Connection between Particle Models and Field Theories, I

---The Case Spin 1/2---

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(Received February 19, 1951)

If we neglect, in Feynman-Dyson's rule of calculation, the contributions due to all closed loops of fermion line, we may deal with the one-particle theory for the fermion. The transformation found by Pryce, by Foldy and Wouthuysen and also independently by the author enables us to get some intuitive explanation of fermion's behavior when it is free. In this paper we will show that also when the fermions are interacting with other fields, this transformation is rather suitable to get a physical interpretation of the results. For example, we are able to compute the effects of the Zitterbewegung. Although it is profitable to work in the one-particle theory, we must interpret the negative energy states in accordance with the positron theory. So we will discuss how far we should modify the one-particle theory in order to maintain the straight parallelism to the quantized field theory. A remark is added concerning with the interpretation of charge operators, i.e., Dirac operators $\rho$'s. Several results of the transformation are tabulated for the sake of reference in future.

§ 1. Introduction

In the relativistically invariant field theory we may often neglect the effect of vacuum fluctuation of a fermion field. This means to omit the effect of all closed loops of the fermion; in this case every fermion line is continuous and the number of fermion lines will be never changed. So we may start with the one-particle theory if we only pay attention to the treatment of the negative energy states. In the present paper we shall treat the case of spin 1/2. The cases of boson with spin 0 or 1 may be treated in an analogous way, which will be shown in the following paper.

Let us consider a single fermion interacting with some type of meson field. (Of course, the meson field should be quantized, for the mesons are emitted or absorbed.) There is no difficulty in making a generalization to cases of two or more fermions. We start from the Hamiltonian formalism in which time plays a special role. We prefer this to the relativistically symmetrical formalism, because here we concern ourselves with the intuitive explanation of the results rather than the mathematical aspects of Lorentz-covariance.

The Hamiltonian of a free fermion $H_0$ is given by

$$H_0 = m_0 \gamma_0 + (p_0)\gamma_1.$$  (1)
In the usual matrix representation of \( \rho \) and \( \sigma \), this Hamiltonian is diagonal when \( p = 0 \), and in this case the operator \( \rho_3 \) can be directly interpreted as the sign of the energy. In order to maintain this property also for the case of \( p \neq 0 \), it is more convenient to eliminate the "perturbation" term \( (p\sigma)p_3 \) and diagonalize \( H_0 \) by means of a transformation, which has been proposed by Pryce, by Foldy and Wouthuysen and also independently by the author. Then the Hamiltonian has the form:

\[
U^{-1}H_0 U = \sqrt{m^2 + p^2} \cdot \rho_3 ,
\]

where

\[
U = \exp \left( -\frac{i}{2} \frac{p\sigma}{\rho} \rho_3 \tan^{-1} \left( \frac{\rho}{m} \right) \right).
\]

After the transformation the wave function is also reduced to a very simple form, (for pos. energy states)

\[
\exp \left[ i(pX - E_p \tau) \right] \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left( (p\sigma) = +\rho \right), \quad \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left( (p\sigma) = -\rho \right),
\]

(for neg. energy states)

\[
\exp \left[ i(pX + E_p \tau) \right] \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left( (p\sigma) = +\rho \right), \quad \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left( (p\sigma) = -\rho \right),
\]

here and later on making use of the abbreviation,

\[
E_p = \sqrt{m^2 + \rho^2}.
\]

The transformation defined by (3) makes it possible to interpret the free fermion's behavior semi-classically; As will be discussed in the next paragraph, we can readily see the velocity dependence of operators \( \rho \)'s or \( \sigma \)'s in the operator equations in the new representation. Also we can separate out the Zitterbewegung amplitude from the coordinate operator, which behaves just in the same way as the position of a classical particle treated relativistically. In this paper we are interested mainly in the benefit of making this transformation, which seems not to have been fully appreciated by other authors; and we will show that by virtue of this transformation we are able to find a semi-classical model of the fermion, which is interacting with some type of meson field.

§ 2. Transformation \( U \) Defined by Eq. (3)

At the beginning, one should notice to the fact that if one writes the operator

* In the following we employ the \( p \)-representation rather than the \( X \)-representation, because the momentum of a particle is more useful in the physical interpretation of the results than its position.
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$U$ given by (3) explicitly as a 4-4-matrix, then one will find that each column is nothing but one of four normalized orthogonal solutions of the usual Dirac equation respectively.

As far as we are concerned with the operators commutable with $p$, we may look upon $p$ as a parameter. Let $A$ denote an operator in the new representation transformed by means of $U$ and let $A^o$ be its old and usual representation. Then, they are connected with each other by the relation,

$$A = U^o - 1 A^o U^o,$$

where $U^o$ means the operator given by (3) in which spin operators $\rho^o$'s and $\sigma^o$'s are inserted. As the result of this transformation, the new representation $A$ becomes dependent of the velocity of the particle also when the dynamical variable $A^o$ does not depend on it. On the other hand, the spin part of the wave function is velocity independent (see (4)), so the velocity dependence of $A$ gives directly the information how the expectation values of various spin quantities depend on the velocity of the particle.

In the following we reproduce the results of the transformation for several dynamical quantities.

(i) The $\rho$-operators

$$U^{-1} \rho_1 U = \frac{m}{\sqrt{m^2 + p^2}} \rho_1 + \frac{(p \sigma)}{\sqrt{m^2 + p^2}} \rho_3,$$  \hspace{1cm} (A, 1)

$$U^{-1} \rho_2 U = \rho_2,$$  \hspace{1cm} (A, 2)

$$U^{-1} \rho_3 U = -\frac{(p \sigma)}{\sqrt{m^2 + p^2}} \rho_1 + \frac{m}{\sqrt{m^2 + p^2}} \rho_3.$$  \hspace{1cm} (A, 3)

An interpretation of these equations for the charge operators will be investigated later. Here we give only the following remark. As has been discussed by Fock, if we imagine a fictitious 3-dimensional $p$-space, we shall find this $p$-space rotating about a certain axis in the Heisenberg representation, the angular velocity of rotation being $2 \sqrt{m^2 + p^2}$. The axis of rotation in the usual representation of $\rho$'s lies in the $\rho_1 - \rho_2$-plane and deviates from the $\rho_3$-axis by the angle $\sin^{-1} ((p \sigma)/\sqrt{m^2 + p^2})$. In our representation this axis is fixed and represented by $\rho_3$; but when $p$ grows greater $p$-space rotates also about the $\rho_3$-axis. This fact is illustrated in Fig. 1.

(ii) The spin operators

The transformation for the spin vector is given by

$$U^{-1} \sigma U = \sigma_{\parallel} + \frac{m}{\sqrt{m^2 + p^2}} \sigma_{\perp} + \frac{1}{\sqrt{m^2 + p^2}} [p \times \sigma] \rho_2,$$  \hspace{1cm} (B)

* The equations of motion for $\rho$'s show that their motion is of spherical character. This has a certain connection with the use of exclusion principle when quantized. This point will be discussed in Part II where we shall meet with the hyperbolic cases corresponding to symmetrical statistics.
where
\[ \sigma_{||} = \frac{(p \sigma)}{\rho^2} \]  
(7)
and
\[ \sigma_\perp = \sigma - \sigma_{||} \]  
(8)
denotes the longitudinal and transverse part of spin-vector respectively.
The velocity dependence of the spin, which is shown in several textbooks by computing its expectation value, is shown in (B) as an operator relation. The transverse part becomes smaller by the Lorentz factor \( \sqrt{1 - \gamma^2} = \frac{m}{\sqrt{m^2 + p^2}} \), while the longitudinal part remains unchanged.

(iii) The position operator
The transformation for the position operator is given by
\[ U^{-1}X U = \tilde{X} + \hat{X}, \]  
(C)
where
\[ \tilde{X} = \frac{1}{2} \sqrt{m^2 + p^2} \left( \sigma_\perp + \frac{m}{\sqrt{m^2 + p^2}} \sigma_{||} \right) \rho_\gamma, \]  
(C, 1)
and
\[ \hat{X} = \frac{1}{2} \left( 1 - \frac{m}{\sqrt{m^2 + p^2}} \right) \frac{p}{\rho_\gamma} \rho_\gamma. \]  
(C, 2)

\( X \) is the coordinate operator in the semi-classical sense; in other words, it is canonically conjugate to the momentum \( p \), and its time derivative (in the new representation) corresponds to the relativistic velocity of the particle; that is,
\[ \dot{X} = \frac{1}{i} [X, \sqrt{m^2 + p^2} \rho_\gamma] = \frac{p}{\sqrt{m^2 + p^2}} \rho_\gamma. \]  
(9)

\( \hat{X} \) denotes the position shift due to Schroedinger's Zitterbewegung. The longitudinal magnitude is \( 1/2m \) times the Lorentz factor. The transverse-longitudinal ratio is also \( \sqrt{1 - \gamma^2} \). Its time derivative is equal to that part of \( \rho_\gamma \sigma \) which gives rise to pair formation and destruction. (Cf. Eq. (D, 1) in (iv)). That is, we have
\[ \dot{\hat{X}} = \rho_\gamma \left( \sigma_\perp + \frac{m}{\sqrt{m^2 + p^2}} \sigma_{||} \right). \]  
(10)

\( \hat{X} \) can be interpreted as a shift of the centre of motion. This term has the
effect to compensate the decrement of the transverse spin angular momentum and conserve the total angular momentum against an adiabatic change of the translational momentum. This point of view is justified as follows: The total angular momentum is a constant of motion and invariant against the transformation \( U \). That is,

\[
U^{-1}\left([X \times p] + \frac{1}{2} \sigma \right) U = [X \times p] + \frac{1}{2} \sigma.
\]

But the left hand side is

\[
[X \times p] + [\tilde{X} \times p] + [\tilde{X} \times p] + \frac{1}{2} \sigma_1 + 2 \frac{m}{\sqrt{m^2 + p^2}} \sigma_2 + \frac{1}{2} \frac{1}{\sqrt{m^2 + p^2}} \eta_2 [p \times \sigma],
\]

according to (B) and (C). Each term in (12) is velocity dependent; in other words, gets changes by the transformation \( U \). So there should be some pairs of terms to compensate with each other and keep the whole expression invariant. The first of these pairs is the "orbital angular momentum associated with the Zitterbewegung" and the odd part* of the spin angular momentum. They are connected by

\[
[X \times p] = -\frac{1}{2} \frac{1}{\sqrt{m^2 + p^2}} [p \times \sigma] \rho_2.
\]

The second is the "increase of the orbital angular momentum by virtue of the shift \( \tilde{X} \)" and the decrease of the transverse spin angular momentum which has been discussed in (ii). Because there is the relation,

\[
[\tilde{X} \times p] = \frac{1}{2} \left(1 - \frac{m}{\sqrt{m^2 + p^2}} \right) \sigma_1,
\]

which justifies the interpretation of \( \tilde{X} \) stated above. It should be remarked that its time derivative vanishes, i.e.,

\[
\dot{\tilde{X}} = 0.
\]

The term \( X \) together with \( \tilde{X} \) constitute the even part (i.e. the part which does not give rise to pair formation and destruction) of \( X \) in the usual representation. This even part may be identified to the position of the centre of gravity in the relativistic mechanics of a classical particle. This point has been fully discussed by Pryce [1] and Möller.[2]

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* We use the term "odd part" for that part of any operator which contains either \( \rho_1 \) or \( \rho_2 \), and "even part" for that containing neither of them.

** Möller has availed himself of the transformation of variables which may be defined by means of \( A \rightarrow A = U^{-1} A U \),

where

\[
U = \exp \left[ \frac{i}{2} \tan^{-1} \left( \frac{\rho}{\eta} \right) \cdot \rho (1 - \frac{\rho}{\eta}) \right],
\]

and essential points in his results are the same as ours.
(iv) **Supplement**

\[ U^{-1} \rho_1 \sigma U = \frac{p}{\sqrt{m^2 + p^2}} \rho_3 + \left( \frac{m}{\sqrt{m^2 + p^2}} \sigma \right) \rho_1, \]  
\( (D, 1) \)

\[ U^{-1} \rho_2 \sigma U = \frac{1}{\sqrt{m^2 + p^2}} \left[ p \times \sigma \right] + \left( \frac{m}{\sqrt{m^2 + p^2}} \sigma \right) \rho_2, \]  
\( (D, 2) \)

\[ U^{-1} \rho_3 \sigma U = \left( \frac{m}{\sqrt{m^2 + p^2}} \sigma \right) \rho_3 - \frac{p}{\sqrt{m^2 + p^2}} \rho_1. \]  
\( (D, 3) \)

\( (D, 1) \) is the transformation for the operator \( \rho_1 \sigma \) which appears in the expression of the current in the case of electro-magnetic interaction.

By summing up Eqs. (D, 2) above and (C) in (ii), we see that the operators \( \sigma \) and \( \rho_2 \sigma \) together form a six-vector and their transformation character with respect to the Lorentz transformation is shown in the right hand side of these equations. For most of other quantities, e.g. the scalar density \( \rho_3 \) or the time-component density of the 4-vector \( I \), our transformation equations show their Lorentz properties only when integrated over the whole space, but not the transformation properties of the densities themselves.

§ 3. **The treatment of the interaction**

In this paragraph we show that the transformation defined by (3) is much useful in treating the problem of field reaction.

In the ordinary representation of fermion operators, the interaction Hamiltonian \( H' \) of a fermion with meson field is of the form

\[ H' = \sum \sum \alpha_{k\lambda}^{*} e^{-i(k \cdot x)} \langle \text{interactor} \rangle + \text{hermitian conjugate}, \]  
\( (16) \)

where \( \alpha_{k\lambda}^{*} \) denotes the emission operator of the meson with momentum \( k \) and polarization \( \lambda \), and in the place of \( \langle \text{interactor} \rangle \) one should put a suitable operator according to what type of meson and coupling is considered. (E.g. in the case of scalar meson and scalar coupling, put \( g(2\pi/e_k)\rho_3 \) in this place.) We denote the coupling constant by \( g \) and abbreviate as

\[ \epsilon_k = \sqrt{\mu^2 + k^2}, \]  
\( (17) \)

\( \mu \) being the meson mass. Naturally the total Hamiltonian is given by

\[ \mathcal{H} = H_0 + H' + H_{\text{mes.}}, \]  
\( (18) \)

where \( H_{\text{mes.}} \) is the Hamiltonian of the isolated meson field which is given by

\[ H_{\text{mes.}} = \sum \sum \epsilon_k \alpha_{k\lambda}^{*} \alpha_{k\lambda} \]  
\( (19) \)

Now in order to treat the field reaction, let us go over into the interaction representation of the total system. Then the Hamiltonian function \( (\mathcal{H})_{\text{int.}} \) has

\[ ( )_{\text{int.}} \]  
denotes the respective operator in the interaction representation.
the form

$$\langle \text{interactor} \rangle \cdot \exp \left[-i(m\rho_3 + (p\sigma)\rho_1)\tau_1\right] + \text{herm. conj.} \quad (20)$$

In interaction representation, the operators $\rho$'s or $\sigma$'s behave in the same way as in the Heisenberg representation of free motion. Though the equations of motion for $\rho$'s and $\sigma$'s are integrated by Fock, the answers look too much complicated to be written down as concise operator relations. Moreover, the factor $-i(kX)$ behaves much more unharmonically in time due to the existence of the Zitterbewegung. Thus, in the representation hitherto used, the time dependence of the right hand member of (20) is far from a simple exponential one; so we must make use of the familiar technique of projection operators, and write down the matrix elements of $\langle \text{interactor} \rangle$, abandoning the exclusive use of the operator form. These troubles are caused by the fact that the operator $m\rho_3 + (p\sigma)\rho_1$ is not yet diagonalized explicitly. But after the transformation $U$ we have a simple exponential time factor, as will be revealed in the following.

If we perform the transformation $U$ we have

$$U^{-1}\langle \text{interactor} \rangle U = \sum_\mathcal{K} \sum_k a_{k\mathcal{K}} \exp (i\epsilon \mathcal{K}^\prime) \cdot \exp \left(i \sqrt{m^2 + p_\mathcal{K}^2} \tau_1\right) \cdot U^{-1} \cdot e^{-i(kX)} \cdot U \times$$

$$U^{-1}\langle \text{interactor} \rangle U \cdot \exp (-i\sqrt{m^2 + p_\mathcal{K}^2} \tau_1) + \text{herm. conj.} \quad (21)$$

At first sight, the right hand side of (21) may look rather complicated; however, it is arranged in a physically interesting way. In the new representation of the interactor, $U^{-1}\langle \text{interactor} \rangle U$, one can see the velocity dependence of the interactor, as was shown in the preceding paragraph. Let us consider the operator $e^{-i(kX)}$ in the new representation and use the following notation

$$U^{-1}e^{-i(kX)}U = e^{-i(kX + \vec{X} + \vec{X}')} = e^{-i(kX)} \cdot \langle \text{recoil} \rangle, \quad (22)$$

where the second statement is due to the results of (iii) of the preceding paragraph. The term $\langle \text{recoil} \rangle$ turns out to be unit operator either if the recoil momentum $k$ might vanish or if the fermion had no inner degree of freedom other than that of translation. We will analyze this term in the next paragraph. As will be shown there, we can consider $\langle \text{recoil} \rangle$ as expressing the disturbance of inner degree of freedom $\rho$ and $\sigma$ caused by the recoil. It consists of two types of expressions, one containing $p_2$ and the other free of $\rho$'s. Now we should classify the operator $\langle \text{recoil} \rangle \cdot U^{-1}\langle \text{interactor} \rangle U$ into two parts, one of which contains either $\rho_1$ or $\rho_2$ and the other does not contain. The term containing $\rho_1$ or $\rho_2$ will gives rise to a pair formation or destruction. The distinction of the sign of energy is self-evident after the transformation, and only $\rho_1$ and $\rho_2$ causes the change of the sign of energy. Consequently one will never feel it necessary to use the projection operators. It is noticeable that the equations of
motion for \( \rho \)'s and \( \sigma \)'s are very simple in our representation. Only \( \rho_1 \) and \( \rho_2 \) show the dependence on time, which is given by

\[
(\rho_1)_{\text{int}} = \rho_1 \cdot \exp \left[ -i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] = \exp \left[ i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] \cdot \rho_1,
\]

\[
(\rho_2)_{\text{int}} = \rho_2 \cdot \exp \left[ -i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] = \exp \left[ i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] \cdot \rho_2.
\]

The operator of momentum transfer, \( e^{-i(k \cdot X)} \), also behaves in a simple manner, which was not the case hitherto; that is to say, we have

\[
(e^{-i(k \cdot X)})_{\text{int}} = e^{-i(k \cdot X)} \cdot \exp \left[ i(\sqrt{m^2 + (p + k)^2} - \sqrt{m^2 + p^2}) t \cdot \rho_3 \right] = \exp \left[ i(\sqrt{m^2 + p^2} - \sqrt{m^2 + (p - k)^2}) t \cdot \rho_3 \right] \cdot e^{-i(k \cdot X)}.
\]

As the consequence of these simplification brought about by \( U \), we get a single exponential time factor, after uniting the time factors of the term \( \langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U \rangle_{\text{int}} \) and of the term \( (e^{-i(k \cdot X)})_{\text{int}} \). This procedure is equivalent to shift the factor \( \exp \left[ -i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] \) in (21) from the right to the left and \( \exp \left[ +i2 \sqrt{m^2 + p^2} t \cdot \rho_3 \right] \) from the left to the right till they are unified into a single exponential function, thereby taking account of the commutation relations for \( p \)'s and \( e^{-i(k \cdot X)} \).

Finally, we have the Hamiltonian in the interaction representation, which is of the form

\[
U^{-1}(\bar{\psi})_{\text{int}} U = \sum \sum \sum a^\dagger_{k\lambda} \exp \left[ i(\sqrt{m^2 + p^2} \rho_3 \pm \sqrt{m^2 + (p - k)^2} \rho_3 + \epsilon_k) t \right] \times
\]

\[
\cdot e^{-i(k \cdot X)} \cdot \langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U + \text{herm.conj.}
\]

Here the double sign in the time factor should be settled to be positive when we deal with that part of \( \langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U \) containing \( \rho_1 \) or \( \rho_2 \) and to be negative for the other part. (The summation indices \( \pm \) denote the summation over both parts in Eq. (25).) In our new representation we find interaction operators of somewhat complicated appearance, but the physical meaning of these operators is clear and able to be read directly. Moreover, the interaction Hamiltonian shows a simple dependence on time in the interaction representation. Thus we can readily find a physical interpretation of field reaction on the fermion as far as we take the free motion as the basis of describing the effect of the meson field. If we desire, we can at once make a non-relativistic or extreme-relativistic approximation to any order in \( p/m \) or \( m/p \).

In computing the perturbation effects caused by \( H' \) by means of the contact transformation method, we meet with the integral of the form,

\[
\int U^{-1}(\bar{\psi})_{\text{int}} U \, dt.
\]

Then, as is evident from (25), we have the following factors after integration

\[
\exp \left[ i(E p \rho_3 \pm E p - k \rho_3 + \epsilon_k) t \right] \over i(E p \rho_3 \pm E p - k \rho_3 + \epsilon_k)
\]

(27)
Comparing this with the corresponding expression of the quantized field theory, we find that in our calculation the energy differences are as well exact for transitions accompanied by pair formation or destruction as for those not accompanied by them; this is due to the existence of the factor $\rho_3$ and the discrimination of the double sign in (25). Such correspondence to the quantized theory is automatically established also in any higher order processes, which should be computed from the following type of integrals.

\[
\left( \frac{1}{i} \right)^n \int dt' \int dt'' \cdots \int dt_n \langle \tilde{\phi}(t') \rangle \cdot \langle \tilde{\phi}(t'') \rangle \cdots \langle \tilde{\phi}(t_n) \rangle.
\]

This is due to the fact that the requirement of the momentum conservation in higher order processes are automatically fulfilled by virtue of the spatial exponential factor $e^{\pm i(kX)}$. But if we want to maintain a perfect parallelism of our scheme to that of the quantized theory from which all closed loops are dropped out, we must deviate a little from the formal procedure: That is, instead of computing with

\[
\left( \frac{1}{i} \right)^n \int dt' \int dt'' \cdots \int dt_n U^{-1} \cdot (\tilde{\phi})_{\text{int}} \langle t' \rangle \cdot (\tilde{\phi})_{\text{int}} \langle t'' \rangle \cdots \cdot (\tilde{\phi})_{\text{int}} \langle t_n \rangle \cdot U,
\]

which results from merely substituting $U^{-1}(\tilde{\phi})_{\text{int}} U$ given by (25) into (28), we must employ the following expression

\[
\left( \frac{1}{i} \right)^n \int dt' \int dt'' \cdots \int dt_n \rho_3 U^{-1} \cdot (\tilde{\phi})_{\text{int}} \langle t' \rangle \cdot \rho_3 \cdot U^{-1} (\tilde{\phi})_{\text{int}} \langle t'' \rangle \cdot U \times \\
\times \cdots \cdots \times \rho_3 \cdot U^{-1} \cdot (\tilde{\phi})_{\text{int}} \langle t_n \rangle \cdot U.
\]

This prescription involves nothing new, because it corresponds to the use of projection operator technique in the usual method of calculation and distinguishing the sign of energy in order to take only the holes of the negative energy sea as giving the true contributions. Let $^+A(p)$, $^−A(p)$ respectively denote the projection operator into the positive or negative energy state for a fermion with momentum $p$. In our representation the operator which represents the operator $^+A(p)−^−A(p)$ is nothing but $\rho_3$ irrespective of $p$. So by inserting $\rho_3$ our results will become in conformity with that of position theory. In the Appendix I, we shall check these points more closely.

§ 4. Recoil of a spin 1/2 particle

As was already mentioned in the preceding paragraph, we meet in the interaction Hamiltonian with a factor of the type given by (22).

Here we give the term $\langle \text{recoil} \rangle$ explicitly as

* N.B. $U^{-1} e^{-i(kX)} U = e^{-i(kX)} \cdot e^{i(kX)} U^{-1} e^{-i(kX)} U = e^{-i(kX)} U^{-1}(p − k) \cdot U(p)$. 
\[ \langle \text{recoil} \rangle = \exp \left\{ -i \Phi \left( \rho_{\pm}(\sigma \mathbf{z}) + (\sigma \mathbf{r}) \right)/\sqrt{\mathbf{z}^2 + \mathbf{r}^2} \right\}, \]  
where
\[ \mathbf{z} = (\sqrt{m^2 + p^2 + m}) \mathbf{k} + (\sqrt{m^2 + (p - k)^2} - \sqrt{m^2 + p^2}) \mathbf{p}, \]
\[ \mathbf{r} = [\mathbf{k} \times \mathbf{p}], \]

and
\[ \sin \Phi = \left\{ \frac{\mathbf{z}^2 + \mathbf{r}^2}{4E_p(m + E_p)E_{p-k}(m + E_{p-k})} \right\}^{1/2}. \]

In (34) the variable \( \Phi \) vanishes in case of \( k = 0 \). The \( \langle \text{recoil} \rangle \) would have been reduced to unit operator if the particle had no inner degree of freedom so that the charge operators \( \rho_{\pm} \)'s and the spin operators \( \sigma \)'s disappeared and the effect of the recoil was only to change the momentum from \( \mathbf{p} \) to \( \mathbf{p} - \mathbf{k} \). Thus we are justified in interpreting the term \( \langle \text{recoil} \rangle \) as due to the recoil disturbance of the inner degree of freedom. Expanding the exponential in (31) we have
\[ \langle \text{recoil} \rangle = \cos \Phi + i \sin \Phi \cdot \frac{\rho_{\pm}(\sigma \mathbf{z}) + (\sigma \mathbf{r})}{\sqrt{\mathbf{z}^2 + \mathbf{r}^2}}. \]

The term containing \( \mathbf{z} \) may be interpreted as the disturbance of the Zitterbewegung caused by the recoil, for the factor \( \rho_{\pm}(\sigma \mathbf{z}) \) has a similar construction as the Zitterbewegung in the free motion given by (C, 1). The term containing \( \mathbf{r} \) corresponds to the disturbance of the shift of the centre of motion given by (C, 2). In fact, such correspondence is justified in the non-relativistic limit. In this case the term \( \mathbf{X} \) and \( \mathbf{X} \) is given by
\[ \mathbf{X} \approx \frac{1}{2m} \rho_{\pm} \sigma \]
and
\[ \mathbf{X} \approx \frac{1}{4m^2} [\mathbf{p} \times \sigma], \]
respectively to the first order of \( \mathbf{p}/m \). Putting these expressions into (22) we have the expression
\[
\varepsilon^{-i(k, X + \mathbf{X} + \mathbf{X})} \approx \exp \left\{ -i \left( \frac{1}{2m} \rho_{\pm}(\mathbf{r}) + \frac{1}{4m^2} (\sigma \cdot [\mathbf{k}, \mathbf{p}]) \right) \right\}
\]
\[
= \varepsilon^{-i(kX)} \left[ -i \left( \frac{1}{2m} \rho_{\pm}(\sigma \mathbf{k}) + \frac{1}{4m^2} (\sigma \cdot [\mathbf{k}, \mathbf{p}]) \right) \right],
\]
which checks our statement.

\( \Phi \) is the safety measure how far we may neglect the disturbance of inner degrees of freedom due to the recoil. The behavior of \( \Phi \) as a function of \( p/m, k/m, \) and \( \theta \) (the angle between \( \mathbf{p} \) and \( \mathbf{k} \)) is much complicated: and it is hard to make a general discussion about this behavior. Here we restrict
ourselves to the limiting case of $k \rightarrow \infty$. In this case (34) reduces to

$$\sin \Phi_{k, \infty} = \frac{1}{2} \left\{ \frac{(1 + \sqrt{1 + (p/m)^2} - (p/m) \cos \theta)^2 + (p/m)^2 \sin^2 \theta}{(1 + \sqrt{1 + (p/m)^2}) \sqrt{1 + (p/m)^2}} \right\}^{1/2}$$

(39)

It should be remarked that it is only when $p \rightarrow \infty$ and $\theta = \pi$ that $\Phi_{k, \infty}$ reaches the value $\pi/2$ and the recoil effects on the spin quantities are expected to change the character of the interactor entirely.

§ 5. Some examples of application

In our system of calculation, operators appear a little more complicated than those in the current use, but they readily furnish us with physical models and are suitable for a qualitative discussion. For a precise numerical evaluation the resort to the Feynman-Dyson's scheme is recommended. Some results of applying our method will be surveyed in the following.

(i) Compensation effect in the self-energy problem

As was explained at the end of § 3, we will insert $\rho_3$'s in the train of the transition operators in the transformation function by which the field reaction is taken into account. We might expect that in computing the 2nd order self-energy, the contributions due to pair formation in the intermediate state and those which do not come from intermediate pairs cancel each other. In higher order processes, it is not a simple matter to make a general discussion on the classification of compensating terms. But in each practical case, the computation is not so hard. As will be observed in Appendix II, there are two groups of interaction type, i.e. $\rho_1 - \rho_1$ group and $1 - \rho_2$ group. $\rho_3 - \rho_1$ group comprises those interaction type with $\rho_3$ for their even part and $\rho_1$ for their odd part. As long as the fermion undergoes a single type of interaction, the mixing of $\rho_3 - \rho_1$ type and $1 - \rho_2$ type in higher order processes does not occur. Then, for classification of compensating terms, one should compute the sign of a train like either

$$\rho_3 \rho_3 \rho_3 \rho_3 \cdots$$

and

$$\rho_3 \rho_3 \rho_3 \rho_3 \rho_3 \cdots$$

etc. (40)

or

$$\rho_3 \rho_3 \rho_3 \rho_3 \rho_3 \cdots$$

and

$$\rho_3 \rho_3 \rho_3 \rho_3 \rho_3 \cdots$$

etc. (41)

(Here in order to distinguish the inserted $\rho_3$ from $\rho$'s contained in the original expression, we have put a bar upon them.)

Now in a process where the ultra-violet contribution plays a main role, the compensation effect will be stressed for that type of interaction which contains $\rho_3$ in the interactor of the usual representation, as compared to $\rho_1$ type; for, as the consequence of recoil by ultra-violet mesons, the fermion have a large virtual momentum, and $\rho_3$ goes over into $\rho_1$. (Cf. § 2, Eqs. (A), the velocity dependence of $\rho$-operators). A similar reasoning shows us that 1-type interaction will be
much more compensated than $\rho_2$-type. In fact, as is well known, the negative value of the second order self-energy in cases of neutral scalar and pseudovector mesons were found and investigated by several authors. \cite{9,10}

(ii) Anomalous magnetic moment

Let us consider a fermion of electric charge $e$ in a homogeneous magnetic field of strength $H$ and the direction $e$ (an unit vector). Then we have the interaction Hamiltonian of the fermion with this magnetic field

$$eU^{-1} \rho_1 (\sigma A) U,$$

with

$$A = \frac{1}{2} H [e \times X].$$

As the results of the transformation shown in § 2, we have

$$\frac{eH}{2} \left( \frac{p}{\sqrt{m^2 + p^2}} \rho_3 + \left( \frac{m}{\sqrt{m^2 + p^2}} \sigma_\parallel \right) \rho_1 \right) \cdot \left( [e \times X] + [e \times X] + [e \times X] \right).$$

The second part of the first factor is proportional to $\dot{X}$, i.e. the current associated with the Zitterbewegung. In the non-relativistic limit we can put, according to (36) and (37)

$$\dot{X} = \frac{1}{2m} \rho_2 \sigma, \quad \dot{X} = \frac{1}{4m^2} [p \times \sigma],$$

and have the reduced Hamiltonian

$$\frac{eH}{2} \left( \frac{p}{\sqrt{m^2 + p^2}} \rho_2 + \rho_2 \sigma \right) \cdot \left( [e \times X] + \frac{1}{2m} [e \times \sigma] \rho_2 + \frac{1}{4m^2} [e \times [p \times \sigma]] \right),$$

in which we find the interaction energy due to the normal magnetic moment,

$$\frac{eH}{4m} (\rho_2 \sigma \cdot [e \times \sigma]) \rho_2 = \frac{eH}{2m} \rho_3 (\sigma e)$$

Thus we recognize the position shift of Zitterbewegung and the current accompanied by it as the origin of the normal magnetic moment. If the fermion interacts with some kind of meson field, then the Zitterbewegung amplitude will be modified and we shall have

$$e^{-is} \dot{X} e^{is}$$

in the place of $\dot{X}$ with no mesonic interaction. Here $e^{is}$ denotes the transformation function to take the virtual meson cloud into account. How to construct this transformation function in its interaction representation was discussed in § 3. We may bring it back into its Schroedinger representation by merely dropping the exponential time factor. By actual calculation it turns out that this transformation function must modify the factor $\rho_2$ in the unperturbed Zitter-amplitude.
in order that we might have a positive sign of anomalous magnetic moment. Because the modification of the $\sigma$ factor (in the unperturbed amplitude) will always decrease the effective value of the spin component in a certain direction. These points were fully discussed by Welton\textsuperscript{11} and Koba.\textsuperscript{12}

In the transformation function the operators $S$ has the same character as the interactor in the first approximation. From this fact, one may expect a positive sign of anomalous magnetic moment in the case of neutral scalar meson\textsuperscript{13,14} or of neutral vector meson (photon), and a negative sign in the case of neutral pseudoscalar.\textsuperscript{15} In the former cases the interactor have a factor $\rho_3$ or $\rho_1$ respectively; so these cases are expected to give rise to a modification of $\rho_2$ in the unperturbed amplitude, for this modification is determined to be great or small essentially according to the order of magnitude of $[\rho_2, S]$. On the other hand, in the latter case of pseudoscalar, the interactor is $\rho_2$, which is obviously commutable with $\rho_3$ in the unperturbed Zitter-amplitude; then the modification of $\sigma$ dominates and the result becomes negative.

By making use of the calculational technique developed in this paper, one can find Koba's model of the anomalous magnetic moment of an electron out of the usual quantum electrodynamical scheme; but we will not go into the details of this problem any further.

(iii) \textit{Minute remarks on nuclear interactions}

At the end of § 4, we have noticed that in the expression

$$\langle \text{recoil} \rangle = \cos \phi + i \sin \frac{\rho_3 (\sigma \mathbf{Z}) + (\sigma \gamma)}{\sqrt{Z^2 + T^2}}$$

$\cos \phi$ vanishes only in the special case of $k \to \infty$, $p \to \infty$, $\theta = \pi$. Thus, generally speaking, it is rather small region of the integration domain where the type of the effective interactor becomes altered very much from the original expression in which no account is taken of the disturbance of $\rho$'s and $\sigma$'s. From this fact we may say that interaction types containing derivative of the meson field amplitude (in their usual representation) will yield higher singularity compared to those free from derivative. However, this cannot be considered as contradicting to the equivalence theorem\textsuperscript{16} in nuclear force problem, because in the $g^2$-order and Born approximation the even part of interactor of the vector-type coupling is proportional to the respective operator of the scalar-type coupling, as is verified from the results in Appendix II. The equivalence is guaranteed in such a problem as we retain only the lowest order contributions, but the deviation occurs in higher order contributions, especially when we take the intermediate pair creations into account.

At the end we add that the recoil disturbance of Zitterbewegung yields the tensor force without $r^{-3}$-singularity in the case of pseudoscalar meson and pseudoscalar coupling;\textsuperscript{17} and that the recoil shift of centre $\hat{X}$ yields the spin-orbit coupling in the case of scalar meson and scalar coupling, which was discussed by Dancoff.\textsuperscript{18}
§ 6. Interpretation of $\rho$-operators

We have met in § 2 the rotation of $\rho$-space. So here we add some points of interest in the physical interpretation of $\rho$'s, by the way.

The expectation value of $\rho_3$, $\rho_5$, $I$, and $\rho_1$ are scalar, pseudoscalar, 4-component of vector and of pseudovector respectively. These four operators constitute a closed set, in which every product of their linear combinations is again such a linear combination of them. The multiplication law in this closed set suggests us how these operators should be interpreted physically. However, we take a different way and for some while consider the relativistically symmetrical formalism, and then come back to the formalism initially adopted. This way might be longer but more smooth than the short-cut mentioned above.

Now, in the relativistically symmetrical formalism, the above mentioned density operators correspond to $I$, $\gamma_{1234}$, $\gamma_4$, and $\gamma_{125}$ respectively. They are generators of reflexions in the representation of the full Lorentz group. But we can interpret them as generators of reflexions of world line in Feynman diagram, instead of reflexions of coordinate axis. This will be clearly seen in the $S$-matrix of Feynman's theory. The $S$-matrix is constituted of a sum of terms like

$$\langle \text{Green function} \rangle \langle \text{interactor} \rangle \langle G.F. \rangle \langle \text{Intr.} \rangle \langle G.F. \rangle \langle \text{Intr.} \rangle \cdots$$

Here the energy differences corresponding to some integral of $D$-functions are inessential for our purpose and not explicitly written down. In computing the transition probability of a certain process we shall multiply the $S$-matrix of the interested process with its hermitian conjugate from the left. Thus we shall have operators of the following type,

$$\langle G.F. \rangle \langle \text{Intr.} \rangle \langle G.F. \rangle \langle \text{Intr.} \rangle \cdots$$

Let us take an example to fix our idea, and suppose that there stands a factor $\rho_3$ as a part of the $(n-1)$-th interactor on each side of the position, "centre"; then we can make this $(n-1)$-th interactor free of $\rho_3$, putting two $\rho_3$-factors into the centre, i.e.

$$\text{Centre}$$

$$\cdots \langle \text{Intr.} \rangle \langle G.F. \rangle \langle \text{Intr.} \rangle \langle G.F. \rangle \rho_3 \langle G.F. \rangle \langle \text{Intr.} \rangle \langle G.F. \rangle \langle \text{Intr.} \rangle \cdots$$

* $G.F.$ stands for Green function and Intr. stands for interactor. The suffix attached to the interactor is showing its order in the train.
Of course, by virtue of \( p_3^2 = 1 \), the \( p_3 \) factors in the centre are cancelled. But they have left their trace in the remainder of (50): In view of the constitution of the Green's functions, \( 1/\sum_{\mu=1}^{4} q_\mu - im \), where \( q_\mu \) denotes the propagation 4-vector for respective states, it is evident that there is a change of sign \( q_\mu \rightarrow -q_\mu (\mu = 1, 2, 3) \) where \( p_3 \) has passed over.

By proceeding in this way, one can, in general, omit \( p_1(\rightarrow \gamma_1) \), \( p_2(\rightarrow \gamma_2) \), \( p_3(\rightarrow \gamma_3) \) contained in interactors, but, on the other hand, making the following replacement, \( q_\alpha \rightarrow -q_\alpha \),

\[
\begin{array}{c|c|c}
\mu = 1, 2, 3 & \text{in all factors of the train (48) lying on the left-side} & \rho_3 \\
\mu = 4 & \text{of that interactor in which we omitted} & \rho_4 \\
\mu = 1, 2, 3, 4 & \quad & \rho_1.
\end{array}
\]

The above statement in the relativistically symmetrical theory is a little obscured when we come back to the asymmetrical system. But we shall find the analogous conclusion here, too.

By application of \( p_3 \) the sign of energy does not change, while the sign of momentum will be reversed.

\[
\rho_3^{-1} \rho_3 \rho_3 = \rho_3, \quad \rho_3^{-1} U \rho_3 = U^{-1}.
\]

By application of \( p_2 \) the sign of energy will be reversed, while the momentum remains unchanged.

\[
\rho_2^{-1} \rho_2 \rho_2 = -\rho_3, \quad \rho_2^{-1} U \rho_2 = U.
\]

And by application of \( p_1 \) the sign of both energy and momentum will be reversed.

\[
\rho_1^{-1} \rho_3 \rho_1 = -\rho_3, \quad \rho_1^{-1} U \rho_1 = U^{-1}.
\]

The apparent paradox, that we have obtained the same result in both cases of symmetrical formalism and asymmetrical one in spite of the different interpretation of the expectation values of operators, will be solved when we consider that it is the mutual commutation relations between these operators which underlies our discussion.

In this way we are justified in terming \( p \)'s as the charge operators: For the reflexions of world line in a Feynman diagram means physically an occurrence of a pair formation or destruction when the change of the sign of energy is contained in the reflexion. Thus three distinct types of reflexion attributed to each one of \( p \)'s correspond to different types of pair formation or destruction. In actual cases, these pair formation or destruction are accompanied with the simultaneous
change of momentum, and their exact representation in Feynman diagrams are
difficult to be given; but neglecting this fact we sum up the results schematically
in Fig. 2.

In the sense stated above, the rotation of ρ-space which we met with in § 2,
can be interpreted as the change of the character of pair formation and destruc-
tion.

Fig. 2. Schematical representation of the effect of ρ's contained in the interactors.
Here we omit the momentum transfer caused by the interaction.

§ 7. Summary and acknowledgement

We have shown in this paper, that in case when we can neglect the effect
of the vacuum fluctuation of the spin 1/2 field, we can get semi-classical models of
spin 1/2 particle which facilitate us much in qualitative understanding of the
results. We should start from the one-particle theory for this fermion, and make
use of the transformation proposed by Pryce, by Foldy and Wouthuysen, and
also by the author. Then, we find semi-classical interpretation of the behavior
when the particle moves without interaction. The results was summed up in §2.,
and they furnish us with quite suitable basis for treating the interaction. The
treatment of the interaction was described in § 3. The result is in conformity
with the positron theory, if we adopt the prescription stated there.

In concluding, I wish to express my cordial thanks to Prof. Tomonaga for his
interest in this work and valuable discussions.

Appendix I

Here we discuss the correspondence between the one-particle theory and the
quantized field theory.

In constructing the transformation function, we must compute integrals of
following type, (given by (28), § 2)

\[ \left( \frac{1}{i} \right)^n \int H(t') dt' \int H(t'') dt'' \int \cdots \int H(t_n) dt_n. \]  

(a, 1)
In our system each $H(t)$ is of the form, according to (25),

$$
\sum_{\mathbf{k}} \sum_{\lambda} e^{+i\hbar \alpha_{\mathbf{k}\lambda}} \left\{ e^{-i(\mathbf{k} \cdot \mathbf{X})} e^{i(E_{\mathbf{p}} - E_{\mathbf{k}})\nu_{\mathbf{p}}} \cdot \langle U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \rangle_{\text{even}} + e^{-i(\mathbf{k} \cdot \mathbf{X})} e^{i(E_{\mathbf{p}} - E_{\mathbf{k}})\nu_{\mathbf{p}}} \cdot \langle U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \rangle_{\text{odd}} \right\} + \text{hermitian conjugate.}
$$

(a, 2)

Here $\langle \cdot \rangle_{\text{even}}$, $\langle \cdot \rangle_{\text{odd}}$ denotes respectively the even and odd parts of the expressions inside the brackets. On the other hand, in the quantized theory, we should substitute the following expression for $H(t)$ in place of the above expression into (a, 1)

$$
\sum_{\mathbf{k}} \sum_{\mu,\nu=1,2} \sum_{\mathbf{p}} a_{\mathbf{k}\lambda}^{\mu} \left\{ \delta_{\mathbf{p} - \mathbf{k}}(\mathbf{p} - \mathbf{k}) \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) e^{i(\mathbf{p}^{+}E_{\mathbf{p}} - E_{\mathbf{p}}^{+})\nu_{\mathbf{p}}} \phi_{\mathbf{k}}^{\nu}(\mathbf{k}) \langle \text{Intr.} \rangle \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) \right\} + \text{hermitian conjugate,}
$$

(a, 3)

where $\delta^{\mu}$ and $\delta$ denote the creation and annihilation operators of fermions, and $\phi(\mathbf{p})$ and $\phi^{\dagger}(\mathbf{p})$ are eigen-solutions of Dirac equation with momentum and its complex conjugate. $\mu$ is the spin index, and $\pm$ denotes the sign of charge. By comparing (a, 2) with (a, 3), we shall find the term by term correspondence between two cases. For there is the following correspondence of the expressions in both cases.

$$
\sum_{\mu=1,2} \phi_{\mathbf{k}}^{\mu}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) \rightarrow \frac{1}{2} (1 + \rho_{\mathbf{k}}) \cdot U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 + \rho_{\mathbf{p}}),
$$

$$
\sum_{\mu=1,2} \phi_{\mathbf{k}}^{\mu}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) \rightarrow \frac{1}{2} (1 - \rho_{\mathbf{k}}) \cdot U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 - \rho_{\mathbf{p}}),
$$

$$
\sum_{\mu=1,2} \phi_{\mathbf{k}}^{\mu}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) \rightarrow \frac{1}{2} (1 + \rho_{\mathbf{k}}) \cdot U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 + \rho_{\mathbf{p}}),
$$

$$
\sum_{\mu=1,2} \phi_{\mathbf{k}}^{\mu}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle \phi_{\mathbf{p}}^{\mu}(\mathbf{p}) \rightarrow \frac{1}{2} (1 - \rho_{\mathbf{k}}) \cdot U^{-1}(\mathbf{p} - \mathbf{k}) \langle \text{Intr.} \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 - \rho_{\mathbf{p}}).
$$

(a, 4)

Thus the correspondence of the transition elements of both theory is checked with respect to the elementary processes described by the interaction Hamiltonian. This statement is also true in general higher order processes, as is clear from the following reasoning. Here we neglect all vacuum effects, and the train of intermediate states in quantized theory which appear in (a, 1) is continuous, in other words, every particle generated in the preceding step must vanish in the next step. In the unquantized system we do not need to say about this continuity.
of intermediate states, but we notice that here the distinction between process with and without pair formation is automatically established by virtue of $\rho_3$ in the exponential time factor before and after integration over time. The spatial exponential factor takes automatically the momentum conservation in higher order processes into account.

However, we must take care as regards to the sign of a train of terms, each of which is given by $(a, 3)$. For positive energy states the creation operators stand on the left-side, while the annihilation operators on the right. For negative energy states the situation is just reversed. So, if a certain intermediate state is one of positive energy, we have a factor $b_+(p)\beta^+(p)$, which can be identified to unity in the one-particle theory. But for a negative intermediate state we shall have a factor $b_-(p)\beta_-(p) = 1 - b_-(p)\beta^-(p)$, which should be put $-1$; because we should discard all effects due to the negative energy sea. This discrimination of sign of intermediate states can be established in our system of calculation by inserting $\rho_3$ into the middle of two transition operators of adjacent step.

Appendix II

We have purposely separated the term $\langle\text{recoil}\rangle$ in the discussion of §3. The following results will be useful in calculations of any kind of interaction type. By suitable change of the notation, one will readily find the essential part of the term $\langle\text{recoil}\rangle U^{-1}\langle\text{interactor}\rangle U$ for any kind of interaction. Here we shall abbreviate as,

$$\mu_p = m + E_p = m + \sqrt{m^2 + p^2},$$

$$\mu_q = m + E_q = m + \sqrt{m^2 + q^2}.$$

$$2[E_pE_q\mu_p\mu_q]^{1/2}U^{-1}(q)\cdot U(p)$$

$$= \{\mu_p\mu_q + (p \cdot q) + i([q \times p] \cdot \sigma)\} + i\rho_3((q\sigma)\mu_p - (p\sigma)\mu_q), \quad (b, 1)$$

$$2[E_pE_q\mu_p\mu_q]^{1/2}U^{-1}(q)\cdot \rho_3\cdot U(p)$$

$$= \rho_3(\mu_p\mu_q + (q \cdot p) + i([q \times p] \cdot \sigma)\} + i((q\sigma)\mu_p - (p\sigma)\mu_q), \quad (b, 2)$$

$$2[E_pE_q\mu_p\mu_q]^{1/2}U^{-1}(q)\cdot \rho_3\cdot U(p)$$

$$= \rho_3(\mu_p\mu_q - (q \cdot p) - i([q \times p] \cdot \sigma)\} + \rho_3((q\sigma)\mu_p + (p\sigma)\mu_q), \quad (b, 3)$$

$$2[E_pE_q\mu_p\mu_q]^{1/2}U^{-1}(q)\cdot \rho_3\cdot U(p)$$

$$= \rho_3(\mu_p\mu_q - (q \cdot p) - i([q \times p] \cdot \sigma)\} - \rho_3((q\sigma)\mu_p + (p\sigma)\mu_q), \quad (b, 4)$$

$$2[E_pE_q\mu_p\mu_q]^{1/2}U^{-1}(q)\cdot \sigma\cdot U(p)$$

$$= \{(\mu_p\sigma + (p\sigma)q + (q\sigma)p - (p \cdot q)\sigma - i[q \times p])\} + i\rho_3(\mu_pq - \mu_qp - i\mu_q[p \times \sigma] - i\mu_p[q \times \sigma]), \quad (b, 5)$$
where

\[U(p) = \exp\left(-\frac{i}{2\beta}\frac{(p\sigma)}{|p|}\tan^{-1}\left(\frac{|p|}{m}\right)\right) = (2\mu_p \cdot E_p)^{-1/2}[\mu_p - i\beta(p\sigma)],\]

\[U^{-1}(q) = \exp\left(+\frac{i}{2\beta}\frac{(q\sigma)}{|q|}\tan^{-1}\left(\frac{|q|}{m}\right)\right) = (2\mu_q \cdot E_q)^{-1/2}[\mu_q + i\beta(q\sigma)].\]