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A Growth Model for Black Ice, Snow Ice and Snow Thickness in Subarctic Basins

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A model for the growth of ice thickness in autumn and winter is presented. Temperature of the upper surface of the ice sheet, snowfall and heat flux from the water are external input variables, and the thickness of black ice, snow ice and snow and the density of snow are predicted through the model. Thermal inertia is neglected. The results are sensitive to the assumed density of freshly fallen snow and the formulation of snow metamorphism. Comparisons between observations and model calculations are given. The agreement is moderate and the need for field work on snow on ice is recognized. A discussion is given on the maximum annual ice thickness under varying external conditions.

Introduction

The well-known rule of Stefan (1891) states that the squared ice thickness is proportional to the time integral of T_f-T_i, where T_f is the freezing point of the water and T_i the temperature of the upper surface of the ice, integrated from the onset of freezing. The derivation was based on the balance between the released latent heat in freezing and the upward heat flux at the bottom of the ice sheet. In practice T_i must be replaced by the air temperature T_a and hence T_f-T_i by max $\{T_f$-T_a, $0\}$. If snow falls on the ice T_a is no more a reasonable estimator of T_i and, in addition, if the snow income is large enough some of it will be transformed into snow ice. These effects have been tried to take into account through various kinds of empirical modifications of the Stefan's law (e.g., Lebedev 1938, Anderson 1961).

In this work we shall study the growth of the thickness in the presence of snow cover using a simple model but trying to approach the role of snow in a physical way. The model includes packing of snow and snow – snow ice transformation. We consider subarctic regions where the ice grows during autumn and winter. The maximum ice thickness is about one metre or less and the role of snow can turn out to be very important. E.g., in Finland ice grows on average to 35-90 cm and the thickness of snow ice can count to ~ ⅓ of the total (Laasanen 1981, Leppäranta and Seinä 1982); the income of snow is on average 25-45 mm equivalent water per month during December-February (Kolkki 1969).

In the Arctic regions where the ice does not totally melt in summer the effect of snow can be different. The thermodynamic equilibrium thickness of Arctic sea ice is typically a little more than three metres but ice may become much thicker in the course of years if the annual snowfall is abnormally high (Maykut and Untersteiner 1971). This is because the snow cover lowers the melt rate in summer.

Basic Principles

Snow-covered floating ice sheet has three distinct layers of frozen water (Weeks and Lee 1958): black ice, snow ice (frozen slush) and snow (Fig. 1). Black ice is formed from the basin water only. In winter snow ice is formed mainly from a mixture of basin water and snow while in spring melted snow or mixing of rain water and snow may give rise to snow ice formation. We do not consider the latter period here.

We denote the thickneses by h, densities by ϱ and thermal conductivities by k using the subscripts i, si and s for black ice, snow ice and snow, respectively. The vertical coordinate is denoted by z; Fig. 1 shows the z-coordinates for the layer boundaries. We consider the density and thermal conductivity of black ice and snow ice constants and those of snow independent of the z-coordinate. The latent heat of ice formation L is assumed constant. The solar radiation is neglected and at the upper surface the temperature is a given function of time, $T=T_0(t)$. At the bottom of the ice sheet the temperature is at the freezing point of the underlying water, $T=T_f$. Thermal inertia is ignored which means here that the temperature profile becomes linear in each layer. The income of snow is an external variable.

The values chosen for the constants are shown in Table 1. Due to the crudeness of our approach it was thought that the well-known properties of pure ice could be taken for the thermal conductivity of black ice and latent heat of ice formation. In natural water bodies they depend on the temperature, content of air bubbles and (in case of sea ice) salinity (Schwerdtfeger 1963). The density of freshly fallen snow is ~0.1-0.2 g cm^{-3} (e.g., Simojoki and Seppänen 1963) and it increases to ~0.3-0.5 g cm^{-3} during a couple of months or less depending on the external conditions. The thermal conductivity of snow is strongly dependent on the density (e.g., Male 1980).

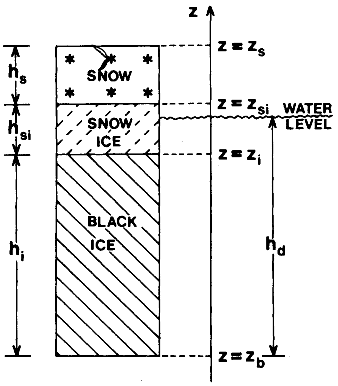

Fig. 1. Cross-section of a floating ice sheet with the vertical coordinate system.

Table 1 – The constants for black ice and snow ice chosen in this work.

Quantity	Notation and value	Comments
Density:		
black ice	$\varrho_i = 0.91 \text{ g cm}^{-3}$	Palosuo (1965)
snow ice	$\varrho_{si} = 0.88 \text{ g cm}^{-3}$	Palosuo (1965)
Thermal conductivitity:		
black ice	$k_i = 0.0224 \text{ J°C}^{-1}\text{cm}^{-1}\text{s}^{-1}$	pure ice (e.g., Pounder 1965)
snow ice	$K_{si} = k_i/2$	assumption (as Uusitalo 1973)
Latent heat of ice formation	$L = 334 \text{ J g}^{-1}$	pure ice (e.g., Pounder 1965)

Black Ice Growth

As said the temperature profile is here piecewise linear. It is determined by the continuity of the heat flow through the layer boundaries, latent heat changes through melting or freezing and by the boundary conditions describing the heat exchange with the atmosphere and the underlying water. At the upper surface the temperature must equal the externally given surface temperature T_0. At the lower surface the temperature is at the freezing point and there is a heat flux Q_w from the water to the ice. Our system of equations reads as (e.g., Simpson 1958, Uusitalo 1973)

$$T(z_s) = T_0 \tag{1a}$$

$$k_s \frac{dT}{dz}\bigg|_{z=z_{si}^+} = k_{si} \frac{dT}{dz}\bigg|_{z=z_{si}^-} \tag{1b}$$

$$k_{si}\frac{dT}{dz}\Big|_{z=z_i^+} = k_i\frac{dT}{dz}\Big|_{z=z_i^-} \tag{1c}$$

$$\rho_i L\frac{dh_i}{dt} = -k_i\frac{dT}{dz}\Big|_{z=z_i^+} - Q_w \tag{1d}$$

$$T(z_b) = T_f \tag{1e}$$

where the superscripts – and + stand for derivative on the left and right, respectively. The derivatives are expressed through $\Delta T/h$ and we obtain the solution (e.g. Uusitalo 1973)

$$\frac{dh_i}{dt} = \frac{1}{\rho_i L}\,\frac{T_f - T_0}{h_i/k_i + h_{si}/k_{si} + h_s/k_s} - \frac{Q_w}{\rho_i L} \tag{2}$$

Snow Ice Formation and Packing of Snow

If the income of snow is large enough the lower boundary of the snow layer tends to submerge according to Archimedes' law. But then the snow will get mixed with the water which means slush formation. This is counted as a sink for the snow layer. When the temperature gets low the slush freezes to snow ice. For the sake of simplicity we include slush in the snow ice layer whether frozen or not, and thus we ignore the freezing of slush and the effect of thermal expansion in the freezing process.

The thickness of snow changes due to three different causes 1) The income of new snow by the rate dh_s'/dt and with density ϱ_s' 2) Snow ice formation is a sink for the snow layer 3) Due to snow metamorphism the density of the snow on ice changes; let π be the rate and assume simply that $\pi = \pi(\varrho_s)$. Note that $\pi > 0$. Now we have

$$\frac{dh_s}{dt} = \frac{dh_s'}{dt} - \frac{dh_{si}}{dt} - \frac{h_s}{\rho_s}\pi(\rho_s) \tag{3}$$

We have made here an important physical assumption: when the lower boundary of the snow layer submerges snow will get mixed with the water in such a way that the rate of increase of snow ice thickness dh_{si}/dt equals the rate of sinking of the snow layer. The change in the snow mass is

$$\frac{dm_s}{dt} = \rho_s'\frac{dh_s'}{dt} - \rho_s\frac{dh_{si}}{dt} \tag{4}$$

and, through $m_s = \varrho_s h_s$, we obtain the equation for the rate of change of snow density

$$\frac{d\rho_s}{dt} = \pi(\rho_s) - \frac{\rho_s' - \rho_s}{h_s}\frac{dh_s'}{dt} \tag{5}$$

The packing function π must be formulated on the basis of empirical data. A simple form would be

$$\pi(\rho_s) = \begin{cases} \pi_0 = \text{constant}, & \text{if } \rho_s < \rho_{s,max}, \\ 0, & \text{else} \end{cases} \tag{6}$$

In addition to snow ice formation, packing effects on the thermal conductivity of snow. Here we take, from Anderson (1976)

$$k_s = 2.09 \times 10^{-4} + 0.025\rho_s^2 \tag{7}$$

where k_s is in $J°C^{-1}cm^{-1}s^{-1}$ and ρ_s in g cm^{-3}.

The draft of the ice sheet h_d (see Fig. 1) is given by

$$h_d = \frac{\rho_s}{\rho_w} h_s + \frac{\rho_{si}}{\rho_w} h_{si} + \frac{\rho_i}{\rho_w} h_i \tag{8}$$

where ρ_w is the density of water. The principle of slush belonging to the snow ice layer means that

$$h_d \leqq h_i + h_{si} \tag{9}$$

Once we have here equality snow ice formation may rise. In that case we must compare the weight of new snow with the change of net buoyancy in the black ice layer. If

$$\rho'_s \frac{dh'_s}{dt} \leqq (\rho_w - \rho_i) \frac{dh_i}{dt} \tag{10}$$

the lower boundary of the snow remains above the water surface, i.e. $dh_{si}/dt = 0$. But if the inequality (10) is false, the snow layer starts to sink leading to slush formation. The transformation process decreases the weight of snow and increases the net buoyancy of the ice as much as the equality below is obtained

$$\rho'_s \frac{dh'_s}{dt} - \rho_s \frac{dh_{si}}{dt} = (\rho_w - \rho_i) \frac{dh_i}{dt} + (\rho_w - \rho_{si}) \frac{dh_{si}}{dt} \tag{11}$$

This closes the system of equations.

Mechanical Growth

External conditions may cause relative movement in an ice cover resulting in overthrusting. When the ice is still thin, the overriding sheets do not break but form together a new level ice sheet with doubled thickness. It is out of the scope of the present paper to consider the mechanics of the overthrusting (for that, see Parmerter 1975) but its effect on the final ice thickness at the end of the growth season can be easily studied with an ice growth model.

An external input variable describing whether overthrusting occurs or not can be added to the model. In the positive case ice thickness is doubled. Through a number of simulations the overthrusting may occur at different times. To obtain realistic results one should note that overriding without breaking is very rare for ice thicker than 10-20 cm.

The Growth Model

The principles and equations given in the previous section provide a basis for a simple growth model for black ice, snow ice and snow thickness. The differential equations must be treated in discrete form. To obtain reasonable accuracy the time-step must be of the order of one hour in the very early growth season, and when the thickness of ice has exceeded about 20 cm one day is a convenient step. The external input variables driving the growth are the temperature of the upper surface (usually replaced by min $\{T_f, T_a\}$ where T_a is the air temperature), snowfall and the heat flux from the water to the ice. The model predicts the thicknesses of the three layers, the density of snow and through that the thermal conductivity of snow.

At each time step black ice thickness is changed through Eq. (2). The criteria in the inequalities Eq. (9) and Eq. (10) tell whether snow ice formation occurs and in the positive case snow ice thickness is changed through Eq. (11). Finally, the thickness and density of snow are changed through Eqs. (3) and (5).

Test of the Model

During the ice season 1976/77 snowfall was exceptionally large in Finland providing suitable data to test the model. We chose the comparison data for the model output from the ice observation station Virpiniemi situated on the Finnish coast of the Bothnian Bay (Inst. Mar. Res. 1982). The atmospheric input data were obtained from the observations of the air temperature at the altitude of two metres and the water equivalent values of snowfall at the meteorological station in Oulu airport (Finnish Meteorol. Inst. 1976-1977). The sea water near the station Virpiniemi has very low salinity (less than 3 ‰) and consequently the ice is quite similar to fresh water ice. The density of the water is 1.00 g cm^{-3}. We chose, more or less arbitrarily, $\varrho_s' = 0.15\text{-}0.20$ g cm^{-3} and for the packing function (Eq. 6) $\varrho_{s,\max} = 0.45$ g cm^{-3} and $\pi_o = 0.005\text{-}0.01$ g cm^{-3} day^{-1}.

The heat flux from the sea was treated climatologically. We applied the principle of Uusitalo (1973) and McPhee and Untersteiner (1982) according to which the heat flux can be estimated from observed changes in black ice thickness. The long-term averages of black ice, snow ice and snow thickness (Leppäranta and Seinä 1982) and of air temperature (Finnish Meteorol. Inst. 1963) were taken as

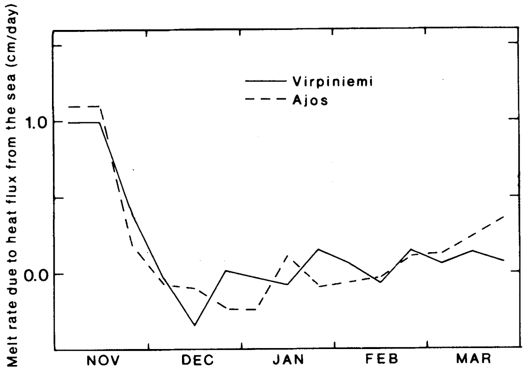

Fig. 2.
Estimated climatological heat flux from the water to the ice.

input data and the heat flux from the sea was calculated from Eq. (2). The thermal conductivity of snow was linearly increased from 1.2×10^{-3} for the first snowfall to $5.3\times10^{-3}\,J^{o}C^{-1}cm^{-1}s^{-1}$ for the end of the growth season (these values correspond to ϱ_s = 0.2 and 0.4 g cm^{-3} in Eq. (7)). For comparison the calculations were also made for the ice observation station Ajos situated by Kemi in the northern Bothnian Bay (the air temperature in Oulu was used). The heat flux Q_w was found to be important until early December being able to melt the ice by up to 1 cm per day (Fig. 2). After that it was small and varied around zero. The negative values are probably due to the uncertainty in the thermal conductivity of snow. Taking a rough eyefit gives us

$$\frac{Q_w}{\rho_i L} = \begin{cases} 1 \text{ cm day}^{-1} - t/40 \text{ cm day}^{-2} & , t \leq 40 \text{ day} , \\ 0 & , t > 40 \text{ day} \end{cases} \tag{12}$$

where t = 0 corresponds to November 1.

The results of the model calculations for the ice season 1976/77 are shown in Fig. 3. First of all we see the high sensitivity of the black ice thickness curve to the assumptions of the snow characteristics. This is due to the strong dependence of the thermal conductivity of snow on density. Clearly, both the density of new snow and the form of the packing function are of great importance. Furthermore, the properties of snow have much effect on the amount of snow ice to be formed.

Discrepancies are seen between the real and calculated graphs and there are several reasons for that. First of all, the snowfall at Oulu airport may not represent very well the snowfall on ice at Virpiniemi and there can be different kinds of local features present in Virpiniemi snow and ice data. Anyway, we feel that the results are promising and further work with snow metamorphism could lead to a reasonable model for simulation and prediction. The present model could be useful in studying the role of different factors in the growth of black ice and snow ice thickness. One could, e.g., gain understanding of how the bearing capacity of an ice cover developes under various external conditions.

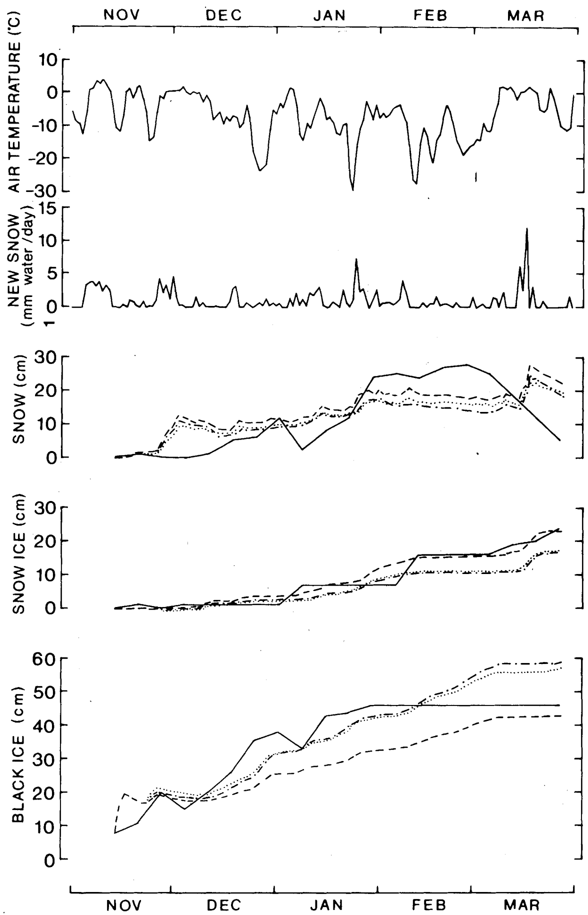

Fig. 3. Model calculations for Virpiniemi 1976/77 with various pairs of density of new snow (g cm^{-3}), packing rate (g cm^{-3} day^{-1}): 0.15, 0.005 (– – –); 0.20, 0.005 (...); 0.15, 0.01 (·– ·–). Solid lines show observed values.

Maximum Annual Ice Thickness

It is interesting to study the importance of various factors on the maximum annual ice thickness. E.g., Palosuo (1981) showed on the basis of observations that the maximum thickness along the Finnish coast of the Baltic Sea occurs in the skerries well away from the coastline. We take now the long-term average air temperature data for Oulu and estimate the thickness of ice near Oulu on March 15 in different hypothetical conditions. Statistics of ice and snow thickness (in cm) for Virpiniemi on March 15 are given below.

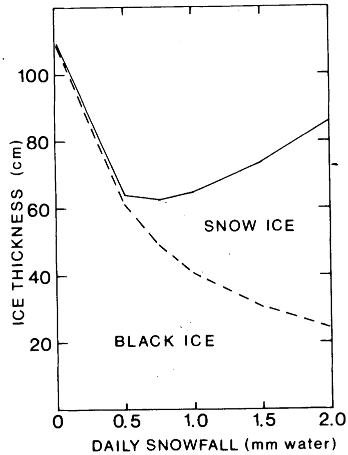

Fig. 4. Ice thickness calculated with climatological air temperature (Oulu) as a function of assumed constant daily snowfall.

	min	mean	max
black ice	41	58	85
snow ice	0	13	31
total ice	53	71	100
snow	0	20	39

The simplest case, no snowfall and no heat flux from the sea, gives h_i = 110 cm. Then we add snow by having a constant daily snowfall with $\varrho_s' = 0.2$ g cm^{-3} and π_o = 0.005 g cm^{-3}. The climatological average is about 0.9 mm equivalent water per day (EWD) in December – February (Kolkki 1969). Black ice thickness decreases fast with snowfall being 45 cm at the average snowfall (Fig. 4). Snow ice thickness becomes notable only when the snowfall is greater than about 0.5 mm EWD and is 18 cm at the average snowfall. The calculated thicknesses at the average snowfall deviate rather much from the average thicknesses above. This is probably due to that the present average snowfall overestimates the average snow income on ice and that we are averaging nonlinear processes. The calculations are also sensitive to the values of ϱ_s' and π_o as will be seen below. Note that there is a minimum total ice thickness at ~0.7 mm EWD. Thus positive and very large negative deviations from the average snowfall tend to increase the total ice thickness.

The sensitivity of the model to the density of new snow and packing function was studied through several simulations using the climatological averages at Oulu for snowfall and air temperature (Fig. 5). A wide range was obtained for the thickness of black ice and snow ice on March 15. We note that an increase in ϱ_s' can be compensated by a decrease in π_o as should be exprected.

Overriding of ice sheets in early winter double the ice thickness locally. Let us now ignore snow and heat flux from the water and let t_1 be the time of overriding. Eq. (2) is then the classical Stefan's equation and can be integrated to

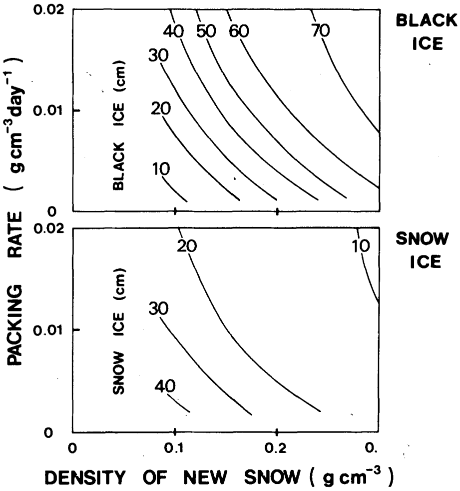

Fig. 5. Ice thickness on March 15, at Oulu, calculated from climatological weather data and with varying density of new snow and packing rate of snow.

$$h_i^2(t_2) = h_i^2(t_1) + \frac{2k_i}{\rho_i L}\int_{t_1}^{t_2}(T_f - T_0)\,dt \qquad (13)$$

for the ice thickness at t_2 without overriding. The thickness with overriding, $h_{i,o}(t_2)$, is obtained through multiplying $h_i(t_1)$ by 2 above. Thus we have

$$\frac{h_{i,0}(t_2)}{h_i(t_2)} = \sqrt{1 + 3\left[\frac{h_i(t_1)}{h_i(t_2)}\right]^2} \qquad (14)$$

The overriding thickness is generally less than 10-20 cm for mechanical reasons (Parmerter 1975) and hence the effect of overriding on the maximum annual ice thickness is small unless the ice does not grow to much more than 20 cm; e.g., with $h_i(t_1)$ = 15 cm, $h_i(t_2)$ = 50 cm we have $h_{i,o}(t_2)$ = 56 cm. This is due to the negative feedback mechanism in ice growth. We can also understand that if the heat flux from the water is significant only in the early ice season, its effect on the maximum annual ice thickness will be small.

Concluding Remarks

We have presented here a growth model for black ice, snow ice and snow thickness driven by the externally given surface temperature, snowfall and heat flux from the water. Emphasis was laid upon snow metamorphism and transformation of snow into snow ice since these processes are of great importance in subarctic regions such as Scandinavia and Finland. The snow layer was acted by a packing function and the snow ice layer grew by the rate of snow sinking beneath the waterline. The classical formula which ignores thermal inertia was taken for the black ice thickness. The results are found promising.

The model includes various physical assumptions and parameters and we consider those involved with snow the least understood. The properties of black ice are rather well known. So is the density of snow ice but its thermal conductivity is questionable. Replacing the surface temperature by the air temperature at the altitude of two metres brings in an error term which can have significance in midwinter when the stratification in the atmospheric surface layer is often stable. The effect of inaccuracies is, however, damped by the negative feedback mechanism in the growth of black ice thickness: underestimation of growth rate leads to reduced insulating effect of the ice, or vice versa. In the case of snow ice there is also a negative feedback mechanism since its underestimation leads to reduced net buoyancy of the ice. But this case is rather complicated because snow – snow ice transformation changes the insulating effect of the material on black ice.

We strongly feel that an intensive field work considering snow on ice should be begun. At least time series on the snowfall and vertical distribution of the density and thermal conductivity of snow should be observed and the packing of snow should be tried to formulate as a function of weather conditions. Another problem is the transformation of snow into snow ice: is our hypothesis of the rate of slush formation correct and how much would it influence if unfrozen slush were treated separately? Once we have got a real touch of these phenomena we can start to consider more advanced modelling – with thermal inertia included and ice and snow temperature coupled with the atmospheric boundary layer.

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