A New Relativistic Mechanics for Two-Particle System

Masashi FUJIGAKI and Seiji KOJIMA

Department of Physics, Nagoya University, Nagoya 464

(Received October 7, 1977)

We present a new theory for two relativistic particles interacting through an action-at-a-distance force. The masses of the constituent particles are well defined. We treat the equal and unequal mass cases simultaneously. In the interaction case we get the conditions which suppress the internal time component in the rest-frame. Using the conditions as strong equations, we quantize the system canonically. A generalized Klein-Gordon equation is obtained. The internal space is three-dimensional and Euclidian but the wave equation is Lorentz covariant. The harmonic oscillator system is solved. The leading Regge trajectory is linear rising in the case of the equal constituent mass. In the unequal mass case, however, it has a leap in an unphysical region and deviate from a straight line near the leap. The extent of the leap and the deviation depends on the value of the mass-difference of the constituent particles.

§ 1. Introduction and summary

Many years ago, the bilocal-field was proposed by Yukawa as a simple example of the non-local field. The bilocal and the trilocal fields are used as the prototype of meson and baryon. For these years, the mechanical models which lead to the bilocal models have been proposed by several authors. However, up to now, there is no convincing treatment for the system composed of the particles which have unequal masses.

In this paper we present and study a new relativistic mechanics of two point particles interacting through an action-at-a-distance force. The theory must be invariant under reparametrization of each world line. Moreover, in order to define well the masses of the constituent particles, we require that, in the no interaction limit, the theory describes the system of the free relativistic particles. Due to this requirement, it will be possible to treat the scattering processes of the relativistic point particles. Needless to say, by a suitable choice of the potential function, we can confine the constituent particles.

In a previous paper we did not require the existence of the simple free-limit but required the exchange-invariance, i.e., the invariance under the exchange of each particle at arbitrary instance. It turned out there that the theory gives the meaningful models only for two- and three-particle systems. However, it was also pointed out that the exchange-invariance has to be violated in order to include radial excitations. The present theory actually gives the model which permits the
A New Relativistic Mechanics for Two-Particle System

radial motion. Simultaneously it will be possible to describe a many-particle system.

The action is constructed by extending the one which describes the two free relativistic particles. The masses of the constituent particles are definitely defined. We treat the unequal as well as the equal mass case. The action gives non-linear equations of motion. They lead to the conditions which suppress the time component of the internal motion and determine the total mass corresponding to the internal motion. By taking a suitable gauge, the equations of motion are also reduced to those which are the same type as in the non-relativistic two-body problem. It should be noticed that, in the interaction case, the classical center-of-mass coordinate is definitely defined and it depends not only on each constituent mass but also on the total mass. In the special case of the Hooke-type potential, we get a mass-spin relation which is generally different from the usual one. The leading Regge trajectory has a leap in the unphysical region and deviates from a straight line near the leap. The extent of the anomalous behavior depends on the mass difference of the constituent particles. In the limit of the equal constituent mass, it becomes entirely linear.

When the system is quantized, we impose strongly the conditions which suppress the time component of the internal motion as the operator equations. Then we need not use the indefinite metric space and are not troubled with the appearance of the unphysical state in calculating the scattering amplitude of the composite systems. It is different from the traditional treatment of the bilocal model in which the conditions are imposed weakly after the method of Gupta-Bleuler in the electromagnetic field theory. It is also a merit of our approach that, for any internal potential, we need not worry about the consistency between the wave equation and the subsidiary conditions. On account of the strong conditions, we cannot carry out the manifest covariant canonical quantization. We seek for the independent canonical variables at the expense of the manifest covariance. The resultant quantum theory is described only by a generalized Klein-Gordon equation with the three-dimensional internal Euclidian space. The wave equation does not permit space-like solutions.

In the following section, we shall set up the action, and solve the Euler equations. In § 3, we describe the Hamiltonian formalism with the constraint conditions. After rewriting it in terms of the center-of-mass and relative variables, the independent canonical variables are found in § 4. In § 5, the theory is quantized. The final section is devoted to discussion.

§ 2. The Lagrangian theory

First of all, let us summarize the notations and postulates which are stated in the previous paper and are necessary to set up the dynamics. The space-time coordinates of two point-particles are denoted as $x_{1}(\tau_{1})$ and $x_{2}(\tau_{2})$, where $\tau_{n}$ are
arbitrary evolution parameters with the properties
\[ \frac{dx_a^b}{d\tau_a} > 0 . \quad (a = 1, 2) \tag{1} \]

The velocity of each particle is required to be less than that of the light:
\[ (dx_a)^2 \leq 0 . \quad (a = 1, 2) \tag{2} \]

The evolution parameters of the interacting points satisfy the monotonous increasing relation
\[ \tau_i = f_{ia}(\tau_s), \tag{3} \]

where \( f_{ia} \) is an arbitrary function. The interaction takes place at space-like distance; the relative coordinate
\[ x^a = x_1^a(\tau_s) - x_2^a(\tau_i) \tag{4} \]

of the interacting points satisfies
\[ x^2 > 0 . \tag{5} \]

We choose the action of the interacting two particles as
\[ I = \int \left[ -\sqrt{U_1(x^b)} - \sqrt{(dx_1)^2} - \sqrt{U_2(x^b)} - \sqrt{(dx_2)^2} \right], \tag{6} \]

where the each potential function \( U_a \) is arbitrary but satisfies
\[ U_a > 0 . \tag{7} \]

This action is a simple one which ensures our requirement stated in the previous section. Actually, in the no interaction limit \( \sqrt{U_a} \to m_a \) (const), it is reduced to
\[ I = \int \sum_{a=1}^2 (-m_a) \sqrt{-\frac{dx_a}{d\tau}} , \tag{8} \]

which is the free action for the two relativistic particles with the masses \( m_1 \) and \( m_2 \). The \( \tau_a \)-invariance of the action (6) is obvious, so that we can conveniently choose the gauge \( \tau_1 = \tau_2 (\equiv \tau) \) to represent the action in the form of the parameter integration \( I = \int d\tau L \). Then
\[ L = -\sum_{a=1}^2 \sqrt{U_a(x^b)} \sqrt{\frac{\dot{x}_a}{\dot{\tau}}}, \tag{9} \]

where \( \dot{x}_a = dx_a/d\tau \) and
\[ x^a = x_1^a(\tau) - x_2^a(\tau). \tag{10} \]

\( ^* \) Differing from the previous paper, we use the suffices \( a, b \), to label the particles, and the suffices \( i, j \), etc. to represent the space components.

\( ** \) We use the metric \( g_{ab} = -g_{ab} = 1 \).

\( *** \) Even if we take the labor of working in the general gauge with \( f_{ia} \), the same results are obtained.
The Lagrangian gives the Euler equations

$$\dot{p}_a^\mu = (-1)^a \sum_{\beta=1}^{2} U_\beta' (U_\beta)^{-1/2} \sqrt{-\dot{x}_\beta^2} x^\mu \quad (a = 1, 2)$$  \hspace{1cm} (11.a)$$

with the canonical momenta

$$p_a^\mu = (\sqrt{U_a'/\sqrt{-\dot{x}_a^2}}) \dot{x}_a^\mu, \quad (11.b)$$

where $U_a' = dU_a(x^\beta)/dx^\beta$.

In order to see what kind of motion follows from the non-linear equations (11), we shall take advantage of the primary constraints

$$p_a^\mu - \sqrt{p_a^\mu + U_a} = 0, \quad (a = 1, 2)$$  \hspace{1cm} (12)$$

which are obtained from Eq. (11·b) by noting the inequalities (1). Let us note that Eq. (12) are equivalent to the following:

$$p_a^\mu > 0, \quad (13)$$

$$p_a^\mu + U_a = 0. \quad (14)$$

Equation (14) insist that $p_a^\mu$ are time-like vectors by the aid of (7). As a result the conserved total momentum

$$P_a = p_1^a + p_2^a$$  \hspace{1cm} (15)$$

becomes a time-like vector with a positive time-component.

The $\tau$-derivative of Eq. (14) gives the secondary constraint

$$(U_a' \dot{p}_1 + U_1' p_a) \cdot x = 0. \quad (16)$$

Except in the special case $U_1' = U_2'$, the subsequent procedure is too complicated to get the solution. Thus we set

$$U_a = U(x^\beta) + m_a x^\beta \quad (a = 1, 2)$$  \hspace{1cm} (17)$$

hereafter. In this case, as we shall see later on, we get a suitable set of constraints which suppress only the time-component of the internal motion. Equation (16) is trivial in the free region ($U' = 0$), but in the interaction region ($U' \neq 0$) it is reduced to

$$P \cdot x = 0. \quad (18)$$

Differentiating Eq. (18) with respect to $\tau$ and using Eqs. (11), we get the constraint

$$\varepsilon_1 \sqrt{-\dot{x}_1^2} / U_1 - \varepsilon_2 \sqrt{-\dot{x}_2^2} / U_2 = 0$$  \hspace{1cm} (19)$$

with the definition

$$\varepsilon_a = P \cdot p_a / p_2. \quad (20)$$

It turns out that $\varepsilon_a$ satisfy
The conservation of $\varepsilon_a$ follows from Eqs. (11·a) and (18). The positivity is easily checked in the rest-frame ($P=0$) which can always be taken due to the time-like property of $P_a$.

Owing to (19), it is possible to choose $\tau$ that the relations*1
\[ \varepsilon_a = (1/\lambda) \sqrt{U_a/\sqrt{-(\dot{\varepsilon}_a)^2}} \]  
(22)
hold for arbitrary but positive constant $\lambda$, without introducing any constraint which is not reserved by Eqs. (11); Eq. (22) are gauge conditions. Under that gauge, the equations of motion are
\[ \dot{p}_a^\mu = (-1)^a (\lambda \varepsilon_\mu \varepsilon_a)^{-1} U^\alpha \partial_x x^\alpha = (2\lambda \varepsilon_\mu \varepsilon_a)^{-1} \partial_x x^\alpha, \]  
(23·a)
with
\[ p_a^\mu = \lambda \varepsilon_\mu \dot{x}_a^\mu. \]  
(23·b)
This form is analogous to the usual equations of motion in the non-relativistic two-body problem. In these ways, we obtain (22) and (23) with (20) as the basic equations. Let us note that Eq. (22) can be replaced by (12) or equivalently by (13) and (14). It is easy to check that Eq. (18) is again derived from them.

In order to go over to the center-of-mass and relative variables, we now introduce, other than $x^a$ and $P^a$, the center-of-mass space-time coordinate $X^a$ and the relative four momentum $p^a$:
\[ X = \varepsilon_1 x_1 + \varepsilon_2 x_2, \quad x = x_1 - x_2, \]  
(24·a, b)
\[ P = p_1 + p_2, \quad p = \varepsilon_1 p_1 - \varepsilon_2 p_2. \]  
(24·c, d)
Equation (24·d) identically yields the constraint
\[ P \cdot p = 0. \]  
(25)
The correspondence between $(x_a, p_a)$ and $(X, P, x, p)$ is one-to-one only in the restricted region where it holds the primary constraint
\[ p_1^2 - p_2^2 = m_2^2 - m_1^2, \]  
(26)
which is obtained by subtracting of each equation of (14). Actually, using (26), we can easily show that $\varepsilon_a$ coincide with $\bar{\varepsilon}_a$ which are defined as
\[ \bar{\varepsilon}_a = 1/2 + (-1)^a (m_1^2 - m_2^2)/(2P^2). \]  
(27)
Thus Eqs. (24) are solved inversely in the restricted region
\[ x_1 = X + \bar{\varepsilon}_2 x, \quad x_2 = X - \bar{\varepsilon}_1 x, \]  
(28·a, b)
\[ p_1 = \bar{\varepsilon}_1 P + p, \quad p_2 = \bar{\varepsilon}_2 P - p. \]  
(28·c, d)
It is easy to check that Eqs. (25) and (28) with (27) inversely yield Eqs. (26)

*1 Generally $\tau$ needs not to be a scalar, but the relations impose $\tau$ to be scalar.
A New Relativistic Mechanics for Two-Particle System

The rest condition of (14) leads to

\[ P^a + (\xi_1, \xi_2)^{-1} p^a + \xi^{-1}_{-} U_1 + \xi_2^{-1} U_2 = 0, \]  

which is obtained by summing each equation of (14) with the weights \( \xi_a^{-1} \). Substituting (27) to (29) and manipulating a little, we get

\[ -P^a = 2p^a + U_1 + U_2 + 2\sqrt{(p^2 + U_1)(p^2 + U_2)}. \]  

This determines the total mass of the system corresponding to the internal motion. Here the sign before the root is decided to satisfy the condition

\[ -P^a > |m_1^2 - m_2^2|, \]  

which follows from Eqs. (27), (21·c) and \( P^a < 0 \).

The equations of motion are transformed into

\[ P = \lambda X, \quad \dot{P} = 0, \]  
\[ \dot{p} = \lambda \xi_1 \xi_2 \dot{x}, \quad \dot{p} = - (U'/\lambda \xi_1 \xi_2) x. \]  

Differentiating (25) regarding \( \tau \) and using (32), we get (18) again. The differentiation of (18) returns to (25). From Eqs. (27) and (32), we again see that \( \xi_a \) are conserved. Differentiating (29), we get no new condition. Thus the whole set of equations in terms of \( P, X, p \) and \( x \) are (18), (25), (30) and (32) with (27).

Equations (32) show that the center-of-mass coordinate moves uniformly. The internal motion is determined by

\[ \ddot{x}^a = - U' (\lambda \xi_1 \xi_2)^{-1} \dot{x}^a, \]  

and in the rest frame the time components vanish owing to (18) and (25).

In the case of the Hooke-type potential

\[ U = \kappa^2 x^2, \]  

where \( \kappa \) is a positive constant with a dimension of mass square, Eq. (33) becomes

\[ \ddot{x}^a = - \omega^2 x^a \]  

for \( \omega = \kappa / \lambda \xi_1 \xi_2 \). In the rest-frame (\( \mathbf{P} = 0 \)), there is the solution

\[ x^a = (l \cos \omega \tau, l \sin \omega \tau, 0, 0). \]  

This represents a rotation with a constant relative distance \( l \) in a plane. The spin vector \( \mathbf{x} \times \mathbf{p} \) takes the value \( s = \kappa \mathbf{F} \) in this motion. Thus we get the mass-spin relation

\[ s = \frac{m_1^2 - m_2^2}{2k}, \quad s = \frac{m_1^2 + m_2^2}{2k}, \]  

\[ s = \frac{m_1^2 - m_2^2}{4k}, \quad s = \frac{m_1^2 + m_2^2}{2k}. \]  

Fig. 1. Leading Regge-trajectory of the two-particle oscillator system in the case \( m_1 > m_2 \).
$M^2 = 4\xi s + m_1^2 + m_2^2 + 2\sqrt{(2\xi s + m_1^2)(2\xi s + m_2^2)}$ (37)

for the rotational motion. This equation gives the leading trajectory. For the unequal constituents mass case ($m_1 \neq m_2$), this trajectory is curved. The shape is drawn in Fig. 1. For the higher spin states, the trajectory approaches linear. In the equal constituents mass case ($m_1 = m_2 = m$), it is entirely linear:

$M^2 = 8\xi s + 4m^2$. (38)

§ 3. The Hamiltonian formalism with constraint conditions

Since our Lagrangian is a first order function with respect to the velocity, the canonical Hamiltonian vanishes. On the other hand, there are the primary constraints (12). The Hamiltonian for this system is

$H = \sum_{a=1}^{3} \nu_a (p_a^0 - \sqrt{p_a^2 + U_a})$, (39)

where $\nu_a$ are arbitrary functions of $x_a$ and $p_a$. We define the Poisson bracket (PB)

$\{A, B\} = \sum_{a} [ (\partial A/\partial x_a) (\partial B/\partial p_a) - (\partial A/\partial p_a) (\partial B/\partial x_a^a)]$ (40)

and assume the canonical equation of motion

$\dot{A} = \{A, H\}$. (41)

From the requirement that the $\tau$-derivative of (12) vanishes, we generate (18) as the secondary constraint. If we introduce the term corresponding to (18) into the Hamiltonian, the consistency of (12) leads to $P \cdot x = 0$. Then only massless states ($P^a = 0$) are permitted. To avoid the massless case, we drop the $P \cdot x$ term from the Hamiltonian and use (39) itself. From $\{P \cdot x, H\} = 0$ we find that

$\nu_a = \nu \varepsilon_a^{-1} (p_a^0 + \sqrt{p_a^2 + U_a})$, (42)

where $\nu$ is an arbitrary function. We need no more constraints. Substituting (42) into (39) and putting $\nu = -(2\lambda)^{-1}$ by adjusting $\tau$, we get the Hamiltonian

$H = (2\lambda)^{-1} \sum_{a=1}^{3} (p_a^0 + U_a) \varepsilon_a^{-1}$. (43)

This Hamiltonian leads to the equations of motion (23) again. Thus we can develop the Hamiltonian theory with Eqs. (12), (18), (40), (41) and (43).

* In the Hamiltonian formalism with constraints, we must differentiate between identities and constraints. The former always holds in the phase space, but the latter only in the physical region of the phase space. The constraint equations can be used only after calculating Poisson bracket, since the calculation needs the information in the unphysical region.
§ 4. Canonical formalism in terms of the center-of-mass and relative variables

We now represent the Hamiltonian formalism in terms of the center-of-mass and relative variables defined by (24). By using (28), the Hamiltonian becomes

$$H = (2\lambda)^{-1} \left( P^2 + (\tilde{z}_1, \tilde{z}_2)^{-1} p^2 + \tilde{z}_1^{-1} U_1 + \tilde{z}_2^{-1} U_2 \right).$$

(44)

The primary constraint (12) is reduced to (25) and (30) as was shown in the previous section. If we note that the Hamiltonian (44) is factorized as

$$H = (8\lambda \tilde{z}_1, \tilde{z}_2 P^2)^{-1} \left[ P^2 + 2p^2 + U_1 + U_2 - 2\sqrt{(p_1^2 + U_1)(p_2^2 + U_2)} \right]$$

$$\times \left[ P^2 + 2p^2 + U_1 + U_2 + 2\sqrt{(p_1^2 + U_1)(p_2^2 + U_2)} \right]$$

(45)

and the constraint (30) gives the relation

$$2\tilde{z}_1, \tilde{z}_2 = (P^2 + 2p^2 + U_1 + U_2)/P^2,$$

(46)

we see that the Hamiltonian (45) is transformed to

$$H' = (2\lambda)^{-1} \left[ P^2 + 2p^2 + U_1 + U_2 + 2\sqrt{(p_1^2 + U_1)(p_2^2 + U_2)} \right].$$

(47)

This Hamiltonian gives the same equations of motion as (44). Then the constraint (30) is represented equivalently as

$$H' = 0.$$

(48)

A straightforward calculation using (24) and (40) gives the Poisson brackets among $X, P, x$ and $p$:

$$\{ P', P'' \} = 0, \quad \{ X', P'' \} = g^{''} r', \quad \{ X', X'' \} = -S'^{''}/P^2,$$

(49·a, b, c)

$$\{ P', r'' \} = 0, \quad \{ P', x'' \} = 0,$$

(49·d, e)

$$\{ X', r'' \} = -p'' P''/P^2, \quad \{ X', x'' \} = -x'' P''/P^2,$$

(49·f, g)

$$\{ p', r'' \} = 0, \quad \{ x'', r'' \} = g^{'''} - P'' P''/P^2, \quad \{ x'', x'' \} = 0,$$

(49·h, i, j)

where

$$S^{''} = x^{'''} p''.$$

(50)

To sum up, the basic set of equations in this stage is (18), (25), (41), (47), (48) and (49). It is directly checked that they reproduce the equations of motion (32).

However, it is not convenient to use Eqs. (49) as the basic PB for the purpose of quantization, because they are not in a canonical form. Whereas it is easily seen that not only $P \cdot P$ but also $P \cdot x$ have zero PB with $X, P, x, and p$. This fact shows that Eqs. (18) and (25) can be considered as identities; the independent internal variables become six. We have to find a set of independent canonical

---

* Strictly speaking, there is a difference between (43) and (44). We can show that the difference has the form $\Sigma C_{aa}(p_1^2 + U_1)(p_2^2 + U_2)$, where $C_{aa}$ are definite functions of $p_0$. Thus the Hamiltonian (44) leads to the same equations of motion as (43) in the physical space.
variables. It is done with the aid of the technique of the polarization decomposition of the massive vector field. Noting the canonical nature of $P^0$ in Eqs. (49), we include $P^0$ among the canonical variables. The other canonical variables are found in Appendix A; the internal canonical variables $\bar{x}_i$ and $\bar{p}_i$ are defined by the relations

$$\bar{x}_i = x_i - x^0 (P^0 + \sqrt{-P^2})^{-1} P_i,$$

$$\bar{p}_i = p_i - p^0 (P^0 + \sqrt{-P^2})^{-1} P_i,$$

while the canonical center-of-mass coordinate $\bar{X}^a$ is

$$\bar{X}^a = X^a - S^{0a} (P^0 + \sqrt{-P^2})^{-1}.$$

They satisfy the basic PB algebra

$$\{\bar{X}^a, P^b\} = g^{ab}, \quad \{\bar{x}_i, \bar{p}_j\} = g_{ij},$$

(52)

other PB = 0.

Let us note that $\bar{X}^0 = X^0$ and $\bar{X}^i$ are similar to the Pryce-Newton-Wigner variables. Using (51), we can easily check that

$$\bar{x}^a = x^a, \quad \bar{p}^a = p^a,$$

(53)

so that the Hamiltonian represented in terms of the independent canonical variables is

$$H' = (2\lambda)^{-1} [P^0 + 2\bar{p}^0 + U_i + U_s + 2\sqrt{(\bar{p}^0 + U_i)(\bar{p}^0 + U_s)}],$$

(54)

where

$$U_s = U(x^0) + m_s^2.$$

(55)

We have thus got the canonical formalism: the canonical PB (52), the Hamiltonian (54), the canonical equation (41) and the constraint (48).

Noting (48), we get the equations of motion

$$\dot{\bar{p}}^a = 0, \quad \dot{\bar{X}}^a = \lambda^{-1} P^a,$$

$$\dot{\bar{x}}_i = - (\lambda \bar{x}_i \bar{x}_i)^{-1} U' \bar{x}_i, \quad \dot{\bar{p}}_i = (\lambda \bar{x}_i \bar{x}_i)^{-1} \bar{p}_i.$$

By using Eqs. (B·3), (B·4) and (B·7) in Appendix B, it can be checked that these equations reproduce (32). It is shown that $\bar{x}_i$ and $\bar{p}_i$ behave complicatedly under Lorentz transformation in Appendix C. But the Hamiltonian (54) is invariant, because $\bar{x}^a$ and $\bar{p}^a$ are Lorentz-scalar as is evident from (53).

§ 5. Quantum theory without subsidiary conditions

We now go over to the quantum theory by the usual prescription: replace the PB by the commutator multiplied by $-i$. Then we get the Heisenberg equations of motion.
A New Relativistic Mechanics for Two-Particle System

\[ \dot{A} = i[H', A] \]  

with the Hamiltonian (54) and the equal \( \tau \) commutation relations

\[ [\tilde{X}^x, P^y] = ig^{xy}, \quad [\tilde{x}_i, p_j] = ig_{ij}, \quad \text{other CR} = 0. \]  

(58)

The condition (48) is ensured by imposing the following condition on the physical state \( |\psi\rangle \):

\[ H' |\psi\rangle = 0. \]  

(59)

By going over to the Schrödinger representation by the transformation

\[ |\psi\rangle_s = e^{-iH't} |\psi\rangle, \]  

(60)

\( \tau \) is completely eliminated from the theory on account of (59). Thus we get the basic equations for the quantum theory: The canonical commutation relations which have the same form as (58) among \( \tau \)-independent variables and the generalized Klein-Gordon equation

\[ (P^2 + \tilde{m}^2) |\psi\rangle = 0 \]  

(61)

with

\[ \tilde{m}^2 = 2(\tilde{p}^2 + U) + m_1^2 + m_2^2 + 2\sqrt{(\tilde{p}^2 + U + m_1^2)(\tilde{p}^2 + U + m_2^2)}. \]  

(62)

The Lorentz-transformation property of \( \tilde{x} \) and \( \tilde{p} \) are the same as in the classical theory. Thus \( \tilde{m}^2 \) is scalar and Eq. (61) is Lorentz invariant. It is to be noted that we have constructed a quantum theory for the arbitrary positive potential \( U \). In the traditional treatment with the subsidiary conditions which eliminate ghost states, it is difficult to ensure the consistency between the wave equation and the subsidiary conditions for arbitrary potential functions.

When the potential is given by (34), it is convenient to introduce the oscillation variables \( \tilde{a}_i^+ \) and \( \tilde{a}_i \) which satisfy the CR

\[ [\tilde{a}_i, \tilde{a}_j^+] = g_{ij} \]  

(63)

by the definition

\[ \tilde{a}_i = (2\kappa)^{-1/2} \tilde{p}_i - i(\kappa/2)^{1/2} \tilde{x}_i, \]  

\[ \tilde{a}_i^+ = (2\kappa)^{-1/2} \tilde{p}_i + i(\kappa/2)^{1/2} \tilde{x}_i. \]  

(64)

The mass operator becomes

\[ \tilde{m}^2 = 4\kappa \tilde{a}_i^\dagger \cdot \tilde{a} + m_1^2 + m_2^2 + 2\sqrt{(2\kappa \tilde{a}_i^\dagger \cdot \tilde{a} + m_1^2)(2\kappa \tilde{a}_i^\dagger \cdot \tilde{a} + m_2^2)}, \]  

(65)

where we dropped the zero point mass understanding the normal ordered product.

Equation (61) can be easily solved in the rest-frame \( (P = 0) \). Introducing a ground state vector \( |0\rangle \) which satisfies \( \tilde{a}_i |0\rangle = 0 \), we get the eigenstate \( |m_\alpha\rangle \) of \( \tilde{m}^2 \) represented as

\[ |m_\alpha\rangle = \prod_{i=1}^3 (\tilde{a}_i^\dagger)^{n_i} |0\rangle. \]  

(66)
This has the mass value

\[ m_n = 4\kappa \left( \sum_{i=1}^{3} n_i + m_i^2 + m^2 \right) + 2 \left[ (2\kappa \sum_{i=1}^{3} n_i + m_i^2) \left( 2\kappa \sum_{i=1}^{3} n_i + m_i^3 \right) \right]^{-1/2}. \]  

(67)

The total state vector of the rest system is represented as

\[ |0, m_n\rangle = |m_n\rangle e^{-i\omega_n t}. \]

(68)

By boosting the rest state vector, the moving state vector is obtained. Then we use the Lorentz-generators

\[ M_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu} \]

with the radial parts

\[ \hat{L}_{\mu\nu} = \hat{X}_{[\mu\nu]} P_{\nu} \]

(69)

and the spin parts

\[ \hat{S}_{\phi} = -i\hat{a}^\dagger \hat{a}_{\phi}, \]

\[ \hat{S}_{\phi} = \frac{\hat{S}_{\phi} P^\mu}{P^\mu + \sqrt{-P^2}}. \]

(71)

The algebra among \( \hat{L}_{\mu\nu} \) and \( \hat{S}_{\mu\nu} \) are given in the Appendix C. They are complicated but \( M_{\mu\nu} \) satisfy the usual Lorentz algebra. Mass-spin relation is investigated with the help of the Pauli-Lubanski operator \( \hat{W}_\mu \). A little manipulation gives

\[ W^2 = (-P^\mu) [ \hat{a}^\dagger \cdot \hat{a} (\hat{a}^\dagger \cdot \hat{a} + 1) - \hat{a}^\dagger \hat{a}^\dagger]. \]

(72)

The highest spin state in each multiplet satisfies

\[ \hat{a}^\dagger |m_n\rangle = 0 \]

(73)

and it lies on the trajectory (37).

§ 6. Discussion

It seems accidental that the PB of (49) are consistent with (18). However, the same algebra can be derived by the Dirac’s method\(^5\) in which the conditions (18) and (26) are used to eliminate one internal degree of freedom. The quantum theory which is obtained by replacing the Dirac bracket of the independent canonical variables by the commutator are the same as the previous one.

In the usual bilocal model, space-like solutions are permitted.\(^5\) We removed the space-like solutions by the assumption that \( P_n^\alpha \) are positive and \( P_n^\nu \) are time-like vectors.

If we intend our theory to apply for mesons, the experimental evidence of our model will be checked by the anomalous behavior of the trajectory. This check is difficult in the resonance region, because the deviation from the straight line is little in that region for the non-charmed meson. The trajectory of the
A New Relativistic Mechanics for Two-Particle System

charmed meson has not yet been established. Thus it is important to investigate the effect of the anomaly to the scattering amplitude.

In order to compare the theoretical result definitely with the experimental one, it is indispensable to include the half-integer spins of the constituent particles, since they must contribute not only to the total spin but also to the total mass.

In our formalism we can treat the scattering of point particles. When \( U \) becomes constant at large distance, the constituent particles can dissociate with each other and move freely. Then the masses of the particles in the free motions are different from \( m_a \).

We developed the theory choosing the gauge in which \( \tau \) is scalar. Alternatively we can work in the non-covariant gauge defined by

\[
\tau_i = x^{\phi}_1(\tau) = x^{\phi}_2(\tau) = t .
\]  

(74)

Then the Lagrangian is

\[
L = -\sum_{a=1}^{2} \sqrt{U_a} \sqrt{1 - (dx_a/dt)^2} .
\]

(75)

In this case the correspondence to the non-relativistic theory becomes clearer.

The generalization of our formalism to \( N \)-particle system is straightforward. The action is given by

\[
I = -\int \sum_{a=1}^{N} \sqrt{U_a} \sqrt{1 - (dx_a)^2} ,
\]

(76)

where \( x_a^{\mu}(\tau_a) \) represents the space-time coordinate of each constituent particle. In the three-particle case the model will describe the prototype of baryon. A detailed investigation of the model will be given in the near future.

Acknowledgements

We would like to thank Professor Y. Ohnuki and the members of the elementary particle group of Nagoya University for valuable discussions. Thanks are also due to Professor S. Ogawa for critical reading of this manuscript. One of us (M.F.) thanks Professor H. Katsumori and the members of his laboratory for useful discussions. One of us (S.K.) would like to thank Professor T. Takabayasi for encouragements.

Appendix A

—Independent Canonical Variables—

Let us first introduce three orthonormal vectors \( e^a(i) \) which are normal to \( P^a \):

\[
P \cdot e(i) = 0 , \ e(i) \cdot e(j) = g_{ij} . \quad (i,j = 1, 2, 3)
\]

(A \cdot 1 \cdot a, b)

They are explicitly represented as
We write the projection of $x^a$ and $p^a$ to $e(i)$ as

$$
\vec{x}_i = x \cdot e(i), \quad \vec{p}_i = p \cdot e(i). \tag{A\cdot3}
$$

They become the internal canonical variables which satisfy

$$
\{\vec{x}_i, \vec{p}_j\} = g_{ij}, \quad \{\vec{x}_i, \vec{x}_j\} = \{\vec{p}_i, \vec{p}_j\} = 0. \tag{A\cdot4}
$$

It can be checked that $X^0$ has vanishing PB with $\vec{x}$ and $\vec{p}$, but $X_i$ does not. Thus $X^0$ itself can be taken as the canonical center-of-mass time coordinate. In order to find the canonical center-of-mass space coordinate, we require that the angular momentum tensor in terms of the canonical variables can be represented in the same form as in the non-canonical ones:

$$
M_{ij} = X_{(i}P_{j)} + x_{(i}p_{j)} = \vec{x}_{(i}P_{j)} + \vec{x}_{(i}\vec{p}_{j)}. \tag{A\cdot5}
$$

By substituting (A\cdot3) to (A\cdot5) and rearranging, we see that $\vec{x}_i$ is chosen as indicated in Eq. (51\cdot c). We can directly check that the center-of-mass variables $\vec{X}^a$ have vanishing PB with each other and with $\vec{x}$, $\vec{p}$.

**Appendix B**

---Inverse Transformation of (51) and Expressions for Lorentz Generators---

From Eqs. (18) and (25), we get

$$
\begin{align*}
\vec{x}^0 &= \vec{x} \cdot P^0 / P^0, \\
\vec{p}^0 &= \vec{p} \cdot x / P^0.
\end{align*} \tag{B\cdot1\cdot a, b}
$$

A little manipulation using (51\cdot a,b) and (B\cdot1, a,b) gives

$$
\begin{align*}
\vec{P} \cdot \vec{x} &= \vec{P} \cdot x \sqrt{-P^2 / P^0}, \\
\vec{P} \cdot \vec{p} &= \vec{P} \cdot p \sqrt{-P^2 / P^0}. \tag{B\cdot2\cdot a, b}
\end{align*}
$$

Equations (B\cdot1) and (B\cdot2) yield

$$
\begin{align*}
x^0 &= \vec{P} \cdot \vec{x} / \sqrt{-P^2}, \\
p^0 &= \vec{P} \cdot \vec{p} / \sqrt{-P^2}. \tag{B\cdot3\cdot a, b}
\end{align*}
$$

From (B\cdot3) and (51\cdot a,b), we get

$$
\begin{align*}
x_i &= \vec{x}_i + \vec{P} \cdot (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2} P_i, \tag{B\cdot4\cdot a} \\
p_i &= \vec{p}_i + \vec{P} \cdot (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2} P_i. \tag{B\cdot4\cdot b}
\end{align*}
$$

Substituting (B\cdot3) and (B\cdot4) to (50) and rearranging, we get

$$
\begin{align*}
S'^0 &= \vec{S}'^0 - (P^0 \vec{S}'^1 + P^1 \vec{S}'^0) P_i (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2}, \tag{B\cdot5\cdot a} \\
S'^0 &= \vec{S}'^0 P_i (-P^2)^{-1/2}, \tag{B\cdot5\cdot b}
\end{align*}
$$

where

$$
\vec{S}'^0 = \vec{x}^0 \vec{P}^3. \tag{B\cdot6}
$$

Equations (51\cdot c) and (B\cdot5\cdot b) give
A New Relativistic Mechanics for Two-Particle System

\[ X^0 = \tilde{X}^0, \quad (B \cdot 7 \cdot a) \]
\[ X^i = \tilde{X}^i - \tilde{S}^{ij} P_j (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2}, \quad (B \cdot 7 \cdot b) \]

Equations (B \cdot 3), (B \cdot 4) and (B \cdot 7) constitute the inverse transformation of (51). Substituting (B \cdot 7) to the radial part

\[ L^{\mu} = X^{\nu \mu} P^\nu \]

of the Lorentz generators \( M^{\mu} \) and rearranging, we get

\[ L^0 = \tilde{L}^0 + (P^i \tilde{S}^{0i} - P^0 \tilde{S}^{0i}) P_i (P^0 + \sqrt{-P^2}) (-P^2)^{-1/2}, \quad (B \cdot 9 \cdot a) \]
\[ L^i = \tilde{L}^i - P^0 \tilde{S}^{0i} P_i (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2} \]

for

\[ \tilde{L}^{\mu} = \tilde{X}^{\nu \mu} P^\nu. \quad (B \cdot 10) \]

Defining \( \tilde{x}^0 \) and \( \tilde{p}^0 \) by the relations

\[ \tilde{x}^0 = P \cdot \tilde{\mathbf{x}} (P^0 + \sqrt{-P^2})^{-1}, \quad \tilde{p}^0 = P \cdot \tilde{\mathbf{p}} (P^0 + \sqrt{-P^2})^{-1}, \]

we see that Eq. (B \cdot 5, b) are written as

\[ S^{0i} = \tilde{S}^{0i} + P^0 \tilde{S}^{0i} P_i (P^0 + \sqrt{-P^2})^{-1} (-P^2)^{-1/2} \]

for

\[ \tilde{S}^{0i} = \tilde{x}^{0i} P^0. \]

Thus \( M^{\mu} = L^{\mu} + S^{\mu} \) can also be written as

\[ M^{\mu} = \tilde{L}^{\mu} + \tilde{S}^{\mu}. \]

Appendix C

—Lorentz-Transformation Property of Dynamical Variables—

As is seen from the definition (B \cdot 11), \( \tilde{x}^0 \) and \( \tilde{p}^0 \) satisfy a little complicated PB with other variables. A straightforward calculation using (51) gives

\[ \{ \tilde{x}^i, \tilde{x}^j \} = -[\tilde{x}^i + P^i \tilde{x}^j (-P^2)^{-1/2}] (P^0 + \sqrt{-P^2})^{-1}, \]
\[ \{ \tilde{p}^i, \tilde{x}^j \} = -[\tilde{p}^i + P^i \tilde{p}^j (-P^2)^{-1/2}] (P^0 + \sqrt{-P^2})^{-1}, \]
\[ \{ \tilde{x}^i, \tilde{x}^j \} = -\tilde{x}^0 (P^0)^{-1/2}, \quad \{ \tilde{p}^i, \tilde{x}^j \} = -\tilde{p}^0 (P^0)^{-1/2}, \]
\[ \{ \tilde{x}^i, \tilde{p}^j \} = \{ \tilde{x}^i, \tilde{p}^j \} = P^i (P^0 + \sqrt{-P^2})^{-1}, \]
\[ \{ \tilde{x}^0, \tilde{p}^0 \} = P^0 (P^0 + \sqrt{-P^2})^{-1}, \]
\[ \{ \tilde{x}^0, \tilde{p}^0 \} = \{ \tilde{p}^0, \tilde{p}^0 \} = \{ \tilde{x}^0, \tilde{x}^0 \} = \{ \tilde{p}^0, \tilde{p}^0 \} = 0. \]

From (B \cdot 10), (B \cdot 6) and (52), we get the usual PB

\[ \{ \tilde{L}^{\mu}, P^\nu \} = g^{\nu \mu} P^\mu - g^{\nu \mu} P^\mu, \quad (C \cdot 2 \cdot a) \]
\[ \{ \hat{L}^{\alpha}, \hat{X}^{\beta} \} = g^{\alpha\beta} \hat{x}^{\gamma} - g^{\alpha\beta} \hat{x}^{\gamma}, \]  
(C·2·b)

\[ \{ \hat{L}^{\alpha}, \hat{x}^{\beta} \} = \{ \hat{L}^{\alpha}, \hat{p}^{\beta} \} = \{ \hat{S}^{ij}, \hat{P}^{\alpha} \} = \{ \hat{S}^{ij}, \hat{X}^{\alpha} \} = 0, \]  
(C·2·c)

\[ \{ \hat{S}^{ij}, \hat{x}^{k} \} = g^{ik} \hat{x}^{j} - g^{jk} \hat{x}^{i}, \]  
(C·2·d)

\[ \{ \hat{S}^{ij}, \hat{p}^{k} \} = g^{ik} \hat{p}^{j} - g^{jk} \hat{p}^{i}. \]  
(C·2·e)

While, using (C·1), (B·6) and (B·10), we get

\[ \{ \hat{L}^{\alpha}, \hat{x}^{\beta} \} = 2\alpha\beta \hat{P}^{\gamma} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·3·a)

\[ \{ \hat{L}^{\alpha}, \hat{p}^{\beta} \} = 2\beta\alpha \hat{P}^{\gamma} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·3·b)

\[ \{ \hat{S}^{ij}, \hat{x}^{k} \} = - \hat{x}^i \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·3·c)

\[ \{ \hat{S}^{ij}, \hat{p}^{k} \} = - \hat{p}^{i} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·3·d)

and, using (B·13), (52) and (C·1),

\[ \{ \hat{S}^{10}, p^{\alpha} \} = 0, \quad \{ \hat{S}^{10}, \hat{X}^{\beta} \} = - \hat{S}^{10} (-p^{2})^{-1/2}, \]  
(C·4·a, b)

\[ \{ \hat{S}^{10}, \hat{X}^{\beta} \} = - [ \hat{S}^{ij} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}] (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·4·c)

\[ \{ \hat{S}^{10}, \hat{x}^{i} \} = \hat{x}^{i} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·4·d)

\[ \{ \hat{S}^{10}, \hat{p}^{i} \} = \hat{p}^{i} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·4·e)

\[ \{ \hat{S}^{10}, \hat{p}^{i} \} = - \hat{p}^{i} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}, \]  
(C·4·f)

\[ \{ \hat{S}^{10}, \hat{p}^{i} \} = - \hat{p}^{i} \hat{p}^{j} (p^{0} + \sqrt{-p^{2}})^{-1}. \]  
(C·4·g)

These relations show that \( \hat{X}^{i}, \hat{x}^{i}, \hat{p}^{i} \) are 3-vectors but \( \hat{X}^{\beta}, \hat{x}^{\beta}, \hat{p}^{\beta} \) are not 4-vectors. By using (C·2·4), we can calculate the PB among the internal and external Lorentz-generators:

\[ \{ \hat{L}^{\alpha}, \hat{L}^{\beta} \} = g^{\alpha\beta} \hat{L}^{\gamma} - g^{\alpha\beta} \hat{L}^{\gamma}, \]  
(C·5·a)

\[ \{ \hat{L}^{\alpha}, \hat{S}^{ij} \} = 0, \]  
(C·5·b)

\[ \{ \hat{L}^{\alpha}, \hat{S}^{10} \} = (\hat{S}^{10} p^{\alpha} - \hat{S}^{10} p^{\alpha}) / (p^{0} + \sqrt{-p^{2}}), \]  
(C·5·c)

\[ \{ \hat{S}^{ij}, \hat{S}^{lm} \} = g^{ij} \hat{S}^{lm} + g^{lm} \hat{S}^{ij} - g^{jm} \hat{S}^{il} - g^{im} \hat{S}^{jl}, \]  
(C·5·d)

\[ \{ \hat{S}^{10}, \hat{S}^{ij} \} = (p^{0} - \sqrt{-p^{2}}) \hat{S}^{ij} + p^{i} \hat{S}^{10} - p^{j} \hat{S}^{10} / (p^{0} + \sqrt{-p^{2}}), \]  
(C·5·e)

\[ \{ \hat{S}^{ij}, \hat{S}^{10} \} = g^{ij} \hat{S}^{10} - g^{ij} \hat{S}^{10} + (\hat{S}^{10} p^{i} - \hat{S}^{10} p^{j}) / (p^{0} + \sqrt{-p^{2}}). \]  
(C·5·f)

These equations give

\[ \{ M^{\alpha}, M^{\beta} \} = g^{\alpha\beta} M^{\gamma} + g^{\alpha\beta} M^{\gamma} - g^{\alpha\beta} M^{\gamma} - g^{\alpha\beta} M^{\gamma} \]  
(C·6)

as expected. The quantum mechanical algebra are given by replacing the PB by the commutator multiplied by \(-i\).
References

4) M. Kalb and P. V. Alstine, preprint, COO-3075-146.
7) P. A. M. Dirac, Lecture on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York, 1964).