published in the December, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 526–528.

[2] "Tavola di radici di basso modulo di un'equazione interessante la Scienza delle Costruzioni," by L. Ricci, Pubblicazioni N. 296, Istituto per le Applicazioni del Calcolo, Rome, Italy, 1951.