A Simple Hadron Cascade Model and Its Application to Electron-Positron Annihilation

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(Received October 24, 1977)

We discuss a simple hadron cascade model in which we assume a diffusion equation for hadron spectrum (mean multiplicity and momentum spectrum) and show such an equation reduces to a kind of bootstrap equation in the statistical bootstrap models. We apply our model to $e^+e^-$ annihilation process and discuss the mean multiplicity and the scaling property of the inclusive spectrum.

§ 1. Introduction

Recently Fukuda and Iso analyzed the momentum spectrum of several reactions at high energy in terms of their new type of quark cascade model. In this model a diffusion equation is assumed which determines the time dependent quark spectrum.

In the parton model in $e^+e^-$ annihilation a time-like photon produces a parton pair or a quark-antiquark pair, the hadrons in the final state are considered as the fragments of these constituents and it predicts $1/s$ behaviour of the total cross section, which is consistent with the recent experimental results for $\sqrt{s} \leq 3.5 \text{ GeV}$ and $5 \leq \sqrt{s} \leq 7.8 \text{ GeV}$. The rising of $R$ for $3.5 \leq \sqrt{s} \leq 5 \text{ GeV}$ can be interpreted as due to a threshold for production of a new quark-antiquark pair.

On the other hand, by Generalized Vector Meson Dominance Model (GVMD) we can also successfully describe the behaviour of the total cross section in $e^+e^-$ annihilation as well as a global feature of deep inelastic lepton-hadron scattering and the rising of $R$ in $e^+e^-$ annihilation can be directly connected with the breaking of the Bjorken scaling. In GVMD picture a time-like photon of mass $\sqrt{s}$ is coupled to vector meson or $J^P=1^-$ continuum state of mass $\sim \sqrt{s}$. Then if we neglect the interference effect among the different vector mesons the final hadronic state in $e^+e^-$ annihilation can be considered to be produced through vector meson of mass $\sim \sqrt{s}$.

When we combine the above GVMD picture and the idea of new type of quark cascade model, we may obtain a diffusion equation for hadrons as a basic assumption, which implies a hadron cascade picture. As for hadron cascade scheme there are already some analyses by the statistical bootstrap models and the chain emission models.

In this paper we take a hadron cascade scheme that multihadron production...
can be described by a succession of the strong two-body decay of resonances (or fireballs) (Fig. 1). And in the following sections we consider only meson system which is, for simplicity, assumed to contain only one species of resonances. We take a mass spectrum which consists of a stable particle of mass $\mu$ and a continuum state of hadron (fireball or resonance) of mass $m \geq 2\mu$, and we call each state "particle".

In § 2 we formulate the hadron cascade scheme using the diffusion equation and show that such a formulation reduces to a kind of bootstrap equation. In §§ 3-5, we investigate various properties of the final state initiated by one fireball: The expressions for physical quantities in $e^+e^-$ annihilation are given in § 3, assuming the isotropic distribution of decay products in each decay step. In § 4 we discuss the scaling property of the inclusive spectrum in $e^+e^-$ annihilation. In § 5 we show that the requirement of scaling in inclusive spectrum leads to the logarithmic or power behaviour of mean multiplicity with $s$. Finally, in § 6 conclusion and discussion are given.

![Fig. 1. General cascade decay.](image)

§ 2. Diffusion equation for one particle inclusive spectrum and bootstrap equation

According to our hadron cascade picture that the multihadron production occurs through the succession of the strong two-body decay of resonances and that any interference can be neglected, we introduce a vertex function $G(p; q, r)$ of each decay step:

$$G(p; q, r) = G(p; r, q) = \frac{1}{\lambda} \sum_{\text{spin}} |\langle q, r | T | p \rangle|^2,$$

where

$$\lambda = \int d^4q d^4r \theta(q_0) \theta(r_0) \delta^4(p - q - r) \sum_{\text{spin}} |\langle q, r | T | p \rangle|^2,$$

*) Even if the system contains more than one species of mesons, the extension is easily done for ordinary mesons.

***) The relation between Fukuda and Iso's model and the bootstrap idea was shown in Ref. 8).
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that is, \( G(p; q, r) \delta^4(p - q - r) d^4q d^4r \) represents a probability that a resonance with four-momentum \( p \) decays into two particles with four-momenta \( q \) and \( r \) in the phase volume \( d^4q \) and \( d^4r \). In Eq. (2.1) \( \sum_{\text{spin}} \) denotes the sum and/or average over all the spin states.

Denoting the average number of particles in the phase volume \( d^4q \) at time \( t \) as \( M(q, t) \), and using the definition of Eq. (2.1), we can write the diffusion equation for the momentum spectrum:

\[
\frac{\partial M(q, t)}{\partial t} = -\lambda(q) \theta(q^2 - 4\mu^2) M(q, t) + 2 \int d^4p \theta(p_0) \lambda(p) G(p, q) M(p, t)
\]

where

\[
G(p, q) = \int d^4r \theta(r_0) G(p; q, r) \delta^4(p - q - r).
\]

In the above equation the first term on the right-hand side represents the decrease due to the decay into two lower mass particles and the second term gives the increase coming from the decay of higher mass particles.

With an initial condition at time \( t = 0 \),

\[
M(q, t = 0) = M_0(q),
\]

we can express the formal solution of Eq. (2.2) as

\[
M(q, t) = \int d^4p e^{i(p \cdot L(p, q) t)} M_0(p).
\]

Since all the resonances which have higher mass than \( 2\mu \) must have successively decayed into the stable particles of mass \( \mu \) in a sufficiently long time, the momentum distribution in the final state is given by the solution of Eq. (2.2) at \( t = \infty \). As \( t \to \infty \), the exponential factor in Eq. (2.5) becomes (see the Appendix)

\[
\lim_{t \to \infty} e^{i(p \cdot L(p, q)t)} = \delta(q^2 - \mu^2) D(p, q)
\]

\[
= \theta(4\mu^2 - p^2) \delta^4(p - q) + \theta(p^2 - 4\mu^2) \sum_{l=1}^{l_{\text{max}}} 2^l G_l(p, q),
\]

where

\[
G_l(p, q) = \int d^4r G_{l-1}(p, r) G(r, q),
\]

\[
G_l(p, q) = G(p, q).
\]

\( D(p, q) \) defined in Eq. (2.6) represents a momentum (\( q \)) distribution in the final state which comes from a resonance with four-momentum \( p \), and the Lorentz invariant spectrum with the initial condition (Eq. (2.4)) is written as

\[
\frac{1}{\sigma_{\text{tot}}} \cdot 2 \cdot d^4q = \int d^4p D(p, q) M_0(p),
\]
where \( \varepsilon = \sqrt{q^2 + \mu^2} \). From Eqs. (2.6) and (2.7) we obtain an integral equation for \( D(p, q) \):

\[
D(p, q) = \theta(4\mu^2 - p^2) 2\delta(p - q) + 2\theta(p^2 - 4\mu^2) \int d^4 r G(p, r) D(r, q).
\] (2.9)

From Eq. (2.9) we get directly an integral equation for the mean multiplicity \( \kappa(p) \):

\[
\kappa(p) = \theta(4\mu^2 - p^2) + 2\theta(p^2 - 4\mu^2) \int d^4 r G(p, r) \kappa(r).
\] (2.10)

We can show that Eqs. (2.9) and (2.10) are kinds of bootstrap equation which was discussed by several authors.\(^{11,12,13}\) For the linear cascade decay scheme in the paper of Engels, Schilling and Satz,\(^6\) they obtained the bootstrap equation for the density of state of the fireball of mass \( \sqrt{Q^2} \). When we take a special form for our \( G(p; q, r) \)

\[
G(p; q, r) = \theta(q^2 - \mu^2) \tilde{G}(p, r) + \delta(r^2 - \mu^2) \tilde{G}(p, q),
\] (2.11)

which implies that at least one of the decay products at each decay step is a stable particle of mass \( \mu \) (Fig. 2), and we put \( \tilde{G}(p, r) \) to be a constant apart from the normalization, our equation (Eq. (2.10)) just corresponds to their bootstrap equation (Eq. (36)) in their paper.\(^6\) Then our equations (Eqs. (2.9) and (2.10)) can be considered as the bootstrap equation of momentum spectrum and mean multiplicity, respectively.

Fig. 2. Linear cascade decay.

We should note here that instead of hadron cascade scheme such as ours, even if we take quark cascade scheme\(^0\) in which diffusion equation is assumed for quark spectrum, the diffusion equation also reduces to the corresponding bootstrap equation.

§ 3. Momentum distribution in \( e^+ e^- \) annihilation

Now we investigate various properties of the final state initiated by one fireball (or resonance) with a given mass \( \sqrt{s} \).

Introducing a new function \( g(s; s_1, s_2) \) through the integration of \( G(p; q, r) \):

\[
g(s; s_1, s_2) = \int d^4 q d^4 r \delta(q_0) \delta(r_0) \delta(q^2 - s_1) \delta(r^2 - s_2)
\]

\[
\times \delta^4(p - q - r) G(p; q, r),
\] (3.1)

in the system of \( p = (\sqrt{s}, 0) \) we rewrite the bootstrap equation for \( D(p, q) \) (Eq.
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(2-9) as

\[ D(s, p = 0, q) = \theta (4\mu^2 - p^2) 2\varepsilon \delta (p - q) + \theta (p^2 - 4\mu^2) 2 \int ds_1 ds_2 g(s; s_1, s_2) \]

\[ \times D(s; p', q) P(\Omega_{p'}) d\Omega_{p'}, \]  

(3.2)

where \( p' \) is a momentum of one of the decay products and

\[ |p'| = \sqrt{\lambda (s, s_1, s_2)/2\sqrt{s}} , \]

(3.3)

and \( P(\Omega_{p'}) d\Omega_{p'} \) represents a probability that the particle has a momentum direction \( \Omega_{p'} \).

In \( e^+e^- \) annihilation we assume the initial state in our scheme is a vector meson with mass \( \sqrt{s} \) and the initial condition, Eq. (2·4), is

\[ M_0(p) = \delta^4(p - Q) , \]

(3·4)

where \( Q = (\sqrt{s}, 0) \) is four-momentum of a time-like photon in the centre-of-mass system. (Then, \( e^+e^- \) annihilation is the simplest example of hadron cascade initiated by one fireball.) Experimentally jet structure is observed and the distribution in inclusive spectrum shows the angular dependence. So it is necessary to introduce the anisotropic effect at some steps in cascade. Now except for the decay of initial vector meson, we assume each resonance decays isotropically in the rest frame,

\[ P(\Omega) = 1/4\pi . \]

(3·5)

In this case, \( D(p, q) \), in which \( p \) is a four-momentum of intermediate resonance, is a function of \( s = p^2 \) and \( |q| \) in the system of \( p = (\sqrt{s}, 0) \) and we denote it by \( W(s, q = |q|) \) hereafter. Taking account of the Lorentz invariance of \( D(p, q) \), Eq. (3.2) is rewritten as

\[ W(s, q) = 2\varepsilon \delta (p - q) \theta (4\mu^2 - s) \]

\[ + \theta (s - 4\mu^2) 2 \int ds_1 ds_2 g(s; s_1, s_2) \]

\[ \times \frac{1}{2v\beta q} \int q'_{\text{max}} dq' W(s_1, q') \frac{q'}{\varepsilon'} , \]

(3.6)

where \( \beta \) and \( \gamma \) are the Lorentz factors:

\[ \beta = |\beta| , \quad \beta = p'_i / E_1 , \quad \gamma = 1/\sqrt{1 - \beta^2} , \]

\[ q' = q + \gamma \beta \left( \frac{\gamma (\beta \cdot q)}{1 + \gamma} - \varepsilon \right) , \]

\[ \varepsilon' = \gamma (\varepsilon - \beta \cdot q) \]

(3·7)

and

\[ q'_{\text{max}} = \gamma (q + \beta \varepsilon) , \]
When we define as jet axis the direction of momentum of a particle which is produced from the initial vector meson at the first decay step, the transverse and longitudinal momentum distributions are given by

\[
\frac{1}{\sigma(s)} \frac{d\sigma}{dq_T} = \int ds_1 s_2 g(s; s_1, s_2) 4\pi q r \int_{q_T}^\infty dq' W(s, q') \frac{q'}{2\varepsilon' \sqrt{q^2 - q_T^2}}, \quad (3.9)
\]

\[
\frac{1}{\sigma(s)} \frac{d\sigma}{dq} = \int ds_1 s_2 \frac{2\pi}{r} g(s; s_1, s_2) \int_{q_T}^\infty dq' \int_{q_T}^{q'} dq'' W(s, q') \frac{q'}{2\varepsilon'}, \quad (3.10)
\]

respectively, where \( q_1 = r|q_3 - \beta \sqrt{q_1^2 + \mu^2}| \) and \( q_2 = r|q_3 + \beta \sqrt{q_1^2 + \mu^2}| \). And the momentum distribution becomes

\[
\frac{1}{\sigma(s)} \frac{d\sigma}{dq} = \frac{4\pi q^2}{2\varepsilon} W(s, q), \quad (3.11)
\]

which is independent of the angular distribution of the jet axis.

To take account of the anisotropic distribution in inclusive spectrum we consider two extreme cases for the angular distribution \( P_\theta(\Phi, \Phi_0) \) of the initial vector meson decay:

Case (I) corresponds to \( 0^- \) pair production and case (II) to \( 0^- \) plus \( 1^- \), i.e.,

\[
P_\theta(\Theta, \Phi) = \frac{3}{8\pi} \sin^2 \Theta \{1 + P^2 \cos 2\Phi\}, \quad (I) \quad (3.12)
\]

\[
P_\theta(\Theta, \Phi) = \frac{3}{16\pi} \{1 + \cos^2 \Theta - P^2 \sin^2 \Theta \cos 2\Phi\}, \quad (II) \quad (3.13)
\]

where \( \Theta \) is the polar angle of the decay product with respect to the incident positron direction, \( \Phi \) is the azimuthal angle measured from the vertical plane containing a beam line, and \( P \) is the magnitude of the polarization of each beam (the polarization direction is vertical). The inclusive momentum spectrum of the final state particles in each case is

\[
\frac{1}{\sigma(s)} 2\varepsilon \frac{d^2\sigma}{dq^2} = \frac{3}{4} (W + C) \left\{1 - \frac{3C - W}{W + C} \cos^2 \Theta + \frac{3C - W}{W + C} P^2 \sin^2 \Theta \cos 2\Phi\right\}, \quad (I) 
\]

\[
\frac{1}{\sigma(s)} 2\varepsilon \frac{d^2\sigma}{dq^2} = \frac{3}{8} (3W - C) \left\{1 + \frac{3C - W}{3W - C} \cos^2 \Theta - \frac{3C - W}{3W - C} P^2 \sin^2 \Theta \cos 2\Phi\right\}, \quad (II) 
\]

where \( W = W(s, q) \),

\[
C = C(s, q) = 2 \int ds_1 s_2 g(s; s_1, s_2) \int_{q_{\text{min}}'}^{q_{\text{max}}'} dq' W(s, q') \frac{\cos^2 \Theta'}{2\varepsilon' \beta q' \varepsilon'}, \quad (3.16)
\]

and \( \cos \Theta' = (\varepsilon' - \varepsilon')/\varepsilon', \beta \) being the angle of the decay product of the initial
vector meson with respect to the final state particle \( q \). From the above equations (Eqs. (3.12) \( \sim \) (3.16)) we can see that the angular distribution of the final state particles shows the weaker angular dependence than that of the initial vector meson decay, i.e., it becomes more isotropic. We note that the coefficient \( a \) in the general form

\[
\frac{d\sigma}{d\Omega} = \sigma_0 (1 + a \cos^2 \theta - P^2 a \sin^2 \theta \cos 2\varphi) \quad (3.17)
\]

is written down by \( C(s, q) \) and \( W(s, q) \) in each case. When we consider the experimental results that \( a \) is positive and increases with momentum, case (II) seems to be favorable.

§ 4. Scaling property of one particle inclusive spectrum in \( e^+e^- \) annihilation

Let us introduce a scaling variable \( x = 2q/\sqrt{s} \) and define \( F(s, x) = sW(s, q) \), and we investigate the behaviour of Eq. (3.6) at large \( s \) and large \( q \). Separating the integration over \( s_1 \) into two parts, \( s_1 = \mu^2 \) and \( 4\mu^2 \leq s_1 \), according to the form of the distribution function

\[
g(s; s_1, s_2) = \delta(s_1 - \mu^2) g^0(s; s_1, s_2) + \theta(s_1 - 4\mu^2) g^1(s; s_1, s_2), \quad (4.1)
\]

we can get an asymptotic form for \( F(s, x) \)

\[
F(s, x) = 2 \frac{2}{\pi x} \int_0^1 d\eta_1 \left[ \int_0^1 (1 - \eta_1)^s \frac{d\eta_2}{s} \int_0^{x_\text{max}} dx' x_\text{min} + \int_x^{x_\text{max}} d\eta_1 \int_0^{x_\text{max}} \frac{d\eta_2}{s} \int_0^{x_\text{max}} d\eta_2 \right] 
\]

\[
+ \int_0^x d\eta' \int_0^{(1-x)(s - \eta')/s} \int_0^{(1-\eta')/s} \frac{d\eta_1 d\eta_2}{s} \int_0^{x_\text{max}} (s \lambda (1, \eta_1, \eta_2)) F(s \eta_1, x', \eta') \quad (4.2)
\]

for \( x^2 > 4\mu^2/s \), where \( x_\text{min} = (1 + \eta_1 - \eta_2 - \sqrt{\lambda}) x/2\eta_1 \), \( x_\text{max} = (1 + \eta_1 - \eta_2 + \sqrt{\lambda}) x/2\eta_1 \), \( \sqrt{\lambda} = \sqrt{\lambda} (1, \eta_1, \eta_2) \), \( \eta_1 = s_1/s \) and \( \eta_2 = s_2/s \). From Eq. (4.2) it can be shown that \( F(s, x) \) has no \( s \)-dependence and is a function of only \( x \), if at large \( s \) the distribution function has an asymptotic form

\[
g^0(s; s_1, s_2) ds_2 = \delta(s_2) d\eta_2, \quad \int_0^x g^0(s; s_1, s_2) d\eta_1 = \delta(s_1) d\eta_1, \quad (4.3)
\]

Using the scaling variable \( x \), the momentum distribution in \( e^+e^- \) annihilation (Eq. (3.11)) is rewritten

\[
\frac{1}{\sigma} \frac{d\sigma}{dx} = \frac{\pi x}{2} F(s, x). \quad (4.4)
\]

We see that this distribution is independent of \( s \), if Eq. (4.3) holds.
When we take the simple chain emission picture, we write \( g(s; s_1, s_2) \) as
\[
g(s; s_1, s_2) = \delta(s_1 - \mu \tau) \tilde{g}(s, s_2) + \delta(s_2 - \mu \tau) \tilde{g}(s, s_1). \tag{4.5}
\]
In this case, if \( \tilde{g}(s, s_1) \) has the following form at large \( s \)
\[
\tilde{g}(s, \eta) = \frac{1}{s} \tilde{h}(\eta), \tag{4.6}
\]
then Eq. (3.6) becomes at large \( s \)
\[
F(s, x) = 2 \int_0^1 d\eta_1 \frac{d\eta_2 \tilde{h}(\eta_2)}{x} \delta(x - 1 + \eta_2) \tilde{h}(\eta_1, \eta_2) \int_0^1 d\eta_1 \int_0^1 d\eta_1 \frac{\tilde{h}(\eta_1)}{1 - \eta_1} - F(s\eta_1, x'). \tag{4.7}
\]
Therefore, the requirement of Eq. (4.6) at large \( s \) also leads to the scaling of the momentum distribution.

The scaling of \( F(s, x) \) leads to the scaling of the transverse and longitudinal momentum distribution, that is, at large \( s \) Eqs. (3.9) and (3.10) become
\[
\frac{1}{\sigma(s)} \frac{d\sigma}{dx_T} = 4\pi x_T \int d\eta_1 d\eta_2 \frac{\tilde{h}(\eta_1, \eta_2)}{1 + \eta_1 - \eta_2} \int_0^1 d\eta_1 \int_0^1 d\eta_1 \frac{F(x')}{\sqrt{x_T^2 - x_T^2/\eta_1}} \tag{4.8}
\]
and
\[
\frac{1}{\sigma(s)} \frac{d\sigma}{dx_i} = 2 \int d\eta_1 d\eta_2 \frac{\tilde{h}(\eta_1, \eta_2)}{1 + \eta_1 - \eta_2} \int_0^1 d\eta_1 \int_0^1 d\eta_1 \left[ \int_0^1 d\eta_1 + \int_0^1 d\eta_1 \right] F(x'), \tag{4.9}
\]
where
\[
x_T = 2q_T/\sqrt{s}, \quad x_i = 2q_i/\sqrt{s}, \quad x_1 = (1 + \eta_1 - \eta_2 - \sqrt{\lambda}) s_1/2q_1,
\]
\[
x_2 = (1 + \eta_1 - \eta_2 + \sqrt{\lambda}) s_1/2q_1
\]
and
\[
\sqrt{\lambda} = \sqrt{\lambda(1, \eta_1, \eta_2)}.
\]

The scaling of transverse momentum distribution with respect to \( x_T \) does not imply the energy independency of the \( q_T \) cutoff, and in fact, a mean transverse momentum \( \langle q_T(s) \rangle \) becomes
\[
\langle q_T(s) \rangle = \frac{1}{\langle n(s) \rangle} 2 \int ds_1 ds_2 g(s; s_1, s_2) \frac{\pi}{4} \int d\eta' 4\pi \eta'^2 W(s, \eta')
\]
\[
= \frac{1}{\langle n(s) \rangle} 2 \int ds_1 ds_2 g(s; s_1, s_2) \frac{\pi}{4} \langle q(s_1) \rangle \langle n(s_1) \rangle. \tag{4.10}
\]
Unless \( \langle q(s) \rangle \) is bounded (in this case apparently \( \langle n(s) \rangle \triangleq \sqrt{s} \)), \( \langle q_T(s) \rangle \) increases with \( s \) in the asymptotic energy. It may be understood that this result is due to
the assumption of the isotropic decay for each cascade step (except for the decay of the initial vector meson) in the rest frame.

For the angular distributions, Eqs. (3·14) and (3·15), when \( s W(s, q) = F(s, x) \) scales, \( s C(s, q) \) also scales and \( (1/\sigma(s)) \cdot (d\sigma/dxdQ) \) comes to be independent of \( s \) at large \( s \).

§ 5. Mean multiplicity in \( e^+e^- \) annihilation

When we require the scaling for momentum spectrum in our model, that is, Eqs. (4·3) are assumed, the bootstrap equation (Eq. (2·10)) for mean multiplicity becomes at large \( s \),

\[
\kappa(s) = 2 \int_0^1 d\eta H(\eta) \kappa(s\eta) \tag{5·1}
\]

with

\[
H(\eta) = \int_0^{(1-\sqrt{\eta})^2} d\eta_2 h(\eta, \eta_2) \tag{5·2}
\]

and Eq. (5·1) has a solution with power of \( s \)

\[
\kappa(s) \sim s^\nu \tag{5·3}
\]

where \( \nu \) is the highest value of the solutions of equation

\[
\int_0^1 d\eta H(\eta) \nu^2 = 1/2 \tag{5·4}
\]

By way of trial we take \( h(\eta_1, \eta_2) = \text{constant} (= 6) \), then Eq. (5·4) gives a value of \( \nu = 0.36 \).

On the other hand, when we take the simple chain emission picture, Eqs. (4·5) and (4·6) lead to the following bootstrap equation for the mean multiplicity:

\[
\kappa(s) = 1 + 2 \int_0^1 d\eta \tilde{h}(\eta) \kappa(s\eta) \tag{5·5}
\]

at large \( s \). In this case we get a logarithmic solution

\[
\kappa(s) = b \cdot \ln s + c \tag{5·6}
\]

with

\[
b^{-1} = -2 \int_0^1 d\eta \tilde{h}(\eta) \ln \eta \tag{5·7}
\]

By way of trial we take \( \tilde{h}(\eta) = \text{constant} (= 1/2) \), then Eq. (5·7) gives \( b = 1 \).

The recent data of mean charged multiplicity in \( e^+e^- \) annihilation can be fitted both by logarithm and by power of \( s \), and considering the new resonance production above the threshold it is difficult to determine the type of \( g(s; s_1, s_2) \) by multiplicity at this stage.
§ 6. Conclusion and discussion

In this paper, starting with the diffusion equation for hadron, we could get a kind of bootstrap equation for mean multiplicity and momentum spectrum. Further we have investigated the properties of final state such as scaling, angular distribution and mean multiplicity which is initiated by a single fireball with a given mass $\sqrt{s}$.

We left free the distribution function $G(p; g, r)$ or equivalently $g(s; s_1, s_2) \times P(\Omega)$, on which the dynamics of hadron cascade essentially lies in our model. In $e^+e^-$ annihilation, however, a simple assumption of distribution function $g(s; s_1, s_2) ds_1 ds_2 = \delta(s_1 - \mu^2) \delta(s_2 - \mu^2) \times P(\Omega) = 1/4\pi$ assures the scaling of one particle inclusive spectrum and gives a power growth of mean multiplicity with energy. And when we assume a special form:

$$
g(s; s_1, s_2) = \delta(s_1 - \mu^2) \delta(s_2 - \mu^2) \times P(\Omega) = 1/4\pi \prod_{i} \left( \frac{1}{2\pi} \right)^{3/2} e^{-\frac{s_i}{2\mu}}
$$

we also get the scaling of the momentum spectrum but in this case mean multiplicity has a logarithmic increase with energy.

Though our diffusion equation in this paper is limited to a meson system with one species of mesons, it is easy to extend to the system which contains many species of mesons, if we assume the probability

$$
P_{ijk} \text{ is a probability that a meson } i \text{ decays into two mesons } j \text{ and } k.
$$

In that case the diffusion equation comes to have a matrix form and can be solved by diagonalization of the matrix $\sum_{k} P_{ijk}$. Equation (2·2) corresponds to the case that the eigenvalue of the matrix $\sum_{k} P_{ijk}$ is 2. Since the scaling property does not depend on the eigenvalue (which corresponds to $\alpha$ in the Appendix), the momentum distribution of each meson scales if the one which corresponds to the eigenvalue 2 scales.

Experimentally the inclusive spectrum in $e^+e^-$ annihilation $sd\sigma/dx$ shows the scaling above the centre-of-mass energy 4.8 GeV for $x \geq 0.3$ and many new resonances are found which can be interpreted as the charmonia. Above the threshold of the new channel the particles in the final state contain the ones which come from the decay of the new resonances via weak interaction as well as strong interaction. When we consider the effects of the new resonances in our scheme the factorizability (Eq. (6·1)) fails and we need more general distribution function. However since in the momentum distribution in the final state the particles due to the decay of the new resonances are not expected to appear for $x \geq 0.5$, our discussion of the scaling can be applied for $x \geq 0.5$ even above the threshold.

Further an important observation in $e^+e^-$ annihilation is the jet structure. Although our hadron cascade model can lead to jet-like structure, the transverse momentum distribution with respect to the jet axis would not so rapidly decrease.
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with \( q_r \), if the jet axis determined experimentally were coincident with our jet axis defined in \( \S \, 3 \). However as for the angular distribution in the final state particles, if we take the case (II) for \( P_\theta (\Omega) \) in \( \S \, 3 \), it would be possible to describe the experimental tendency that \( a \) (coefficient of \( \cos^2 \theta \)) in Eq. (3.17) is positive and increases with \( x \).

In order to fit to the data on the detail of experimental results it is necessary to fix an explicit form for the distribution function \( G(p; q, r) \), which can include the case of \( P(\Omega) \neq 1/4\pi \) in general, but that is beyond the scope of this paper.

Acknowledgements

The authors would like to thank Professor H. Fukuda and Professor C. Iso for helpful comments and their colleagues at TIT high energy group for discussions.

Appendix

---Momentum distribution \( D(p, q) \)---

We treat here the simplest case that \( \lambda(p) \) in Eq. (2.5) is a constant. Even though \( \lambda \) has a \( p \)-dependence we can show that the final results are not changed, but it is rather lengthy.

The exponential factor in Eq. (2.5) is defined as

\[
e^{i L(p,q)} = \sum_{n=0}^{\infty} \frac{(\lambda p)^n}{n!} L_n(p, q),
\]

(A.1)

where \( L_n(p, q) \) is given by a recurrence formula:

\[
L_n(p, q) = \int d^k L_{n-1}(p, k) L(k, q),
\]

(A.2)

\[
L_0(p, q) = \delta^4(p-q)
\]

(A.3)

and

\[
L(p, q) = \theta(p^2 - 4\mu^2) \{ - \delta^4(p-q) + \alpha \cdot G(p, q) \}
\]

(A.4)

(\( \alpha = 2 \) in the text, but here we allow an arbitrary value for \( \alpha \).)\(^{**} \) We define \( G_i(p, q) \) by the following recurrence formula

\[
G_i(p, q) = \int d^k L_{i-1} (p, k) G(k, q), \quad (i \geq 1)
\]

(A.5)

\[
G_0(p, q) = \delta^4(p-q)
\]

(A.6)

here we should note that because of the finiteness of the lowest mass \( \mu \) in our

---

\(^* \) The coefficient corresponding to \( \alpha \) is misprinted and dropped in Eq. (4) in our previous letter.
particle spectrum,

\[ G_{l}(p, q) = 0 \text{ for } l > (\sqrt{p^2 - q^2})/\mu. \]  

(A·7)

Then, \( L_n(p, q) \) can be rewritten

\[
L_n(p, q) = \theta(p^2 - 4\mu^2) \left\{ \theta(q^2 - 4\mu^2) \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{l} (-1)^{n-l} \alpha^l G_l(p, q) 
+ \theta(4\mu^2 - q^2) \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{l-1} (-1)^{n-l} \alpha^l G_l(p, q) \right\} .
\]  

(A·8)

Substituting Eq. (A·8) into Eq. (A·1) and summing over \( n \), we get

\[
\sum_{n=0}^{\infty} \frac{(\lambda \mu)^n}{n!} L_n(p, q) = \theta(4\mu^2 - p^2) \delta^4(p - q)
+ \theta(p^2 - 4\mu^2) \theta(q^2 - 4\mu^2) e^{-\lambda t} \sum_{l=0}^{\lfloor \lambda t \rfloor} \frac{(\alpha \mu)^l}{l!} G_l(p, q)
+ \theta(p^2 - 4\mu^2) \theta(4\mu^2 - q^2) \sum_{l=1}^{\lambda t} (-\alpha)^l G_l(p, q) S_l(-\lambda t),
\]  

(A·9)

where

\[
S_l(z) = e^{\sum_{r=0}^{l-1} (-1)^r \frac{z^{1-l}}{(l-1-r)!}} + (-1)^l.
\]  

(A·10)

In order to get the momentum spectrum in the final state we take the limit \( (t \to \infty) \), then Eq. (A·9) becomes as follows:

\[
\lim_{t \to \infty} \sum_{n=0}^{\infty} \frac{(\lambda \mu)^n}{n!} L_n(p, q) = J(p, q)
= \theta(4\mu^2 - p^2) \delta^4(p - q)
+ \theta(p^2 - 4\mu^2) \theta(q^2 - 4\mu^2) \sum_{l=1}^{\lambda t} \alpha^l G_l(p, q),
\]  

(A·11)

In general we can write the distribution function as

\[ G(p, q) = \delta(q^2 - \mu^2) G^0(p, q) + \theta(q^2 - 4\mu^2) G^1(p, q), \]  

(A·12)

from which we can write \( J(p, q) \) as

\[ J(p, q) = \delta(q^2 - \mu^2) D(p, q), \]  

(A·13)

then from Eq. (A·11) we can show that \( D(p, q) \) satisfies the following integral equation:

\[
D(p, q) = 2q_0 \delta(p - q) \theta(4\mu^2 - p^2)
+ \theta(p^2 - 4\mu^2) \alpha \int d^4k G(p, k) D(k, q).
\]  

(A·14)
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References

   See also, O. Sawada and G. Takeda, Preprint, TU/76/42.
2) See for example, R. P. Feynman, Photon-Hadron Interactions (Benjamin, Reading, Mass., 1972).
   A. Bramon, E. Etim and M. Greco, Phys. Letters B41 (1972), 609. See also Ref. 4).