Recoil Effect on Electron-Proton Forces
and Inapplicability of Energy Law

Gentaro ARAKI and Sigeru HUZINAGA

Faculty of Engineering, Kyoto University

(Received July 16, 1951)

Bethe and Fermi showed that Breit's formula for the two-electron interaction transmitted by photons can be obtained according to the quantum electrodynamics if the recoil effect is neglected. It was shown by Breit himself that his formula was not in complete agreement with experiment. The discrepancy was shown to come from a term which was proportional to $AE$ in the Pauli approximation, but its true origin has not yet fully been known. In the present paper it is shown that this surplus term is due to the neglect of the recoil effect and the discrepancy disappears when the interaction is derived correctly taking into account the effect of the electron and proton recoils according to the method of the unitary transformation. Further it has generally been known that the electron-proton interaction transmitted by photons takes the form which was previously derived by Möller for the two-electron interaction if the conservation law is applied to the total energy of the two particles in the free state. In the present paper it is shown that the application of the energy law gives rise to an incorrect result in case of hydrogen atom. According to Möller's formula the interval between $2\,S_{1/2}$ and $2\,P_{1/2}$ levels of the hydrogen atom becomes too large. Its value amounts to 7273 Mc/s in contradiction to the Lamb-Retherford experiment. Therefore we arrive at the conclusion that the energy law can not be made use of in general when forces between Fermions in the bound states are derived.

Introduction and Summary

Recently one of the present authors studied the effect of the nucleon recoil on nuclear forces. Although it may bring one into compliance that the effect is important in its theoretical aspect as a general problem on two-Fermion forces transmitted by Bosons, any definite conclusion can not be drawn from the result because the related phenomena have not yet been studied experimentally as well as theoretically in a sufficient exactness and even the meson theory has not yet arrived at its satisfactory form. There is no ambiguity of such a kind in case of two-electron or electron-proton forces. The fine structure of hydrogen levels was precisely analysed by the recent experiment of Lamb and Retherford. We can therefore examine the effect in question by comparing theory with experiment in this case as a test of the general problem on forces between two Fermions. This is the purpose of the present paper.

We shall seek for the answers to two questions: (i) What is the difference between the new and conventional formulas when we correctly take into account the effect of electron and proton recoils on their interaction? (ii) What effect
arises when we apply the conservation law to the total energy of two Fermions in their free state in deriving their interaction?

A formula for the electromagnetic interaction between two electrons in the configuration space was derived by Breit\(^3\) by considering a quantum-mechanical correspondence of the classical formula. Another formula for the interaction in the form of matrix elements was deduced by Möller\(^4\) according to Klein's method. Bethe and Fermi\(^5\) showed on the basis of quantum electrodynamics that Breit's formula can be obtained if the effect of electron recoil is neglected and further that Möller's formula is equivalent to Breit's when the conservation law of energy is valid.

It seems to have been believed by many authors that the neglect of the recoil effect\(^6\)\(^7\)\(^8\) and the application of the energy law\(^9\)\(^10\) are adequate, or the former effect was considered in the wrong way\(^5\)\(^7\). However the neglect of the former effect can not be justified for the reason which was discussed in A\(^*\). In fact it was shown by Breit\(^11\) himself that his formula can not be in complete agreement with the experiment on the He triplet. The discrepancy comes from a term which is proportional to \(\epsilon^4\) in the Pauli approximation. Recently Ishidzu\(^12\) has shown in case of the electron-proton interaction that a similar term can not be reconcilable with the experiment on the hyperfine structure of hydrogen.

Explanations\(^13\)\(^14\)\(^15\) of the justification for omitting this surplus term of \(\epsilon^4\) were tried, but they could not convince us in a satisfactory way. On the standpoint of quantum electrodynamics, Breit's formula can be considered as a term of \(\epsilon^4\) in the two-electron interaction transmitted by photons. One may think on this basis, as was noted by Rosenfeld\(^15\), that the \(\epsilon^4\)-term must be omitted in order to consistently take into account only \(\epsilon^2\)-terms. The calculation in A was carried out on this standpoint. As will be shown in the third section below, however, the above mentioned \(\epsilon^4\)-term can not distinctly be separated from the remaining terms because we find another \(\epsilon^4\)-term in Breit's formula.

Nambu\(^7\) pointed out that the fourth order term in the \(\epsilon\)-perturbation cancels the second order term in the \(\epsilon^2\)-perturbation. However, this was not yet clear-cut, though it might be thought most convincing, because the Pauli approximation is the second degree term in a power series in \(1/\epsilon\) whereas Nambu's perturbation is a series in \(\epsilon\).

In the present paper the problem is considered from another viewpoint. The electron-proton interaction transmitted by photons is derived taking into account the effect of electron and proton recoils considering both the positive and negative energy states of them. The calculation will be carried out in the second section according to the method mentioned in A. The apparent form of the new formula is entirely different from Breit's, but we find in the Pauli approximation that it

\* The paper cited in Ref. (1) will be referred to as A.
Recoil Effect on Electron-Proton Forces and Inapplicability of Energy Law 675

coincides with the latter except for the above mentioned surplus term. This will be shown in the third section. Consequently we see that Breit's surplus term of \( \epsilon' \) is due to the neglect of the recoil effect and the correct method is capable of avoiding the defect of Breit's formula and that two-electron or electron-proton forces predicted by quantum electrodynamics is in quite agreement with experiment.

As is mentioned above, Møller's formula can not be obtained without assuming the energy law, according to quantum electrodynamics. The application of the law implies the neglect of the term in the form \([H_0 S']\) where \(H_0\) is the Hamiltonian without interaction, \(S'\) is a skew-Hermitian operator, and the bracket denotes a commutator. The neglect of such a term has no validity unless its influence is proved to be negligibly small, except for the case of a scattering problem where the influence exactly vanishes.

If the interaction between two Fermions is very weak the influence of the above mentioned term is a second order correction and it may consequently be neglected compared with a first order correction. In the present case, however, the influence of the Coulomb interaction is so large that the energy eigenstates of an electron-proton system are wholly different from those in case of no interaction. Therefore the effect of \([H_0 S']\) is by no means negligible compared with Breit's term.

It has often been considered\(^{(7)(16)}\) that \([H_0 S']\) can be unitarily transformed into the higher order term in \(\epsilon\) and that the \(\epsilon'\)-forces involve an ambiguity in this sense, but, in our opinion, such a transformation is superfluous in principle. In the present paper the two-Fermion forces transmitted by Bosons are derived by a well-defined unitary transformation which makes the Hamiltonian of a system consisting of Fermions and Bosons, diagonal with respect to the number of Bosons. This method uniquely defines the forces and there is no ambiguity. This is more concretely accounted for in the first section.

The effect of neglecting \([H_0 S']\) will be examined in the fourth section by comparing Møller's formula with the experiment of Lamb and Retherford\(^{(29)}\). It will be shown that the calculated result can not be reconcilable with their experiment. According to Møller's formula both the \(2P_{3/2}\) and \(3/2\) levels of the hydrogen atom are deeper than the position predicted by Dirac's theory while the \(2S_{1/2}\) level is not influenced. The shift amounts to 7273 Mc/s which is wholly inconsistent with the observed value of 1062 Mc/s. Generally speaking, this proves the inadequacy of applying the conservation law of energy to the derivation of two-Fermion forces transmitted by Bosons. Thus we finally obtain the definite answers to the two questions mentioned in the beginning.

§ 1. Problem and Method

We shall first define the method of deriving the electron-proton forces. Consider a system which consists of electrons, protons and photons. The Hamiltonian
of the system is written in the form $H = H_0 + U + H'$ where $H_0$ is the Hamiltonian without interaction, $U$ denotes the Goulomb interaction between the Fermions, and $H'$ is the interaction of the Fermions with photons.

The first two parts of the $H$ are diagonal and the last is non-diagonal with respect to the number of photons. We unitarily transform $H$ into a diagonal form with respect to the number of photons where the possibility of the transformation is of course an assumption. The new Hamiltonian then takes the form $H_0 + U + H''$ and its matrix becomes of a staircase form as is shown in Fig. 1.

![Staircase Form of Hamiltonian](https://academic.oup.com/ptp/article-abstract/6/5/673/1843383)

Let $W$ be the part of $H''$, which has its non-vanishing matrix-elements in the BA-part of Fig. 1 and depends on the relative position between an electron and a proton. Then $W$ represents their interaction transmitted by photons, and the total interaction between an electron and a proton is given by $u + W$ where $u$ denotes the Coulomb interaction between them. ($u$ is not equal to $U$).

We denotes the unitary transformation by $\exp S$ where $S$ is skew-Hermitian and non-diagonal with respect to the number of photons. We again assume that $S$ and the transformed Hamiltonian can be expanded into powers in $\epsilon$:

\[
S = S^{(0)} + S^{(2)} + S^{(4)} + \ldots
\]

\[
(\exp S)H \exp(-S) = H_0 + U + H^{(2)} + H^{(4)} + H^{(6)} + \ldots
\]

where $S^{(n)}$ and $H^{(n)}$ are proportional to $\epsilon^n$. In order that (1.2) is diagonal with respect to the number of photons $S^{(n)}$'s must satisfy the following system of simultaneous equations:

\[
S^{(n)}_{ij} = \begin{cases} 
1 & \text{if } i = j \text{ and } i, j \text{ are even} \\
0 & \text{otherwise}
\end{cases}
\]
Recoil Effect on Electron-Proton Forces and Inapplicability of Energy Law 677

\[ [H_0 S^{(1)}] = H' \]  

(1.3) (a)

\[ H^{(3)} + [H_0 S^{(3)}] = (1/2) [S^{(1)} H'] \]  

(1.3) (b)

\[ [H_0 S^{(3)}] = - (1/2) [H_0 [S^{(1)} S^{(2)}]] + (1/3) [S^{(0)} [S^{(3)} H']] + [S^{(1)} U] \]  

(1.3) (c)

\[ H^{(4)} + [H_0 S^{(4)}] = - (1/2) [H_0 [S^{(1)} S^{(3)}]] - (1/6) [S^{(2)} [S^{(4)} H_0]] \]  

\[ - (1/12) [S^{(1)} [S^{(2)} [S^{(4)} H_0]]] + (1/3) [S^{(3)} [S^{(4)} H']] \]  

\[ + [S^{(2)} U] + (1/2) [S^{(3)} [S^{(4)} U]] \]  

(1.3) (d)

etc.

where the bracket denotes the commutator. The non-diagonal property of \( S \) with respect to the number of photons is capable of avoiding the ambiguity in \( S^{(2)} \). \( H^{(n)} \) represents the diagonal part of the \( e^2 \)-term in (1.2) and \( S^{(n)} \) is so determined that \( [S^{(n)} H_0] \) cancels the non-diagonal part of the rest. In this method \( S^{(n)} \) is unique and there is no ambiguity in it.

The simultaneous equations can be actually solved by introducing any orthonormal set. The correct solution can be obtained provided that the set is complete. We can adopt the eigenfunctions of \( H_0 \) as the orthonormal set. In order to fulfil the requirement of completeness we have to consider all eigenfunctions corresponding to both the positive and negative energy. None of them can be omitted in the present case of interacting two Fermions, as was stressed in A.

We shall consider only the \( e^2 \)-term in \( W \). This comes from \( H^{(3)} \). Its matrix elements are, by (1.3) (b), given by

\[ W_{RA} = (1/2) [S^{(1)} H']_{RA} \]  

(1.4)

where \( A \) and \( B \) are two arbitrary states in the BA-part of Fig. 1. The conventional formula (17)(18)(19) is given by \(-[S^{(0)} H']_{RA}\) in contrast with (1.4), but this does not represent a physical quantity because it is not Hermitian. Therefore the conventional formula is incorrect while (1.4) is Hermitian.

If we insert the explicit formula for \( H' \) in (1.4) we have

\[ W_{RA} = - \frac{2\pi}{\hbar} \epsilon_1 \epsilon_2 \sigma^2 \left( \begin{array}{c} \rho_1^{(1)} \rho_2^{(2)} \delta (dp) + (1 \leftrightarrow 2) \end{array} \right) \]  

(1.5)

where \( \hbar = p - p_0, p = p_2^{(1)}, p_0 = p_4^{(1)}, \epsilon = \epsilon_2^{(1)}, \epsilon_0 = \epsilon_4^{(1)}, \) and \( \Delta p = p_2^{(1)} + p_2^{(2)} - (p_4^{(1)} + p_4^{(2)}) \). Further \( \epsilon_1 \) and \( \epsilon_2 \) are respectively the charges of the first and second Fermions, \( \hbar \) is the volume of the normalization space, \( p \) and \( \epsilon \) are respectively the momentum and energy of a Fermion, and all quantities are measured in \( \hbar = 1 \) units. Quantities in the state \( A \) (or \( B \)) are indicated by the subscript \( A \) (or \( B \)), the quantities of the first (or second) Fermion except for the charge and mass are indicated by the superscript \( (1) \) (or \( (2) \)), and \( (1 \leftrightarrow 2) \) denotes the expression obtained from the preceding one by interchanging the first and second particles. \( \rho_1, \rho_2, \rho_0, \) and \( \sigma \) are Dirac's matrices, and their matrix subscripts for appropriate parts of Dirac's wave functions are omitted for the sake of simplicity. The formula
(1.5) is valid for the two-electron interaction too.

The denominators of (1.5) are different from those of the conventional formula(17) owing to the above-mentioned difference between their general forms. When the energy is conserved the difference disappears. In the present case we are considering the case in which the energy is not conserved. We have therefore to sharply distinguish these two forms. The term \((\varepsilon-\varepsilon_0)^2\) represents the retardation effect according to Klein-Møller’s method,(4)(5) or it represents the effect of the energy change due to the recoil. We shall refer to this term as a recoil term. The first problem in the present paper is to examine the rôle of this term.

Making use of Dirac’s equation which is satisfied by the wave function of electron or proton we can transform (1.5) as follows*:

\[
W_{R_A} = -\frac{2\pi}{V} \varepsilon \varepsilon_1 \left\{ \frac{\rho_1^{(2)}\rho_2^{(2)}\sigma^{(2)} - 1}{k^2 - (\varepsilon-\varepsilon_0)^2} + \frac{1}{k^2} \right\} \delta(dp) + (1\leftrightarrow 2) + [H_0S']_{R_A} \tag{1.6} (a)
\]

where

\[
S'_{R_A} = -\frac{2\pi}{V} \varepsilon \varepsilon_2 \left\{ \frac{(\varepsilon-\varepsilon_0)\delta(dp)}{k^2} + (1\leftrightarrow 2) \right\} \tag{1.6} (b)
\]

In order to obtain Møller’s formula we have to omit \([H_0S']\) in (1.6)(a). The second problem in the present paper is to study the effect of \([H_0S']\), in another word, to study the difference between Møller’s and exact formulas.

§ 2. Electron-Proton Forces

The method of correctly taking into account the recoil effect for deriving two-Fermion forces was accounted for in A. The denominator of (1.5) can be written in the following form:

\[
\frac{1}{k^2 - (\varepsilon-\varepsilon_0)^2} = \frac{1}{2} \left( 1 + \frac{\varepsilon_0 + p\|p_0}{m_1 c^2} \right) \frac{k^2 + k^2 p_0^2 - (k p_0)^2}{m_1^2 c^2} \tag{2.1}
\]

where \(m_1\) is the mass of the first particle. (The mass of the second particle will be denoted by \(m_2\).) The right side is inserted in (1.5). \(\varepsilon\) and \(\varepsilon_0\) are then replaced by \(\rho_1^{(2)}\sigma^{(2)}p + m_1 c p_0^{(2)}\) and \(\rho_1^{(3)}\sigma^{(3)}p_0 + m_1 c p_0^{(3)}\) respectively and the former is placed on the left and the latter on the right of \(\rho_1^{(4)}\) and \(\sigma^{(4)}\) in (1.5). The denominator of the right side in (2.1) is expanded in a power series in \(1/c^2\). This series converges independently of the magnitude of \(p\) provided that \(|p_0| < m_1 c\). We calculate the right side of (1.5) up to the terms of \(1/c^2\) in this way.

* The result obtained on the basis of the conventional formula is different, as was given by Hamilton(19)
Making use of the relations

\[
\frac{4\pi}{V} \frac{\delta(p)}{k^2} = (\varphi_B, r^{-1} \varphi_A) \quad (2.2) (a)
\]

\[
\frac{4\pi}{V} \frac{(vk)(wk)}{k^2} \frac{\delta(p)}{k^2} = 2^{-1} (\varphi_B, r^{-1} \{v \cdot w - r^{-2} (v \times w) \} \varphi_A) \quad (2.2) (b)
\]

we write (1.5) in the form which contains the expressions in the right side of these formulas where \( \mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \), \( r = |\mathbf{x}| \), \( \mathbf{x}^{(1)} \) and \( \mathbf{x}^{(2)} \) are the position vectors of the first and second Fermions respectively, \( \varphi_A \) and \( \varphi_B \) are the space parts of the eigenfunctions of two free Fermions in the states \( A \) and \( B \) respectively, and \( \mathbf{v} \) and \( \mathbf{w} \) are vectors. We next replace remaining \( p \) and \( P_0 \) by \(-iP^{(1)}\) and place the former on the left and the latter on the right of \( r^{-1}, r^{-2}, \) and \( \mathbf{x} \). We have then the interaction between two Fermions transmitted by photons in the configuration space as follows:

\[
W^{(A)} = \frac{1}{4m_c} \rho^{(1)}_A \rho^{(1)}_B \frac{\epsilon^2}{r} \left\{ i \sigma^{(1)} \mathbf{p}^{(1)} + \frac{i}{r^2} (\sigma^{(1)} \mathbf{x}) (\mathbf{x} \mathbf{p}^{(1)}) - \frac{\mathbf{x}}{r^2} [\sigma^{(1)} \sigma^{(2)}] \right\} + (1/z^2) \quad (2.3)
\]

where the thick blacket denotes a vector product of two vectors embraced in it. This formula is exact up to \( 1/c^2 \) in the correct sense.

If we omit the recoil term in (1.5) we have Breit's formula as follows:

\[
W^{(B)} = -\rho^{(1)}_A \rho^{(2)}_B \Gamma \quad (2.4) (a)
\]

where

\[
\Gamma = \frac{\epsilon^2}{2r} \left\{ \sigma^{(1)} \sigma^{(2)} + \frac{\sigma^{(1)} (\mathbf{x}) \mathbf{p}^{(1)}}{r^2} \right\} \quad (2.4) (b)
\]

The apparent forms of (2.3) and (2.4) are entirely different. The difference between them is the contribution of the recoil term. This will be studied in the next section. The neglect of the recoil term is to take the zero degree term in the incorrect expansion as follows:

\[
\frac{1}{k^2 - (\epsilon - \epsilon_0)^2} = \frac{1}{k^2} \sum_{n=0}^{\infty} \left( \frac{\epsilon - \epsilon_0}{k} \right)^{2n} \quad (2.5)
\]

The incorrectness of such an expansion was fully discussed in A. The arguments of Bethe and Fermi and Nambu were based on this incorrect expansion.

### § 3. Recoil Effect

We shall examine the Pauli approximation of \( W^{(A)} \) and \( W^{(B)} \) in order to study the influence of the recoil term. The method was already explained in the previous papers\(^6\)\(^{16}\). When there is no photon the Hamiltonian of the two-Fermion system is, according to (1.2), given by

\[\text{In the present case photons have the vanishing rest mass whereas mesons have the finite mass. Therefore the Pauli approximation was considered in a somewhat different way in A.}\]

\[\text{In the present case photons have the vanishing rest mass whereas mesons have the finite mass. Therefore the Pauli approximation was considered in a somewhat different way in A.}\]
\[ H = -i\epsilon \rho_1^{(0)} \sigma \mathcal{F}^{(1)} - i\epsilon \rho_1^{(0)} \sigma \mathcal{F}^{(2)} + m_1 c^2 (\rho_2^{(0)} - 1) + m_2 c^2 (\rho_3^{(0)} - 1) + u + W \] (3.1)

where \( u = \epsilon \ell_1 / r \) is the Coulomb interaction, and the rest energy of the system (including the self-energy) is subtracted for the sake of later convenience. We write the eigenfunction of the Hamiltonian in the form

\[
\psi = \psi^{++} \chi^{+} (1) \chi^{+} (2) + \psi^{+-} \chi^{+} (1) \chi^{-} (2) + \psi^{-+} \chi^{-} (1) \chi^{+} (2) + \psi^{--} \chi^{-} (1) \chi^{-} (2)
\] (3.2)

where \( \chi^{+} (1) \) and \( \chi^{-} (1) \) are the eigenfunctions of \( \rho_3^{(0)} \) belonging to its eigenvalues +1 and -1 respectively. The coefficient \( \psi^{++} \) etc. are functions of space and spin coordinates. The eigenvalue equation \( H \psi = E \psi \) then splits into simultaneous equations for \( \psi^{++} \) etc.. If we eliminate \( \psi^{+-} \), \( \psi^{-+} \), and \( \psi^{--} \) from them, we have an equation in the form \( H' \psi^{++} = E \psi^{++} \) where \( H' \) involves \( E \).

We can expand \( H' \) in a power series in \( 1/e \). \( E \) in \( H' \) can be eliminated by replacing \( E \psi^{++} \) with \( H' \psi^{++} \). Thus we have

\[
H' = -\frac{1}{2m_1} A^{(0)} - \frac{1}{2m_2} A^{(2)} - \frac{1}{8m_1^2 c^2} (A^{(0)})^2 - \frac{1}{8m_2^2 c^2} (A^{(2)})^2 + \cdots + u + W'
\] (3.3)

where \( u \) is the Coulomb interaction, \( u + W' \) stands for a part depending on the relative position of two particles. Now \( H' \) does not contain \( E \) and the equation \( H' \psi^{++} = E \psi^{++} \) is still valid. We can consider \( H' \) as the effective Hamiltonian because its eigenvalue is \( E \). \( u + W' \) represents an effective interaction. If we reserve only the terms up to the second degree and omit higher degree terms from (3.3), \( H' \) is the Pauli approximation of the Hamiltonian and \( W' \) is the Pauli approximation of \( W \).

In this way we have the Pauli approximation of \( W'^{(a)} \) and \( W'^{(b)} \) as follows:

\[
W'^{(a)} = -\frac{1}{4m_1^2 c^2} \frac{\epsilon \ell_2}{r^2} (x \mathcal{F}^{(0)} + \sigma^{(0)} \mathbf{L}^{(0)}) - \frac{1}{4m_2^2 c^2} \frac{\epsilon \ell_2}{r^2} (-x \mathcal{F}^{(2)} + \sigma^{(2)} \mathbf{L}^{(2)})
\]

\[
+ \frac{1}{2m_1 m_2 c^2} \ell_2 \left( \mathcal{F}^{(0)} \mathcal{F}^{(2)} + \frac{1}{r^2} x (x \mathcal{F}^{(0)}) \mathcal{F}^{(2)} - \frac{1}{r^2} (\sigma^{(0)} \mathbf{L}^{(2)} + \sigma^{(2)} \mathbf{L}^{(0)}) \right)
\]

\[
+ \frac{1}{4m_1 m_2 c^2} \frac{\epsilon \ell_2}{r^2} \left( \sigma^{(0)} \sigma^{(2)} - \frac{3 (\sigma^{(0)} x) (\sigma^{(2)} x)}{r^2} \right)
\] (3.4)

\[
W'^{(b)} = W'^{(a)} + \frac{I^2}{2 (m_1 + m_2) c^2}
\] (3.5)

where \( I \) is given by (2.4) (b), \( x = x^{(0)} - x^{(2)} \), \( \mathbf{L}^{(0)} \) is the orbital angular momentum of the first particle with respect to the second particle, and \( \mathbf{L}^{(2)} \) is that of the second particle with respect to the first:

\[
\mathbf{L}^{(0)} = -i [x \mathcal{F}^{(0)}] \quad \mathbf{L}^{(2)} = i [x \mathcal{F}^{(2)}]
\] (3.6)

The difference between the new formula and Breit's is the last term of (3.5). This is just the surplus term mentioned in the introduction, and the defect of Breit's formula was to contain this term. Breit showed in case of the two-
electrons interaction that if the $I^2$-term in (3.5) was reserved the calculated interval of He $^2P$ was not reconcilable with experiment. On the contrary, according to the formula given by (3.4) the calculated interval was in good agreement with experiment. Further Ishidzu has recently shown in case of the electron-proton interaction that the $I^2$-term is in contradiction to the experiment on the hyperfine structure of hydrogen. Thus we see that the new formula is in good agreement with experiment.

In order to see the nature of each terms and the degree of approximation we consider the case of hydrogen. For the sake of simplicity the electron mass is adopted as a mass unit and the proton mass is assumed to be infinite. Inserting $W'\tilde{A}_J$ into (3.3) we have

$$H'=-\frac{1}{2}\frac{\varepsilon}{r}-\frac{1}{8c^2}\frac{\varepsilon^3}{r^3}+\frac{1}{4c^2}\frac{\varepsilon^2}{r^2}(\mathbf{s}\cdot\mathbf{l})$$

(3.7)

This is the well-known Pauli formula. When its eigenvalue is evaluated considering $\varepsilon^2$-terms as perturbations it can be written in the form

$$H'=-\frac{1}{2}\frac{\varepsilon}{r}-\frac{1}{2c^2}\left(E_0+\frac{\varepsilon^3}{r^3}\right)^2-\frac{\varepsilon^2}{4c^2}\mathbf{r}^2(x\mathbf{p}-\sigma\mathbf{l})$$

(3.8)

where $E_0$ is the unperturbed eigenvalue. Its eigenvalue is given by

$$E=-\frac{\varepsilon}{2n^2}-\frac{\varepsilon}{2n^2c^2}\left(\frac{1}{n+1/2}-\frac{3}{4n}\right)$$

(3.9)

where $n$ and $J$ are the principal and inner quantum number. This is exactly equal to the zeroth and first degree terms in the power series in $\varepsilon^2/c^2$ of Dirac's eigenvalue without the rest energy. Therefore our method is mathematically correct. Since the second degree terms in $\varepsilon^2/c^2$ is very small owing to the smallness of $\varepsilon^2/c^2=137^{-2}$, the error of the Pauli approximation is very small.

The $I^2$-term in Breit's formula (3.5) is proportional to $(\varepsilon^2/r)^2$, as is seen from (2.4)(3.1). Its factor is independent of $\varepsilon$ and $r$. The formula (3.8) contains $(\varepsilon^2/r)^2$ also, but this cannot be omitted in order to obtain the correct result, as we have just seen. The contributions from the various $\varepsilon^2$-terms in (3.8) are all proportional to $\varepsilon^2$ as well as those from the terms in (3.4) and (3.5) including the $I^2$-term, where each $1/r$ or $\partial/\partial r$ gives a factor $\varepsilon^2$. Therefore it can not be a correct reason for omitting the $I^2$-term that is proportional to $\varepsilon^2$.

§ 4. Comparison of Möller's Formula with Experiment

We shall compare the theoretical result deduced from Möller's formula with the experiment on the fine structure of hydrogen in order to examine the effect of omitting $[H_0S']$ from (1.6). The calculation is carried out as before. If we omit $[H_0S']$ from (1.6) we have
The Pauli approximation of this interaction is given by

\[
W_{\text{ICM}} = W_{\text{ICM}}^{(M)} + \rho_{1}^{(\text{p})} \rho_{2}^{(\text{p})} \left\{ i \sigma^{(\text{p})} \left( 2 \mathbf{P}^{(\text{p})} - \frac{x}{r^3} \right) - \frac{x}{r^3} \left[ \sigma^{(\text{p})} \sigma^{(\text{p})} \right] \right\} + (\mathbf{1} \cdot \mathbf{2}) \tag{4.1}
\]

where the following relation is made use of:

\[
- \mathbf{A} + \frac{2}{r^2} \mathbf{x} \mathbf{P} + \frac{1}{r^2} \mathbf{x} \left( \mathbf{x} \mathbf{P} \right) = \frac{1}{r^2} \mathbf{L}^2 \tag{4.2}
\]

The Pauli approximation of this interaction is given by

\[
W_{\text{ICM}} = W_{\text{ICM}}^{(A)} + \frac{\epsilon_{1} \epsilon_{2}}{4c^2 r^3} \left\{ \frac{1}{m_{1}^2} \left( \mathbf{L}^{(1)} \right)^2 + \frac{1}{m_{2}^2} \left( \mathbf{L}^{(2)} \right)^2 \right\}
+ \frac{1}{2m_{1}m_{2}c^2} \frac{\epsilon_{1} \epsilon_{2}}{r} \left\{ \mathbf{P}^{(1)} \mathbf{P}^{(2)} + \frac{x}{r^2} \left( \mathbf{P}^{(1)} - \mathbf{P}^{(2)} \right) - \frac{1}{r^2} \mathbf{x} \left( \mathbf{P}^{(1)} \right) \left( \mathbf{P}^{(2)} \right) \right\} \tag{4.3}
\]

where \( W_{\text{ICM}}^{(A)} \) is given by (3.4). The difference between \( W_{\text{ICM}}^{(M)} \) and \( W_{\text{ICM}}^{(A)} \) is the influence of omitting \([H_2S']\). This difference is by no means negligibly small. It gives rise to a serious contradiction with experiment as will be shown below.

We shall compare (4.3) with the level structure of hydrogen atom. We shall calculate the shift of fine structure levels caused by \( W_{\text{ICM}}^{(M)} - W_{\text{ICM}}^{(A)} \) in atomic units. Let the first particle be the electron and the second be the proton. In this case \( \epsilon_{1} \epsilon_{2} = -1 \), \( m_{1} = 1 \) and \( m_{2} = M = 1840 \). In the centre-of-mass system we have

\[
W_{\text{ICM}}^{(M)} - W_{\text{ICM}}^{(A)} = - \frac{1}{4c^2} \left( 1 + \frac{1}{M} \right)^2 \mathbf{L}^2 \tag{4.4}
\]

Its expectation value in the \( nL \)-eigenstate of the zero-degree Hamiltonian in \( 1/c \) is given by

\[
\langle \left( W_{\text{ICM}}^{(M)} - W_{\text{ICM}}^{(A)} \right) \rangle = 0 \quad \text{for } L=0
\]

\[
= - \frac{1}{4c^2 \left( 1 + \frac{1}{M} \right)^2} \frac{n^2 \left( L + \frac{1}{2} \right)}{2} \quad \text{for } L \geq 0 \tag{4.5}
\]

where \( n \) and \( L \) are respectively the principal and azimuthal quantum numbers of the state.

The shift of levels from the position predicted by Dirac's theory is given by (4.5) because the energy eigenvalue calculated on the interaction \( n + W_{\text{ICM}}^{(A)} \) is equal to Dirac's eigenvalue up to \( 1/c^2 \), as was shown in the preceding section. Therefore \( S \)-levels are unchanged and \( P, D, F, \ldots \)-levels are lowered. For \( 2P \) the shift amounts to \( 7273 \text{ MeV} \), where \( 1/2c^2 \text{au} = 5.822 \text{ cm}^{-1} \) is used.\(^{(29)}\) The interval between \( 2^3P_{3/2} \) and \( 2^3P_{1/2} \) is 1.5 times of this value.\(^{(25,26)}\) Thus, according
Recoil Effect on Electron-Proton Forces and Inapplicability of Energy Law 683

to Møller's formula, we have

\[ E(2^3P_{3/2}) - E(2^3S_{1/2}) = 3637 \text{Mc/s} \]  \hspace{1cm} (4.6) (a)

\[ E(2^3S_{1/2}) - E(2^3P_{1/2}) = 7273 \text{Mc/s} \]  \hspace{1cm} (4.6) (b)

The level structure is shown in Fig. 2. The value given by (4.6) (b) is inconsistent with the observed value \(1062 \pm 5\) Mc/s of Lamb and Retherford. Therefore we see that \([H_0'S']\) in (1.6) can not be omitted.

\[ \begin{array}{c}
3637 \text{ Mc/s} \\
\uparrow \\
7273 \text{ Mc/s} \\
\downarrow \\
1062 \text{ Mc/s}
\end{array} \]

calc. \hspace{1cm} obs.

Fig. 2. Level structure of hydrogen atom according to Møller's formula

References

1) G. Araki, Prog. Theor. Phys. 6 (1951), 379
3) G. Breit, Phys. Rev. 34 (1929), 553
4) C. Møller, ZS. f. Phys. 70 (1931), 786
5) H. Bethe und E. Fermi, ZS. f. Phys. 77 (1932), 296
6) J. R. Oppenheimer, Phys. Rev. 35 (1930), 461
7) Y. Nambu, Prog. Theor. Phys. 5 (1950), 614
8) V. Fock, Sowj. Phys. 6 (1934), 425
11) G. Breit, Phys. Rev. 56 (1930), 383
12) T. Ishidzu Prog. Theor. Phys. 6 (1951), 48, 154
13) G. Breit, Phys. Rev. 39 (1932), 616
14) L. Landau, Sowj. Phys. 8 (1935), 487
15) L. Rosenfeld, ZS. f. Phys. 75 (1931), 253
16) L. van Hove, Phys. Rev. 75 (1949), 1519
17) W. Heitler, Quantum Theory of Radiation (1944), p. 98
18) W. Heitler and L. Nordheim, J. de Phys. Rad. 5 (1934), 449
22) H. Bethe, Geiger-Scheel's Handb. der Phys. 24–1 (1930), 315
23) R. T. Birge, Phys. Rev. Suppl. 1 (1929), 1