A Quatum Electrodynamics on a Lattice

Yuji NAKAWAKI

Faculty of Education, Setsunan University, Ikeda, Neyagawa 572

(Received May 7, 1977)

Aiming at incorporating the so-called “ε-separation method” into a consistent and finite formulation of quantum electrodynamics, we construct a quantum electrodynamics on a lattice. Thus we can put a reasonable interpretation on ε. The fermion field on a lattice is described by adopting the gradient operator of Drell et al. which solves the problem concerning degenerate energy levels of fermions on a lattice. An inverse operator for the gradient is defined to introduce vector potentials on a lattice. It is shown that a vector current similar to that of “ε-separation method” is thus obtained. It is shown that Ward identities of vector and axial vector currents are verified with the aid of sea gull terms and the remarkable property that we can shift origin of any loop integrals in the lattice theory. It is also shown that we cannot obtain usual local amplitudes taking the limit a→0 and that we may utilize loop integrals of the lattice theory as devices to regularize singularities of corresponding loops if we calculate loop integrals only over the region where all integral momenta are confined within their principal values.

§ 1. Introduction

In spite of the fact that quantum electrodynamics (Q.E.D.) is the only field theory whose predictions agree very well with experimental data, it has been pointed out that at the very basis there need certain modifications of the conventional formulation of Q.E.D. which gives rise to the paradoxical problems concerning the Goto-Imamura-Schwinger term (G.I.S. term) and the Adler anomaly. To define singular operator products appearing in currents, one often refers to the so-called ε-separation method which introduces a small separation into \( \bar{\phi} \) and \( \phi \) like

\[
\bar{\phi}(x+\epsilon)\gamma\phi(x)\exp\left[ie\int_{x}^{x+\epsilon}d\xi A_{\mu}(\xi)\right].
\]

We think that the devices, however, cannot be regarded as satisfactory as far as one assumes the usual field equation and quantization which especially provide the local current. The Adler anomaly, for example, can be obtained if one handles gauge invariant ε-separated axial vector current carefully. But the derivation never proves that there exists anomaly in the axial vector Ward identity. It rather demonstrates that the modified Ward identity resulting from calculation of the divergence of the ε-separated axial vector current using the usual field equation gives rise to the anomaly in the local limit ε→0. It may be enough expected that such non-local currents in general require both the field equation and the field
quantization to be changed. In this respect Kitazoe, Mugibayashi and the present author constructed an extended formulation of Q.E.D. before\(^5\) with the aid of which we could define a non-local axial vector current which satisfied

\[ \sum_{\sigma} \bar{\psi}(x; \varepsilon) = 2m\psi(x) \gamma_5 \psi(x). \]

Therefore we suppose that Adler anomaly would not exist in a finite and consistent Q.E.D.

In connection with the search for reliable ways to study strong coupling field theories, gauge theories on a lattice have been considered by several authors\(^6\) who aim at explaining the observed properties of hadrons starting with a field theory of quark constituents. We would rather construct in this paper a Q.E.D. on a lattice since we can thereby formulate a finite and consistent non-local field theory in which lattice spacing \(a\) is expected to play a similar role to \(\varepsilon\) of the "\(\varepsilon\)-separation method". Thus we can put a reasonable interpretation on \(\varepsilon\). Besides it seems noteworthy to investigate how the finite and consistent lattice theory solves diseases of the local theory and whether the local limit of the former agrees with the latter or not.

In formulating gauge theories of fermions, there arise two kinds of problems. First one has to solve the question how to introduce gauge fields on a lattice so that the theory has full gauge invariance. The second problem concerns the prescription for describing the fermion field on a lattice. The usual transcription of the gradient as a difference operator leads to the degenerate fermionic degrees of freedom. To solve the latter question, we adopt the prescription of Drell et al.\(^7\) according to which the gradient operator is defined as an infinite number of sum of differences like

\[ \partial \psi(t, n) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{1}{a} \left\{ \psi(t, n + m\varepsilon_i) - \psi(t, n - m\varepsilon_i) \right\}, \quad (l = 1, 2, 3) \]  

(1·1)

where \(\varepsilon_i\) stands for a unit vector in the direction of the \(l\)-th axis. If we substitute the following Fourier transformation:

\[ \psi(t, n) = \frac{1}{(2\pi)^{3/2}} \int_{-\pi/a}^{\pi/a} d^3p \psi(t, \vec{p}) e^{i\vec{p}\cdot\vec{n}} \]  

(1·2)

for \(\psi(t, n)\) on the right-hand-side of (1·1), we can obtain the energy-momentum dispersion relation like

\[ E = \left\{ M^2 + \sum_{l=1}^{3} \hat{p}_l \right\}^{1/2}, \]  

(1·3)

where \(\hat{p}_l\) is a periodic function with a period \(2\pi/a\) like

\[ \hat{p}_l = \frac{i}{a} \sum_{m=0}^{\infty} \frac{(-1)^m}{m} e^{-i(m+1)p_l} = p_l \quad \text{for} \quad -\frac{\pi}{a} < p_l < \frac{\pi}{a}. \]  

(1·4)
We find it convenient below to write
\[
\frac{1}{a} \frac{(-1)^{m-1}}{m} = \frac{-ia}{2\pi} \int_{-\pi/a}^{\pi/a} dk \, ke^{iakm} = D(m)
\]
with which (1.1) is described as
\[
\partial_t \psi(t, n) = \sum_{m=-\infty}^{\infty} D(m) \psi(t, n-m \xi_i) = \sum_{m=-\infty}^{\infty} D(n_t-m) \psi(t, n + (m-n_t) \xi_i).
\]

As for the former question the temporal potential \( A_t(t, n) \) is introduced in the minimal way since time coordinates are continuous. Spatial potentials are introduced so as to construct the following gauge invariant quantity:
\[
e^{-i\partial_t^{-1}A_t(t, n)} \psi(t, n) = \psi^{(l)}(t, n), \quad (l = 1, 2, 3)
\]
where \( \partial_t^{-1} \) is the inverse operator of the gradient (1.1) defined as
\[
\partial_t^{-1} A_t(t, n) = \frac{i}{(2\pi)^{3/2}} \int_{-\pi/a}^{\pi/a} \frac{dk}{k + i\varepsilon} A_t(t, \hat{k}) e^{ia(k \cdot n)}
\]
\[
= \sum_{m=-\infty}^{\infty} \frac{ia}{2\pi} \int_{-\pi/a}^{\pi/a} dk \frac{1}{k + i\varepsilon} e^{ia(k \cdot n_t - m)} A_t(t, n + (m-n_t) \xi_i)
\]
\[
= \sum_{m=-\infty}^{\infty} D^{-1}(n_t - m) A_t(t, n + (m-n_t) \xi_i), \quad (l = 1, 2, 3)
\]
In fact one can confirm that
\[
\partial_t (\partial_t^{-1} A_t(t, n)) = \partial_t^{-1} (\partial_t A_t(t, n)) = A_t(t, n),
\]
if he notes
\[
\partial_t^{-1} D(n_t - m) = \partial_t D^{-1}(n_t - m) = \delta_{n_t m}.
\]

With the aid of (1.7) we can construct the following gauge invariant equation of motion for the fermion field
\[
(i\gamma^0 (\partial_\mu - ieA_\mu(t, n)) - M) \psi(t, n) + i \sum_{l=1}^{3} e^{(t - l)A_l(t, n)} r_l \psi^{(l)}(t, n) = 0.
\]
Note that (1.7) and (1.11) are invariant under the gauge transformation
\[
\psi(t, n) \rightarrow e^{ieA(t, n)} \psi(t, n), \quad A_t(t, n) \rightarrow A_t(t, n) + \partial_t A(t, n).
\]
can define an axial vector current which satisfies

\[ \partial_\mu J_5^\mu (t, n) + \sum_{i=1}^{3} \partial_i J_5^i (t, n) = 2M \bar{\psi} (t, n) \gamma_5 \psi (t, n). \]  (1.13)

Finally we describe Feynman rules in momentum space.

In § 3, we consider the lattice theory in connection with the local theory. It is indicated that only amplitudes without any loop can give rise to the corresponding amplitudes of the local theory.

We show some features of G.I.S. term of two-dimensional Q.E.D. and amplitudes for two photons annihilating into vacuum via the axial vector current, which are examples of amplitudes we cannot obtain the local amplitudes in the limit \( a \to 0 \).

Remarkable feature is that we can shift origin of any loop integrals in the sense that

\[ \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} d^2p f(p_0, p) = \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} d^2p f(p_0 + k, p + k), \]  (1.14)

which guarantees the canonical Ward identity of axial vector current in contrast with the local theory which lacks the same property due to linearly divergent integrals. We show that if we calculate loop integrals only over the region where all absolute values of space components of any momenta appearing in the loops do not exceed \( \pi/a \) we can obtain in the local limit G.I.S. term in two-dimensional Q.E.D. and a constant similar to the Adler anomaly. Therefore we indicate that the lattice theory may be utilized as devices to regularize singularities of the local theory.

Appendix A is devoted to more detailed description of operators \( \partial_i \) and \( \partial_i^{-1} \). In particular we verify the relations

\[ \sum_{n_1=-\infty}^{\infty} a \{ \partial_i f(n) \cdot g(n) + f(n) \cdot \partial_i g(n) \} = 0, \]  (1.15)

\[ \sum_{n_1=-\infty}^{\infty} a \{ \partial_i^{-1} f(n) \cdot g(n) + f(n) \cdot \partial_i^{-1} g(n) \} = 0, \quad (l = 1, 2, 3) \]  (1.16)

which can be used to derive field equations and lattice conserving quantities.

\section{2. Quantum electrodynamics on a lattice}

We start with the following Lagrangian:

\[ L (t, n) = -\frac{1}{2} \sum_{\mu=0}^{3} \{ \partial_\mu A_\mu^a (t, n) \partial_\mu A_\mu^a (t, n) - \sum_{k=1}^{3} \partial_\mu A_\mu^a (t, n) \partial_k A_\mu^a (t, n) \} \]

\[ -\frac{1}{2} \bar{\psi} (t, n) \{ i \gamma^8 (\partial_\mu - i e A_\mu (t, n)) - M \} \psi (t, n) \]

\[ + \frac{i}{2} \sum_{i=1}^{3} \bar{\psi}^{(i)} (t, n) i \gamma^i \partial_\mu \psi^{(i)} (t, n) + \text{h.c.} \]  (2.1)
Let us apply the action principle to \( I = \int_{t_1}^{t_2} \sum_n a^\dagger(t, n) L(t, n) dt \) to derive the field equations. We thus obtain

\[
(\partial_0^2 - \sum_{k=1}^3 \partial_k^2) A^\mu (t, n) = J^\mu (t, n), \tag{2.2}
\]

\[
\{i\gamma_\mu (\partial_0 - ie A_0 (t, n)) - M\} \psi (t, n) + i \sum_{l=1}^3 e^{ie_l} \gamma_\mu A_l(t, n) i\gamma_l \delta_l \Psi^{(l)} (t, n) = 0, \tag{2.3}
\]

where

\[
J^\mu (t, n) = -e \bar{\psi} (t, n) \gamma^\mu \psi (t, n),
\]

\[
J^\mu (t, n) = -e \sum_{m} D^{-1} (n_l - m) \{ \bar{\Psi}^{(n)} (t, n + (m - n_l) \xi_l) \gamma^\mu \delta_l \Psi^{(n)} (t, n + (m - n_l) \xi_l)
\]

\[+ \partial_t \bar{\Psi}^{(n)} (t, n + (m - n_l) \xi_l) \gamma^\mu \Psi^{(n)} (t, n + (m - n_l) \xi_l) \}. \tag{2.4}
\]

To derive these equations (1.15) and (1.16) are made use of. Current conservation is verified if we note (1.10), namely

\[
\partial_0 D^{-1} (n_l - m) = \partial_{n_l m}.
\]

Hence it follows that

\[
\partial_0 J^\mu (t, n) + \sum_{l=1}^3 \partial_l J^l (t, n) = 0. \tag{2.5}
\]

Energy conservation

\[
\partial_0 T^{00} (t, n) + \sum_{l=1}^3 \partial_l T^{l0} (t, n) = 0 \tag{2.6}
\]

is also verified if we put the following in (2.6):

\[
T^{00} (t, n) = -\frac{1}{2} \sum_{n=1}^3 \{ \partial^\mu A_\mu (t, n) \partial^\nu A^\nu (t, n) + \sum_{k=1}^3 \partial_k A_\mu (t, n) \partial_k A^\mu (t, n) \}
\]

\[+ \frac{i}{2} \{ \bar{\psi} (t, n) \gamma^\mu \partial^\nu \psi (t, n) - \partial^\nu \bar{\psi} (t, n) \gamma^\mu \psi (t, n) \}, \tag{2.7}
\]

\[
T^{00} (t, n) = \sum_{n=1}^3 D^{-1} (n_l - m) F^{00} (t, n + (m - n_l) \xi_l), \tag{2.8}
\]

where

\[
F^{00} (t, n) = -\{ \partial_0^2 A_\mu (t, n) \partial^\mu A^\mu (t, n) + \partial_1 A_\mu (t, n) \partial_1 \partial^\mu A^\mu (t, n) \}
\]

\[= \frac{1}{2} \{ i \partial_1 \bar{\Psi}^{(n)} (t, n) \gamma^\mu \delta_l \Psi^{(n)} (t, n) + i \bar{\Psi}^{(n)} (t, n) \gamma^\mu \delta_l \Psi^{(n)} (t, n) + h.c. \}
\]

\[+ J^1 (t, n) \partial^\mu A^\mu (t, n) + \partial_1 J^l (t, n) \partial_1 \partial^\mu A^\mu (t, n). \tag{2.9}
\]

We thus obtain time independent Hamiltonian
A Quantum Electrodynamics on a Lattice

\[ H = \sum_n a^\dagger T^{\phi}(t, n) = \sum_n a^\dagger \left[ -\frac{1}{2} \sum_{\rho=0}^3 \{ \partial^\rho A_\rho(t, n) \partial^\rho A^\rho(t, n) \\
+ \sum_{\kappa=1}^3 \partial_\kappa A_\kappa(t, n) \partial_\kappa A^\kappa(t, n) \} + J^\phi(t, n) A_0(t, n) + M \tilde{\phi}(t, n) \phi(t, n) \right] \\
- \frac{i}{2} \sum_{\kappa=1}^3 \{ \tilde{\mathcal{T}}^{(\kappa)}(t, n) \gamma^\kappa \tilde{\mathcal{T}}^{(\kappa)}(t, n) - \partial_\kappa \tilde{\mathcal{T}}^{(\kappa)}(t, n) \gamma^\kappa \tilde{\mathcal{T}}^{(\kappa)}(t, n) \} \right]. \]

(2.10)

With the aid of (2.10) and canonical commutation relations, i.e.,

\[ [A_\rho(t, n), A_\sigma(t, n')] = [\partial_\rho A_\rho(t, n), \partial_\sigma A_\sigma(t, n')] = 0, \]

\[ [\partial_\rho A_\rho(t, n), A_\sigma(t, n')] = ig_{\rho\sigma} \frac{\delta^{nn'}}{a^2}, \quad (g_{\rho\sigma} = 1, -1, -1, -1) \]    

(2.11)

\[ \{ \phi_a(t, n), \phi_\beta(t, n') \} = \{ \phi_a^*(t, n), \phi_\beta^*(t, n') \} = 0, \]

\[ \{ \phi_a(t, n), \phi_\beta^*(t, n') \} = \delta_{a\beta} \frac{\delta^{nn'}}{a^2}, \]    

(2.12)

Hamiltonian formalism can be formulated.

Let us proceed to the interaction picture, for it is advantageous to work in that picture for demonstrating features of our present theory in contrast with the local theory.

Interaction Hamiltonian turns out to be

\[ H_I(t) = \sum_n a^\dagger \left[ :J_\phi(t, n): A_0(t, n) - \frac{1}{2} \sum_m D(m) \cdot \frac{i}{2} \sum_{\kappa=1}^3 \{ \tilde{\phi}(t, n), \gamma^\kappa \phi(t, n - me) \} \right] \]

\[ \times \left( \exp \left[ i e \partial_\tau^{-1} A_1(t, n) - i e \partial_\tau^{-1} A_1(t, n - me) \right] - 1 \right) + \text{h.c.}, \]    

(2.13)

where the bilinear products of the spinor field are antisymmetrized to guarantee the charge conjugation invariance of the Hamiltonian. In (2.13) \( \phi(t, n) \) stands for the following solution of the free field equation:

\[ \phi(t, n) = \frac{1}{(2\pi)^{3/2}} \int_{-\pi/a}^{\pi/a} d\tau p \sqrt{\frac{M^2}{\rho_0}} \sum_{r=1}^2 \{ b^{(r)}(\hat{p}) u^{(r)}(\hat{p}) e^{-i(\rho_0 - \rho' - p \cdot n)} \]

\[ + d'^{(r)}(\hat{p}) v^{(r)}(\hat{p}) e^{i(\rho_0 - \rho' - p \cdot n)} \} \]    

(2.14)

with \( \rho_0 = (M^2 + \sum_{i=1}^3 \hat{p}_i^2)^{1/2} \), spinors satisfying

\[ \{ \gamma^\rho \rho_0 + \sum_{\kappa=1}^3 \gamma^\kappa \hat{p}_\kappa - M \} u^{(r)}(\hat{p}) = (\gamma \cdot p - M) u^{(r)}(\hat{p}) = 0, \]

\[ (\gamma \cdot p + M) v^{(r)}(\hat{p}) = 0 \quad (r = 1, 2) \]    

(2.15)

and the operators quantized by

\[ \{ b^{(r)}(\hat{p}), b^{(s)}(\hat{q}) \} = \{ d^{(r)}(\hat{p}), d^{(s)}(\hat{q}) \} = \delta_{rs} \sum_n \delta^{(3)}(p - q + \frac{2\pi}{a} n). \]    

(2.16)
Similarly \( A_\mu(t, n) \) expresses the solution

\[
A_\mu(t, n) = \frac{1}{(2\pi)^{3/2}} \int_{-\pi/\alpha}^{\pi/\alpha} \frac{d^3 k}{\sqrt{2k_0}} \{ a_\mu(k) e^{-i(k\cdot t - a \cdot k \cdot n)} + a_\mu^*(k) e^{i(k\cdot t - a \cdot k \cdot n)} \},
\]

where \( k_0 = \{ \sum_{i=1}^3 k_i \}^{1/2} \). It is quantized by

\[
[a_\mu(k), a_\nu(k')] = [a_\mu^*(k), a_\nu^*(k')] = 0,
\]

\[
[a_\mu(k), a_\nu^*(k')] = -g_{\mu\nu} \sum_n \delta^{(3)}(k - k' + \frac{2\pi}{a} n).
\] (2.18)

Now we can obtain any scattering amplitudes we want by estimating elements of \( S \) matrix

\[
S = \sum_{n=1}^\infty (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n T(H_f(t_1)H_f(t_2)\cdots H_f(t_n)),
\]

(2.19)

features of which are that to the lowest order in the expansion it gives rise to the simultaneous emission or absorption of any number of spacelike polarized photons at a vertex as indicated in Fig. 1(a) and to the tad pole terms, as indicated in Figs. 1(b) and (c), which result from non-vanishing vacuum expectation value of the non-local and non-linear interaction Hamiltonian (2.13). These are colloquially called the sea gull terms and supplement amplitudes derived from terms of higher order in the expansion (2.19) to preserve gauge invariance of a full amplitude. More definitely gauge invariance is maintained by a cancellation between the divergence of the sea gull term and the Schwinger term originating from the divergence of \( T \)-product to currents. We again refer to this point below.

To discuss Adler anomaly besides the vector current there needs an axial vector current on a lattice. We require it to be invariant under the local gauge transformation (1.12) and to satisfy

\[
\partial_\mu J_5^\mu(t, n) + \sum_{l=1}^3 \partial_l J_5^l(t, n) = 2M\bar{\psi}(t, n)\gamma_\mu\gamma^\mu\psi(t, n).
\] (2.20)

We take the following as such a current

\[
J_5^\mu(t, n) = \frac{1}{2} [\bar{\psi}(t, n) i\gamma_5 \gamma^\mu, \psi(t, n)],
\]

\[
J_5^l(t, n) = \frac{1}{2} \sum_m D^{-1}(m) \{ [\partial_l \bar{\psi}^{(l)}(t, n-m\xi_l) i\gamma^{l'} \gamma^l, \bar{\psi}^{(l)}(t, n-m\xi_l)] + [\bar{\psi}^{(l)}(t, n-m\xi_l) i\gamma^{l'} \gamma^l, \partial_l \bar{\psi}^{(l)}(t, n-m\xi_l)] \} \quad (l=1, 2, 3)
\] (2.21)

where bilinear products of the spinor field are antisymmetrized to preserve \( C \)

\[\text{(a)}\]  \[\text{(b)}\]  \[\text{(c)}\]

Fig. 1.
invariance.

Now we can derive Feynman rules to write down the matrix element corresponding to any given Feynman diagram. We describe only rules useful in the next section to demonstrate validity of Ward identity of the axial vector current \((\mathbf{2} \cdot \mathbf{A})\). To simplify the description in momentum space, the following abbreviations are made use of:

1. \( p_{\mu} \) for \((p_\alpha, \tilde{p}_\alpha)\), \((\mu = 1, 2, 3)\)
2. \([\mathbf{p} + \mathbf{k}]_\mu = (p_\alpha + k_\alpha) e^{-i \alpha \mathbf{m} (p_\alpha + k_\alpha)} \) for \(\mu = 0\),
3. \( \delta^{(i)} (\mathbf{p}_2 - \mathbf{p}_1 - \mathbf{k}) \) for \(\sum \delta (\mathbf{p}_{20} - \mathbf{p}_{10} - \mathbf{k}_0) \delta^{(i)} \left( \mathbf{p}_2 - \mathbf{p}_1 - \mathbf{k} + \frac{2\pi}{a} \mathbf{n} \right) \) \((2.22)\)

One gives

(a) a factor
\[
\Gamma^\mu (p_2; p_1 \pm k) = -e \gamma^\mu \frac{p_2 \mp p_1 \mp k}{k_\mu},
\]
for each vertex where one photon line of polarization \(\mu\) and momentum \(k\) is absorbed or emitted (the \(+\) or \(-\) sign is used depending on whether the photon is absorbed or emitted) and \(p_1\) and \(p_2\) are the momenta of the electron line coming in and going out of the vertex respectively;

(b) a factor
\[
\Gamma^{\mu \nu} (p_2; p_1 \pm k_1 \pm k_2) = -\frac{e^2}{\pm k_{1\mu}} \gamma^\mu \left( \gamma^\nu \gamma^0 - \gamma^\nu \gamma^\nu \right) \left( \frac{p_{2\nu} - [p_2 \mp k_2]}{\pm k_{2\nu}} \right) \\
- \frac{[p_1 \mp k_1 \mp k_2]}{\pm k_{2\nu}} \delta^{(i)} (p_2 - p_1 \mp k_1 \mp k_2)
\]
for each vertex where two photon lines of polarization \(\mu\) and \(\nu\) and of the momenta \(k_1\) and \(k_2\) are absorbed or emitted;

(c) a factor
\[
\Gamma_5 (p_2; p_1 \pm k) = \gamma_5 \delta^{(i)} (p_2 - p_1 \mp k)
\]
for each pseudoscalar vertex;

(d) a factor
\[
\Gamma_5^\nu (p_2; p_1 \pm k) = i \gamma_5 \gamma^\nu \frac{p_{2\mu} - p_{1\mu}}{\pm k_{1\nu}} \delta^{(i)} (p_2 - p_1 \mp k)
\]
for each axial vector vertex;

(e) a factor
\[
\Gamma_5^{\nu \mu} (p_2; p_1 \pm k \pm k_1) = -\frac{e}{\pm k_{1\mu}} \gamma^\nu \gamma^0 - \gamma^\nu \gamma^\nu \gamma^\nu \gamma^\nu \gamma^\mu
\]
for each vertex where one axial vector line of polarization $\nu$ and momentum $k$ is absorbed or emitted and one photon line of polarization $\mu$ and momentum $k_1$ is absorbed or emitted;

(f) a factor $(i/(2\pi)^4)(1/\gamma\cdot p - M + i\varepsilon)$ for each internal electron line of momentum $p$;

(g) a factor $(-1)$ for each closed electron loop;

(h) a factor $(-i)^n$ corresponding to the $n$-th order term in the perturbation expansion;

(i) one is to integrate over the momenta of all the internal lines.

Before closing this section we verify the relation which connects one photon vertex (2.23) with that of two photons (2.24), with the aid of which to order of $\epsilon^2$ the cancellation between the divergence of the sea gull term and the Schwinger term can be shown. We make use of the following identities

\[
\sum_{n=-1}^{1} \sum_{n=-1}^{1} \int_{-\pi/a}^{\pi/a} dp \delta\left( p_1 + q_1 - p + \frac{2\pi}{a} n \right) \delta\left( p - p_1 - q_3 + \frac{2\pi}{a} n \right) = \sum_{n=-1}^{1} \delta\left( p_1 + q_1 - p_3 - q_3 + \frac{2\pi}{a} n \right),
\]

\[
\sum_{n=-1}^{1} \sum_{n=-1}^{1} \int_{-\pi/a}^{\pi/a} dp \delta\left( p_1 + q_1 - p + \frac{2\pi}{a} m \right) \delta\left( p - p_1 - q_3 + \frac{2\pi}{a} n \right) = \sum_{n=-1}^{1} \left[ p_1 + q_1 \right] \delta\left( p_1 + q_1 - p_3 - q_3 + \frac{2\pi}{a} m \right)
\]

and similar identities for time components to rewrite (2.24) after some manipulations as

\[
\Gamma^{\nu\mu}(p_1; p_1 \pm k_1 \pm k_2) = -\frac{ie}{\pm k_{\mu}} (g^{\rho\sigma}g^{\nu\sigma} - g^{\rho\nu}) \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} d^3p
\]

\[
\times \left\{ \Gamma^{\nu}(p_2; p \pm k_2) \delta^{(4)}(p_1 + k_1 - p) - \Gamma^{\nu}(p_1 + k_1 - p) \delta^{(4)}(p - p_2 \pm k_1) \right\}.
\]

Note that similar considerations derive

\[
\Gamma_{5\nu}^{\mu}(p_2; p_1 \pm k \pm k_1) = \frac{e}{\pm k_{\nu}} (g^{\rho\sigma}g^{\nu\sigma} - g^{\rho\nu}) \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} d^3p
\]

\[
\times \left\{ i\Gamma_{5}(p_2; p \pm k_1) \Gamma^{\nu}(p_1; p \pm k_1) - i\Gamma_{5}(p_1; p \pm k_1) \Gamma^{\nu}(p_2; p \pm k_1) \right\}
\]

or

\[
\Gamma_{5\nu}^{\mu}(p_2; p \pm k \pm k_1) = \frac{e}{\pm k_{\nu}} (g^{\rho\sigma}g^{\nu\sigma} - g^{\rho\nu}) \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} d^3p
\]

\[
\times \left\{ \Gamma_{5}(p_2; p \pm k) \delta^{(4)}(p - p_1 \mp k_1) - \Gamma_{5}(p_1; p \pm k) \delta^{(4)}(p_2 - p \mp k_1) \right\}.
\]
§ 3. The lattice amplitudes with loops

In this section we show some features of the amplitudes with loops to clarify differences between the lattice amplitudes and those of the local theory. And we indicate possibilities to utilize some of them as devices to regularize singularities of the local theory gauge invariantly.

First of all we note that we can obtain amplitudes of the local theory from those of the lattice theory taking the local limit \( a \to 0 \) only in the case that the latter ones do not contain any internal or external lines with absolute values of space components larger than \( \pi/a \). For example, the Compton scattering amplitude

\[
T'^{m}(p_2 k_2; p_1 k_1) = \frac{1}{\gamma \cdot (p_1 + k_1) - M + i\varepsilon} \frac{1}{\gamma \cdot (p_2 - k_1) - M + i\varepsilon} \Gamma^{m}(p + k_1; p_i k_i) 
+ \Gamma^{m}(p_2; p_2 - k_1 k_1) 
+ \Gamma^{m}(p_2 k_2; p_1 k_1)
\]

turns out to be

\[
T'^{m}(p_2 k_2; p_1 k_1) = \gamma^{\mu} \frac{1}{\gamma \cdot (p_1 + k_1) - M + i\varepsilon} \gamma^{\mu} + \gamma^{\nu} \frac{1}{\gamma \cdot (p_2 - k_1) - M + i\varepsilon} \gamma^{\nu}, \tag{3.1}
\]

provided that \(|p_1| < (\pi/a)\), \(|k_1| < (\pi/a)\), \(|p_1 + k_1| < (\pi/a)\) \((i=1, 2)\) and \(|p_2 - k_1| = |p_1 - k_2| < (\pi/a)\). Note that we can always let external and internal momenta except those of loops satisfy above condition if we take suitably small \( a \). However, momenta of any loop take inevitably values exceeding \( \pi/a \) which thus gives rise to terms not found in the local amplitudes. To clarify this point we demonstrate G.I.S. term of two-dimensional Q.E.D. and the amplitudes for two photons annihilating into vacuum via the axial vector current \((2 \cdot 21)\).

G.I.S. term of two-dimensional Q.E.D. can be obtained by calculating divergence of the amplitude

\[
\Pi^{m}(k) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} dp \text{ Tr} \left[ \Gamma^{m}(pk; p + k) \frac{1}{\gamma \cdot [p + k] - M + i\varepsilon} \Gamma^{m}(p + k; pk) \frac{1}{\gamma \cdot p - M + i\varepsilon} \right]. \tag{3.3}
\]

Making use of the property that we can shift origin of any loop integrals in the sense that

\[
\int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} dp F(p_0, p) = \int_{-\infty}^{\infty} dp_0 \int_{-\pi/a}^{\pi/a} dp F(p_0 + k_3, p + k), \tag{3.4}
\]

it is verified that \( \sum_{\mu=1}^{3} k_{\mu} \Pi^{m}(k) = 0 \). Equation (3.4) holds on account of the property that only time component extends to infinity and space components are periodic function defined by (1.4). For the case \( \nu = 1 \), we obtain
\[
\sum_{\alpha=0}^{1} k_{\alpha} \Pi_{\alpha}(k) = \frac{-ie^2}{2\pi} \int_{-\pi/a}^{\pi/a} dp \left\{ \frac{p - [p+k]}{\sqrt{p^2 + M^2} - \sqrt{(p+k)^2 + M^2}} \right\}.
\] (3.5)

Considering the fact that
\[
[p+k] = \begin{cases} 
  p+k & \text{for } 0 < k < \frac{\pi}{a}, -\frac{\pi}{a} < p < \frac{\pi}{a} - k, \\
  p+k - \frac{2\pi}{a} & \text{for } 0 < k < \frac{\pi}{a}, \frac{\pi}{a} - k < p < \frac{\pi}{a},
\end{cases}
\]
we evaluate (3.5) as follows:
\[
\sum_{\alpha=0}^{1} k_{\alpha} \Pi_{\alpha}(k) = \frac{-ie^2}{2\pi} \int_{-\pi/a}^{\pi/a} dp \left\{ \frac{p - [p+k]}{\sqrt{p^2 + M^2} - \sqrt{(p+k)^2 + M^2}} \right\}
\]
\[
+ \frac{ie^2}{2\pi} \frac{k - (2\pi/a)}{k} \int_{(\pi/a) - k}^{\pi/a} dp \left\{ \frac{p - [p+k] - (2\pi/a)}{\sqrt{p^2 + M^2} - \sqrt{(p+k - (2\pi/a))^2 + M^2}} \right\}
\]
\[
= - \frac{ie^2}{\pi} \left\{ \sqrt{\left(\frac{\pi}{a}\right)^2 + M^2} - \sqrt{\left(\frac{\pi}{a} - k\right)^2 + M^2} \right\} \left(1 - \frac{k - (2\pi/a)}{k}\right).
\] (3.6)

We thus see that though (3.7) diverges as a whole as \(a \to 0\) the term in the curly bracket of (3.7) gives rise to G.I.S. term of the local theory, since
\[
\lim_{a \to 0} - \frac{ie^2}{\pi} \left\{ \sqrt{\left(\frac{\pi}{a}\right)^2 + M^2} - \sqrt{\left(\frac{\pi}{a} - k\right)^2 + M^2} \right\} = - \frac{ie^2}{\pi} k.
\] (3.8)

Now we describe the amplitudes for two photons annihilating into vacuum via the axial vector current (2.21). Diagrams we have to consider are the following.


Fig. 2.

It follows from this that
\[
T^{\rho\sigma}(k; k_1 k_2) = -i \int d^4p \int d^4q \int d^4r.
\]
\[ \times \text{Tr} \left[ \frac{\gamma \cdot p + M}{p^2 - M^2 + i\varepsilon} \Gamma^{\mu}_{\mu}(p + k; q) \frac{\gamma \cdot q + M}{q^2 - M^2 + i\varepsilon} \Gamma^{\nu}(q; r + k) \right] \]

\[ \times \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}(r; p + k) + \frac{\gamma \cdot p + M}{p^2 - M^2 + i\varepsilon} \Gamma^{\mu}_{\mu}(p + k; q) \]

\[ \times \frac{\gamma \cdot q + M}{q^2 - M^2 + i\varepsilon} \Gamma^{\mu}(q; r + k) \left[ \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}\right] \]

\[ -i \int d^4 p \int d^4 q \text{Tr} \left[ \frac{\gamma \cdot p + M}{p^2 - M^2 + i\varepsilon} \Gamma^{\mu}(p + k; q) \frac{\gamma \cdot q + M}{q^2 - M^2 + i\varepsilon} \Gamma^{\mu}(q; p + k + k) \right] \]

\[ -i \int d^4 q \int d^4 r \text{Tr} \left[ \Gamma^{\mu}\Gamma^{\mu}(r + k; q + k) \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}(q; r + k) \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}(r; p + k) \right] \]

\[ + \Gamma^{\mu}\Gamma^{\mu}(r + k; q + k) \left[ \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}(q; r + k) \frac{\gamma \cdot r + M}{r^2 - M^2 + i\varepsilon} \Gamma^{\mu}(r; p + k) \right] \]

(3.9)

A) It is not difficult to show gauge invariances and axial vector Ward identity of (3.9) since we can use (3.4) and relations (2.30) \sim (2.32). Therefore Adler anomaly dose not exist in the present lattice theory to the second order in \( e \) in the sense that the perturbative calculation of the axial vector divergence agrees with that of evaluation by means of the formal application of lattice field equations.

B) All sea gull terms of (3.9) vanish exactly. For example, the term on the third line can be redescribed as

\[ -i e \int_{-\infty}^{\infty} d p_0 \int_{-z/a}^{z/a} d^4 p \text{Tr} \left[ \frac{1}{\gamma \cdot p - M + i\varepsilon} \Gamma^{\mu}(p; p + k) \frac{1}{\gamma \cdot \left[ p + k \right] - M + i\varepsilon} \right] \]

\[ \times \frac{g^{\mu\nu}g^{\mu\nu} - g^{\mu\nu}}{k^2} \{ \Gamma^{\mu}(p + k; p + k) - \Gamma^{\mu}(p + k; p + k) \} \],

(3.10)

which equals zero since

\[ 0 = -i \int_{-\infty}^{\infty} d p_0 \int_{-z/a}^{z/a} d^4 p \text{Tr} \left[ \frac{1}{\gamma \cdot p - M + i\varepsilon} \Gamma^{\mu}(p; p + k) \right] \]

\[ \times \frac{1}{\gamma \cdot \left[ p + k \right] - M + i\varepsilon} \Gamma^{\mu}(p + k; p + k) \],

(3.11)

The right-hand-side of the first term of (3.11) is evaluated to be

\[ \sum_{a, \delta = 0} \frac{e^{\mu\nu\alpha\beta}}{a} \int_{-\infty}^{\infty} d p_0 \int_{-z/a}^{z/a} d^4 p \frac{\hat{p}_{\mu} \cdot \hat{p}_{\nu}}{\hat{p}_{\mu}} \frac{\hat{p}_{\mu}}{k_2} \frac{\hat{p}_{\mu}}{k_2} \frac{\hat{p}_{\mu}}{k_2} \]

\[ \times \frac{\hat{p}_{\mu} \cdot \hat{p}_{\mu}}{\left[ \hat{p}^2 - M^2 + i\varepsilon \right]} \left[ \left[ \hat{p} + k \right]^2 - M^2 + i\varepsilon \right] \]
which becomes zero on account of the property

\[ I_{ab} = \int dp_x \int dp_y \frac{p_x[p+k]_x - p_x[p+k]_y}{(p^2 - M^2 + i\varepsilon)(p^2 - M^2 + i\varepsilon)} = 0 \quad (3.12) \]

as is verified in Appendix B. The second term of (3.11) is also proved with the aid of (3.12).

C) Integrals of the local theory corresponding to (3.11) diverge linearly and it matters where we put origin of integrating momentum. It is deserving to note that in the local theory the following linearly divergent integral does not vanish and results in a finite constant like

\[ \int_{-\infty}^{\infty} dp \int_{-\pi/a}^{\pi/a} d^4p \frac{p_x-k_x}{[(p-k)^2-M^2+i\varepsilon]^2} = -\frac{i\pi^2}{2} k_x \quad (3.13) \]

in contrast with

\[ \int_{-\infty}^{\infty} dp \int_{-\pi/a}^{\pi/a} d^4p \frac{[p-k]_x}{[(p-k)^2-M^2+i\varepsilon]^2} = 0 \ , \quad (3.14) \]

which, however, gives rise to a similar constant as \( a \to 0 \) when we integrate (3.14) with \( \mu = i(l=1,2,3) \) only in the interval

\[-\frac{\pi}{a} + k_1 < p_1 < \frac{\pi}{a} \quad \text{for} \quad 0 < k_1 < \frac{\pi}{a} \]

or

\[-\frac{\pi}{a} < p < \frac{\pi}{a} + k_1 \quad \text{for} \quad -\frac{\pi}{a} < k_1 < 0 , \quad (3.15)\]

where the absolute value of argument of the periodic function \( [p-k]_1 \) does not exceed \( \pi/a \); namely for \( 0 < k_1 < \pi/a \)

\[ \lim_{a \to 0} \int_{-\infty}^{\infty} dp_1 \int_{-\pi/a}^{\pi/a} d^3p_1 \frac{p_1-k_1}{[(p-k)^2-M^2+i\varepsilon]^2} = \frac{-i\pi^2}{3} k_1 . \quad (3.16) \]

Note that we cannot derive any constant with \( \mu = 0 \) since time and space components have not an equal footing and only time component extends into infinity. We think that the difference between constants of (3.13) and (3.16) results from the same origin.

Thus we have seen that principal parts of the lattice amplitudes (parts where any internal or external lines do not have their space components with absolute values larger than \( \pi/a \)) may be connected with the corresponding local amplitudes and that we may utilize amplitudes of loops of the lattice theory as the devices to regularize singularities of the usual Q.E.D.

Acknowledgement

It is with great pleasure that the author acknowledges invaluable discussions with Dr. T. Kitazoe to complete this paper.
Appendix A

The gradient operator a la Drell et al. is defined for an exponential function as

\[ i \partial e^{iakn} = \hat{k}(k)e^{iakn}, \quad (n = 0, \pm 1, \pm 2, \cdots) \]  

(A·1)

where the eigenvalue \( \hat{k}(k) \) has to satisfy

\[ \hat{k}(k) = \hat{k}(k + \frac{2\pi}{a}m), \quad (m = \pm 1, \pm 2, \cdots) \]  

(A·2)

since

\[ e^{iakn} = e^{iak(2nm/a)n}. \]  

(A·3)

The eigenvalue serving our purpose is one which satisfies

\[ \hat{k}(k) = k \quad \text{for} \quad -\frac{\pi}{a} < k < \frac{\pi}{a}. \]  

(A·4)

It follows that

\[ \hat{k}(k) = \begin{cases} 
\frac{2}{a} \sum_{m} \frac{(-1)^{m-1}}{m} \sin(akm) & \text{for} \quad k = \frac{\pi}{a} + \frac{2\pi l}{a} \\
\frac{c}{a} & \text{for} \quad k = \frac{\pi}{a} + \frac{2\pi l}{a}. \quad (l = 0, \pm 1, \pm 2, \cdots)
\end{cases} \]  

(A·5)

We can verify that \( c \) has to be zero if we restrict ourselves to the case that \( c \) is a finite constant. For that purpose we require that the gradient operator should satisfy

\[ i \sum_{n=-\infty}^{\infty} \partial f(n) \cdot g(n) = -i \sum_{n=-\infty}^{\infty} f(n) \cdot \partial g(n). \]  

(A·6)

In the case that

\[ f(n) = \frac{1}{\sqrt{2\pi}} e^{iaqn}, \quad g(n) = \frac{1}{\sqrt{2\pi}} e^{iaqn}, \]  

(A·6) becomes

\[ \sum_{n=-\infty}^{n} (\hat{k}(p) + \hat{k}(q)) \frac{a}{2\pi} e^{ia(p+q)n} = \sum_{n} (\hat{k}(p) + \hat{k}(q)) \delta(p + q + \frac{2\pi}{a}) = 0 \]  

(A·7)

which demands

\[ \hat{k}\left(\frac{\pi}{a}\right) + \hat{k}\left(-\frac{\pi}{a}\right) = 2\hat{k}\left(\frac{\pi}{a}\right) = 0 \rightarrow c = 0. \]  

(A·8)

We generalize (A·1) to get a more convenient form for general functions. The gradient operator is applied for a function
\[ f(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/a}^{\pi/a} dk \hat{f}(k) e^{iakn} \]

with
\[ \hat{f}(k) = \hat{f}\left(k + \frac{2\pi}{a}m\right), \quad (m = \pm 1, \pm 2, \ldots) \]  
(A·9)

to get
\[ i\partial f(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/a}^{\pi/a} dk \hat{k} \hat{f}(k) e^{iakn}. \]  
(A·10)

If we substitute
\[ \hat{f}(k) = \frac{a}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-iakm} f(m) \]  
(A·11)

for \( \hat{f}(k) \) in the integrand of (A·10), we obtain
\[ \partial f(n) = \sum_{m=-\infty}^{\infty} -i \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \hat{k} e^{iak(n-m)} \cdot f(m) \]
\[ = \sum_{m=-\infty}^{\infty} D(n-m) f(m) = -\frac{1}{a} \sum_{m=-\infty}^{\infty} (-1)^{m-n} f(m). \]  
(A·12)

We can now verify (A·6) with the aid of (A·12) as follows:
\[ \sum_n \partial f(n) \cdot g(n) = \sum_n \left( \sum_m D(n-m) f(m) \right) \cdot g(n) \]
\[ = -\sum_m f(m) \left( \sum_n D(m-n) g(n) \right) = -\sum_m f(m) \cdot \partial g(m), \]
where we have made use of the property \( D(n-m) = -D(m-n) \).

The inverse operator for the gradient is likewise defined although it requires a few cares. We describe it as
\[ \partial^{-1} f(n) = \frac{i}{\sqrt{2\pi}} \int_{-\pi/a}^{\pi/a} dk \frac{1}{k} \hat{f}(k) e^{iakn}, \]  
(A·13)

where \( \hat{f}(k) \) satisfies besides (A·9)
\[ \lim_{k \to 0} \hat{f}(k) = \hat{f}(0) = 0, \]
\[ \lim_{k \to \pi/a} \hat{f}(k) = \hat{f}\left(\frac{\pi}{a}\right) \quad \text{or otherwise} \quad \hat{f}\left(\frac{\pi}{a}\right) = 0 \]  
(A·14)
in order to get unambiguously
\[ \partial \partial^{-1} f(n) = \partial^{-1} \partial f(n) = f(n). \]  
(A·15)

(A·13) is also rewritten as
\[ \partial^{-1} f(n) = \sum_{m} \frac{ia}{2\pi} \int_{-\pi/a}^{\pi/a} dk \frac{1}{k} e^{iak(n-m)} \cdot f(m) \]
A Quantum Electrodynamics on a Lattice

\[ = \sum_{m} D^{-1}(n - m) \cdot f(m), \quad \text{(A.16)} \]

which can be used to show

\[ \sum_{n} \{ \partial^{-1} f(n) \cdot g(n) + f(n) \cdot \partial^{-1} g(n) \} = 0. \quad \text{(A.17)} \]

Appendix B

— Verification of (3.12) —

What we have to do is to show

\[ I_{ab} = \int dq \int dq = \frac{q_{a}[q+k]_{\beta}-q_{\beta}[q+k]_{a}}{[q^2-M^2+i\varepsilon][q+k]^2-M^2+i\varepsilon]}. \quad \text{(B.1)} \]

whose left-hand-side can be rewritten with the aid of the Feynman parameter integral as

\[ I_{ab} = \int_{0}^{1} dz \int dq = \frac{q_{a}[q+k]_{\beta}-q_{\beta}[q+k]_{a}}{[q^2-M^2+i\varepsilon+z([q+k]^2-q^2)]}. \quad \text{(B.2)} \]

Since the numerator is described as

\[ \begin{align*}
q_{a}[q+k]_{\beta}-q_{\beta}[q+k]_{a} \\
= \{q_{a}(1-z)+z[q+k]_{a}\} \cdot \{q+k]_{\beta}-q_{\beta}\} \\
+ \{q_{\beta}(1-z)+z[q+k]_{\beta}\} \cdot \{q_{a}-[q+k]_{a}\}
\end{align*} \quad \text{(B.3)} \]

to prove (B.1), it is sufficient to show that

\[ I_{c} = \int dq \frac{1}{[q^2-M^2+i\varepsilon+z([q+k]_{a}-q^2)]} \quad (\gamma=\alpha \text{ or } \beta) \quad \text{(B.4)} \]

vanishes. We can rewrite (B.4) as

\[ I_{c} = -\frac{1}{2} \int dq \cdot \frac{1}{dq} \frac{1}{[q^2-M^2+i\varepsilon+z([q+k]_{a}-q^2)]} \]

and hence

\[ \begin{align*}
I_{c} &= -\frac{1}{2} \left[ \frac{1}{[q^2-M^2+i\varepsilon+z([q+k]_{a}-q^2)]} \right]_{-\infty}^{\infty} = 0 \quad \text{for } \gamma = 0 \\
-\frac{1}{2} \left[ \frac{1}{\varepsilon/\alpha} \right]_{-\varepsilon/\alpha}^{\varepsilon/\alpha} &= 0 \quad \text{for } \gamma = 1, 2, 3, \quad \text{(B.5)}
\end{align*} \]

where in the second line we have used the property of the periodical function.

References

3) J. Schwinger, Phys. Rev. 82 (1951), 664.
4) J. Schwinger, Phys. Rev. 82 (1951), 664.

Note added in proof: If we want to derive Dirac equation on a lattice which is free from the problem concerning degenerate energy levels, it seems necessary to introduce an infinite number of difference operators. For example, Klein-Gordon equation usually used on a lattice

\[ (\partial_t^2 - \sum_\mathbf{r} \mathcal{P} \cdot (\mathbf{r} \cdot \mathbf{p}) \cdot \mathbf{r}) \psi(t, \mathbf{n}) = 0 \]

can be factorized a la Dirac as

\[ (i \gamma^0 \partial_t + \sum_\mathbf{r} i \gamma^0 \mathcal{P} \cdot \mathbf{r} \cdot \mathbf{p}) \psi(t, \mathbf{n}) = 0 \]

if we can rewrite \( \mathcal{P} \cdot \mathbf{r} \cdot \mathbf{p} \psi(t, \mathbf{n}) = \mathcal{P} \cdot \mathbf{r} \cdot \mathbf{p} \psi(t, \mathbf{n}) \). The difference operators turn out to be

\[ \mathcal{P} \cdot \mathbf{r} \cdot \mathbf{p} \psi(t, \mathbf{n}) = \sum_{\mathbf{m}} \frac{1}{\sin \theta} (-1)^{n-1} \frac{2m}{m^2 - \frac{1}{4} a^2} \sin \theta \psi(t, \mathbf{n} + \mathbf{m}) - \psi(t, \mathbf{n} - \mathbf{m}) \]

since if we substitute \( \exp(-ia \mathbf{p} \cdot \mathbf{n}) \) for \( \psi(t, \mathbf{n}) \), we obtain

\[ \mathcal{P} \cdot \mathbf{r} \exp(-ia \mathbf{p} \cdot \mathbf{n}) = i \left( \sum_{\mathbf{m}} \frac{1}{\sin \theta} (-1)^{n-1} \frac{2m}{m^2 - \frac{1}{4} a^2} \exp(-ia \mathbf{p} \cdot \mathbf{n}) \right) \exp(-ia \mathbf{p} \cdot \mathbf{n}) \]

\[ = i \left( \frac{2}{a} \sin \frac{a \mathbf{p} \cdot \mathbf{n}}{2} \right) \exp(-ia \mathbf{p} \cdot \mathbf{n}) \]

It can be shown that the same conclusions are obtained if we start with this difference operator.