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Discussion

Discussion: "Zeroth-Order Shear Deformation Theory for Laminated Composite Plates" (Ray, M. C., 2003 ASME J. Appl. Mech., 70, pp. 374–380)

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It is the contention of the authors that the "zeroth-order" shear deformation theory presented by Ray [1] is mathematically equivalent to Reddy's third order theory [2]. The notation of Ref. [1] is used herein. Ray's approximations for the in-plane displacements

$$u=u_0-zw_{,x}+\left(\frac{3z}{2h}-\frac{2z^3}{h^3}\right)\frac{Q_x}{\lambda_x},\quad v=v_0-zw_{,y}+\left(\frac{3z}{2h}-\frac{2z^3}{h^3}\right)\frac{Q_y}{\lambda_y}\tag{1}$$

are identical to those of the Reddy's theory, [2], with

$$\psi_x+w_{,x}=\frac{3Q_x}{2h\lambda_x},\quad \psi_y+w_{,y}=\frac{3Q_y}{2h\lambda_y}.\tag{2}$$

Hence the equations of motion and boundary conditions of Ray's theory are mathematically equivalent to those of the dynamic version of Reddy's theory. This is established explicitly by comparing the governing equations of the two theories.

For Reddy's theory, the equations of motion are

$$N_{x,x}+N_{xy,y}=I_0\ddot{u}_0-I_1\ddot{w}_{,x}+\frac{2h}{3}I_8(\ddot{\psi}_x+\ddot{w}_{,x})\tag{3}$$

$$N_{xy,x}+N_{y,y}=I_0\ddot{v}_0-I_1\ddot{w}_{,y}+\frac{2h}{3}I_8(\ddot{\psi}_y+\ddot{w}_{,y})\tag{4}$$

$$\begin{aligned}Q_{x,x}-\frac{4}{h^2}R_{x,x}+Q_{y,y}-\frac{4}{h^2}R_{y,y}+\frac{4}{3h^2}(P_{x,xx}+2P_{xy,xy}+P_{y,yy})+p\\=I_0\ddot{w}+\frac{4}{3h^2}I_3(\ddot{u}_{0,x}+\ddot{v}_{0,y})+\frac{4}{3h^2}\left(I_4-\frac{4I_6}{3h^2}\right)(\ddot{\psi}_{x,x}+\ddot{w}_{,xx}\\+\ddot{\psi}_{y,y}+\ddot{w}_{,yy})-\frac{4I_4}{3h^2}(\ddot{w}_{,xx}+\ddot{w}_{,yy})\end{aligned}\tag{5}$$

$$\begin{aligned}\left(M_x-\frac{4}{3h^2}P_x\right)_{,x}+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)_{,y}-\left(Q_x-\frac{4}{h^2}R_x\right)\\=\frac{2h}{3}\left[I_8\ddot{u}_0-I_9\ddot{w}_{,x}+\frac{2h}{3}I_7(\ddot{\psi}_x+\ddot{w}_{,x})\right].\end{aligned}\tag{6}$$

$$\begin{aligned}\left(M_y-\frac{4}{3h^2}P_y\right)_{,y}+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)_{,x}-\left(Q_y-\frac{4}{h^2}R_y\right)\\=\frac{2h}{3}\left[I_8\ddot{u}_0-I_9\ddot{w}_{,y}+\frac{2h}{3}I_7(\ddot{\psi}_y+\ddot{w}_{,y})\right].\end{aligned}\tag{7}$$

For Ray's theory, the equations of motion are

$$N_{x,x}+N_{xy,y}=I_0\ddot{u}_0-I_1\ddot{w}_{,x}+I_8\frac{\ddot{Q}_x}{\lambda_x}\tag{8}$$

$$N_{xy,x}+N_{y,y}=I_0\ddot{v}_0-I_1\ddot{w}_{,y}+I_8\frac{\ddot{Q}_y}{\lambda_y}\tag{9}$$

$$\begin{aligned}M_{x,xx}+2M_{xy,xy}+M_{y,yy}+p\\=I_0\ddot{w}+I_1(\ddot{u}_{0,x}+\ddot{v}_{0,y})-I_2(\ddot{w}_{,xx}+\ddot{w}_{,yy})+I_9\left(\frac{\ddot{Q}_{x,x}}{\lambda_x}+\frac{\ddot{Q}_{y,y}}{\lambda_y}\right)\end{aligned}\tag{10}$$

$$\begin{aligned}\left(M_x-\frac{4}{3h^2}P_x\right)_{,x}+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)_{,y}-\left(Q_x-\frac{4}{h^2}R_x\right)\\=\frac{2h}{3}\left[\frac{I_7}{\lambda_x}\ddot{Q}_x+I_8\ddot{u}_0-I_9\ddot{w}_{,x}\right]\end{aligned}\tag{11}$$

$$\begin{aligned}\left(M_y-\frac{4}{3h^2}P_y\right)_{,y}+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)_{,x}-\left(Q_y-\frac{4}{h^2}R_y\right)\\=\frac{2h}{3}\left[\frac{I_7}{\lambda_y}\ddot{Q}_y+I_8\ddot{v}_0-I_9\ddot{w}_{,y}\right]\end{aligned}\tag{12}$$

with $I_7=9I_2/4h^2-6I_4/h^4+4I_6/h^6$. Using Eq. (2), it is observed that Eqs. (8), (9), (11), (12) are identical to Eqs. (3), (4), (6), (7). Forming the combination Eq. (10)–Eq. $(11)_{,x}$–Eq. $(12)_{,y}$ yields Eq. (5).

For Reddy's theory, the boundary conditions are obtained from the following boundary integral formed after using Green's theorem in Hamilton's principle

$$\int\left[N_n\delta u_n+N_{ns}\delta u_s+\hat{M}_n\delta\psi_n+\hat{M}_{ns}\delta\psi_s-\frac{4}{3h^2}P_n\delta w_{,n}\right.$$
$$+\left[\hat{Q}_x n_x+\hat{Q}_y n_y+\frac{4}{3h^2}\{(P_{x,x}+P_{xy,y})n_x+(P_{y,y}+P_{xy,x})n_y\right.$$
$$+P_{ns,s}\}-\frac{4I_3}{3h^2}\left\{\ddot{u}_0+\left(I_4-\frac{4I_6}{3h^2}\right)(\ddot{\psi}_x+\ddot{w}_{,x})-I_4\ddot{w}_{,x}\right\}n_x$$
$$\left.\left.-\frac{4I_3}{3h^2}\left\{\ddot{v}_0+\left(I_4-\frac{4I_6}{3h^2}\right)(\ddot{\psi}_y+\ddot{w}_{,y})-I_4\ddot{w}_{,y}\right\}n_y\right]\delta w\right]ds$$
$$-\sum_i\frac{4}{3h^2}\Delta P_{ns}(s_i)\delta w(s_i) \quad (13)$$

where s_i are locations of plate corners and

$$u_n=u_0n_x+v_0n_y,\quad u_s=u_0s_x+v_0s_y$$
$$N_n=N_xn_x^2+N_yn_y^2+2N_{xy}n_xn_y,$$
$$N_{ns}=N_xn_xs_y+N_yn_ys_y+N_{xy}(n_xs_y+n_ys_x)$$
$$\hat{Q}_\alpha=Q_\alpha-\frac{4}{h^2}R_\alpha\ \ (\alpha=x,y),\quad \hat{M}_\beta=M_\beta-\frac{4}{3h^2}P_\beta\ \ (\beta=x,y,xy) \quad (14)$$

with $s_x=-n_y, s_y=n_x$. The expressions of $\hat{M}_n,\hat{M}_{ns};M_n,M_{ns}$; P_n,P_{ns} are similar to those of N_n, N_{ns}, and of ψ_n,ψ_s; $w_{,n},w_{,s}$ are similar to those of u_n,u_s. For Ray's theory the corresponding boundary integral is

$$\int\left[N_n\delta u_n+N_{ns}+\delta u_s+\frac{3\,\delta Q_x}{2h\lambda_x}\left\{\left(M_x-\frac{4}{3h^2}P_x\right)n_x\right.\right.$$
$$\left.+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)n_y\right\}+\frac{3\,\delta Q_y}{2h\lambda_y}\left\{\left(M_y-\frac{4}{3h^2}P_y\right)n_y\right.$$
$$\left.+\left(M_{xy}-\frac{4}{3h^2}P_{xy}\right)n_x\right\}-M_n\delta w_{,n}-M_{ns}\delta w_{,s}$$
$$+\left\{(M_{x,x}+M_{xy,y})n_x+(M_{xy,x}+M_{y,y})n_y\right.$$
$$+\left(-I_1\ddot{u}_0+I_2\ddot{w}_{,x}-\frac{\ddot{Q}_x}{\lambda_x}I_9\right)n_x$$
$$\left.\left.+\left(-I_1\ddot{v}_0+I_2\ddot{w}_{,y}-\frac{\ddot{Q}_y}{\lambda_y}I_9\right)n_y\right\}\delta w\right]ds. \quad (15)$$

Substituting

$$\frac{3}{2h\lambda_x}\delta Q_x=\delta\psi_x+\delta w_{,x}=(\delta\psi_n+\delta w_{,n})n_x+(\delta\psi_s+\delta w_{,s})s_x$$
$$\frac{3}{2h\lambda_y}\delta Q_y=\delta\psi_y+\delta w_{,y}=(\delta\psi_n+\delta w_{,n})n_y+(\delta\psi_s+\delta w_{,s})s_y \quad (16)$$

in Eq. (15) reduces it to

$$\int\left[N_n\delta u_n+N_{ns}\delta u_s+\hat{M}_n\delta\psi_n+\hat{M}_{ns}\delta\psi_s\right.$$
$$-\frac{4}{3h^2}(P_n\delta w_{,n}+P_{n,s}\delta w_{,s})+\left\{(M_{x,x}+M_{xy,x})n_x\right.$$
$$+(M_{xy,y}+M_{y,y})n_y+\left(-I_1\ddot{u}_0+I_2\ddot{w}_{,x}-\frac{\ddot{Q}_x}{\lambda_x}I_9\right)n_x$$
$$\left.\left.+\left(-I_1\ddot{v}_0+I_2\ddot{w}_{,y}-\frac{\ddot{Q}_y}{\lambda_y}I_9\right)n_y\right\}\delta w\right]ds. \quad (17)$$

Substituting the expressions of $M_{x,x}+M_{xy,y}$ and $M_{y,y}+M_{xy,x}$ from equations of motion (11) and (12) into Eq. (17) and using Eq. (2), reduces it to exactly the same expression as in Eq. (13). Hence Ray's theory is not a new theory since its equations of motion and boundary conditions are mathematically equivalent to those of Reddy's theory. The results of this theory for any boundary conditions will be identical to those of Reddy's theory. The statics results of Ray's theory in Table 1 agree with Reddy's results, [2]. The difference in Table 6 from Reddy's results is due to neglect of some inertia terms by Ray while obtaining Navier's solution. Ray's theory is not a zeroth-order theory but Reddy's third order theory in disguise. Moreover, the displacement approximation of Ray's theory is valid only for the case of cross-ply and antisymmetric angle-ply laminates since for the general lay-up, the given expressions of λ_x, λ_y would not be valid.

References

[1] Ray, M. C., 2003, ''Zeroth-Order Shear Deformation Theory for Laminated Composite Plates,'' ASME J. Appl. Mech., **70**, pp. 374–380.
[2] Reddy, J. N., 1997, *Mechanics of Laminated Composite Plates Theory and Analysis*, CRC Press, Boca Raton, FL.