Relativistic Two-Body Problem in Quantum Theory

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A general method to treat the quantal two-body problem relativistically is developed on the line of Dirac's many-time theory. The case where two charged fermions interact with each other through the electromagnetic field is investigated as an example. We construct a canonical transformation which eliminates the electromagnetic field variables in the original Schrödinger equations, and derive the potential energy between two charged particles. Finally, some properties of the simultaneous wave equations for the system of two particles are discussed, and a method to treat the stationary state of that system is indicated.

§ 1. Introduction

Tomonaga et al. and Schwinger have succeeded in the extension of the field theory of Heisenberg-Pauli into a completely relativistic formalism. By means of this formalism, we may calculate the potential energy between two fermions coupled by the Bose field, but this formalism seems to give difficulty for the consistent treatment of the stationary state of the two-body system. On the other hand, the S-matrix theory of Heisenberg, which is another approach to the problem—integral formalism—, affords a hope to deal with the bound system as well as the scattering process. However, in the present state of the theory, few investigations have been tried on this problem, as far as we know. Thus we had better attempt to make another approach.

As early as 1932, Dirac proposed the many-time theory as a relativistic generalization of Heisenberg-Pauli's. But in this theory, it is to be noted, the second quantization was not applied to the matter field. Thus, for a system of many material particles, the same number of equations as that of particles was required to be considered simultaneously, while the Bose fields as the interaction between matter fields are the free fields. This theory is very useful for the two-body system, in that we can retain the particle picture throughout the calculations.

On the other hand, for the non-stationary problems, Feynman has succeeded in solving the Schrödinger equations without the procedure of the second quantization. That is, he solves the Schrödinger equation with the aid of Green's function which plays the role of the $D$-function (or associated $D$-function $\overline{D}$, $D^{(1)}$ etc.) which appeared in the procedure of the second quantization. Therefore, by means of the many-time formalism, we also deal with the non-stationary state...
for the many-body system, if the structure of the simultaneous equations is known. And we may add that the above structure has been partly discussed by Bloch, on which more details will be given in the second section of this paper. Thus, it is with the many-time formalism, we assert, that both stationary state and non-stationary state can be attacked. It is especially to be noted that we can handle relativistically the stationary state of the two-body system in a consistent manner, preserving the particle aspect.

Recently Thirring proposed a theory on the line of the many-time formalism, and discussed on the mass-defect of a bound system. But the foundation of his field equation is obscure, and our present investigation shows that the potentials in his equation do not seem to be adequate as far as the relativistic treatment of the two-body system is intended.

The second section of the present paper begins with a simple description of the many-time theory for the case where two charged fermions interact through the electromagnetic field. And the conditions for the integrability of equations of motion are investigated. The part (a) of the third section is devoted to a discussion on the covariant elimination of the longitudinal field which is replaced by the Coulomb potential. The part (b) concerns itself with the general derivation of the potential by performing a canonical transformation in the configuration space. The concept of potential seems to be valid to some extent even in the relativistic system. In the fourth section, we discuss the necessary and sufficient condition for the stationary state of the two-body system, retaining the relativistic formalism as perfectly as possible. Then, we construct the convenient equations to solve the stationary state, the potential energy being contained in the equation. This enables us to examine the structure of the Kemmer's equation which he has taken for the deuteron problem.

The present theory can be applied without difficulty to other cases, for example, to the deuteron problem, where nucleons interact through the meson field.

§ 2. Fundamental equations of motion for two-body system

According to Dirac's many-time theory, the wave function \( \psi(x_I, x_{II}) \) for the two-body system satisfies the following equations:*  

\[
\begin{align*}
F_I \psi(x_I, x_{II}) &= 0, \quad \text{(1a)} \\
F_{II} \psi(x_I, x_{II}) &= 0, \quad \text{(1b)}
\end{align*}
\]

where

\[
F_I = i\gamma_I \left( p_I^\mu - \frac{e}{c} A_\mu(x_I) \right) + m_I c^2, \quad \text{(2a)}
\]

* Hereafter Greek and Latin subscripts assume values ranging from 1 to 4 and from 1 to 3, respectively and a repeated index is to be so summed, and the symbol \( I \) or \( II \) signifies the particle \( I \) or \( II \), respectively.
in the case where two charged fermions interact through the electromagnetic field. The notation \( \gamma_\mu \) is used for the Hermitian matrices that obey the following commutation relations

\[
\{ \gamma^\mu, \gamma^\nu \} = 2 \delta^\mu_\nu, \quad \{ \gamma^\mu, \gamma^\nu \}_+ = 2 \delta^\mu_\nu.
\]

The four dimensional coordinate vector \( x^\mu \) is denoted by

\[
x^\mu = (r, x^a) = (r, ix^a), \quad x^0 = ct,
\]

and the energy-momentum four vector \( p_\mu \) is represented by

\[
p_\mu = -i\hbar \frac{\partial}{\partial x^\mu} = -i\hbar \partial_\mu.
\]

The four vector potential \( A_\mu \) of the electromagnetic field satisfies the relations

\[
\Box A_\mu(x) = 0, \quad [A_\mu(x), A_\nu(x')] = i\hbar \delta^\mu_\nu D(x-x'),
\]

where \( D \) is the usual invariant \( D \)-function, and the following supplementary condition must be added to the equations of motion in order to be consistent with the commutation relations

\[
[\partial_\mu A_\nu(x) - eD(x-x_1) - eD(x-x_2)] \Psi = 0.
\]

Now, the integrability conditions for the system of equations (1a) and (1b) become

\[
[\gamma^\mu, \gamma^\nu] = 0, \quad \text{for all } \mu, \nu,
\]

and

\[
D(x_1-x_2) \Psi (x_1, x_2) = 0,
\]

where the latter condition was formerly discussed in details by Bloch.

As is well known, the relations (3) and (6) are satisfied by 16 rows and columns matrices, but not by any 4 rows and columns matrix, and the representation of the former matrix will be conveniently given with the aid of Kronecker's symbol :\( \delta \):

\[
\gamma^k = \sigma_2 \times I \times I \times \sigma_k, \quad \gamma^k_- = I \times \sigma_2 \times \sigma_k \times I,
\]

\[
\gamma^k = \sigma_3 \times I \times I \times I, \quad \gamma^k_- = I \times \sigma_3 \times I \times I,
\]

where \( \sigma \)'s and \( I \) are Pauli's 2-2 spin matrices and 2-2 unit matrix respectively.

As the fundamental relativistic equations for the stationary state of the two-body system, we will employ, for convenience's sake, the following equations instead of (1a) and (1b):

\[
(\gamma^l F_l + \gamma^k_- F_k) \Psi = 0,
\]

\[
(\gamma^l F_l + \gamma^k_- F_k) \Psi = 0,
\]
Then it becomes our task to solve the equations (9a) and (9b) simultaneously under the conditions of (4), (5) and (7). As will be shown later, when we take the time-points of two particles to be the same, (9b) may be regarded as an auxiliary equation which is identically satisfied as the relation between energies of two particles, and thus the equation (9a) corresponds to the equation of Heisenberg-Pauli or that of Kemmer.

§ 3. Canonical transformation in the many-time theory and the potential energy

(a) Separation of the Coulomb potential

Both in the classical and quantal electrodynamics, it was possible to eliminate the scalar potential and the longitudinal part of the vector potential, leaving only the Coulomb potential and the transverse vector potential. This conventional procedure will be preserved even in the many-time theory in a covariant manner. We can, after the procedure of Schwinger, replace the electromagnetic field vector \( A_\mu(x) \) by two scalar fields, \( A(x) \) and \( A'(x) \), together with a restricted vector field \( \mathcal{A}_\mu(x) \), in such a way that the supplementary condition (5) involves only the scalar fields, while the equations of motion contain only \( \mathcal{A}_\mu(x) \). The decomposition will be conveniently expressed with the aid of an arbitrary time-like unit vector \( n_\mu (n_\mu = -1) \);

\[
A_\mu(x) = n_\mu n_\nu \partial_\nu A(x) - (\partial_\mu + n_\mu n_\nu \partial_\nu) A'(x) + \mathcal{A}_\mu(x),
\]

where

\[
n_\mu \mathcal{A}_\mu = 0, \quad \partial_\mu \mathcal{A}_\mu(x) = 0.
\]

The commutation relations of the three fields thus defined are the same as Schwinger's, for example:

\[
[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] = i\hbar c (\partial_{\mu\nu} + n_\mu n_\nu) D(x-x')
\]

\[
-i\hbar c (\partial_\mu + n_\mu n_\nu \partial_\nu) (\partial_\nu + n_\nu n_\mu \partial_\mu) \mathcal{D}(x-x'),
\]

where \( \mathcal{D}(x) \) is defined by

\[
(n_\mu \partial_\mu)^8 \mathcal{D}(x) = D(x), \quad \Delta \mathcal{D}(x) = 0,
\]

and is an odd function of the coordinates.

The supplementary condition involves only the scalar fields and, in fact, only combination \( A(x) - A'(x) \), since

\[
[A(x) - A'(x) - e \mathcal{D}(x-x_1) - e \mathcal{D}(x-x_{11})] \Psi = 0.
\]

Now we introduce the canonical transformation

\[
\Psi = \exp \left(-iG\right) \Phi, \quad G = \frac{\epsilon}{\hbar c} \{ A'(x_1) + A'(x_{11}) \},
\]

\[
(\gamma l F_1 - \gamma l F_{11}) \Psi = 0.
\]
here we must note the fact that
\[ \exp(-iG) \neq \exp\left\{-\frac{i\epsilon}{\hbar c} A'(x_i)\right\} \times \exp\left\{-\frac{i\epsilon}{\hbar c} A'(x_{ii})\right\}. \]

Therefore, we define the operator \( \exp(-iG) \) in the following way:
\[ \exp(-iG) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} G \cdot G \cdot \cdots \cdot G, \tag{15} \]
with
\[ G = \frac{e}{\hbar c} \{ A'(x_i) + A'(x_{ii}) \}. \]

The new equations of motion and supplementary condition are
\[ [\exp(iG) \cdot F \cdot \exp(-iG)] \Phi = 0, \tag{16a} \]
\[ [\exp(iG) \cdot F_{ii} \cdot \exp(-iG)] \Phi = 0 \tag{16b} \]
and
\[ [\exp(iG) \{ A(x) - A'(x) - e\mathcal{D}(x-x_i) - e\mathcal{D}(x-x_{ii}) \} \exp(-iG) ] \Phi = 0. \tag{17} \]

Since the following formula is true for any operator \( F \)
\[ \exp(iG) F \exp(-iG) = F + i[G, F] - \frac{1}{2!} [G, [G, F]] + \cdots, \tag{18} \]
the supplementary condition (17) becomes simply
\[ [A(x) - A'(x)] \Phi = 0 \tag{17'} \]
and the equation of motion (16a)
\[ \left[ i\gamma_{\mu} \left\{ \frac{p_{\mu} - e}{c} \mathcal{A}_{\mu}(x_i) + \frac{e^2}{2c^2} \partial_{\mu} \mathcal{D}(x_{ii} - x_i) + \frac{e^2}{c^n} n_{\mu} n_{\nu} \partial_{\nu} \mathcal{D}(x_{ii} - x_i) \right\} \right. \]
\[ + m \gamma^2 + ic \lim_{x \rightarrow x_i} \left\{ \frac{e^2}{2c^2} \partial^2 \mathcal{D}(x-x_i) + \frac{e^2}{c^n} n_{\mu} n_{\nu} \partial_{\nu} \mathcal{D}(x-x_i) \right\} \]
\[ \left. - i e \gamma_{\mu} n_{\nu} \partial_{\nu} \{ A(x_i) - A'(x_i) \} \right\} \Phi = 0 \tag{16a'} \]
and by interchanging \( I \) for \( II \) in (16a) and vice versa, we have the corresponding equation for (16b). In view of the supplementary condition (17'), Eq. (16a') is reduced to
\[ \left[ i\gamma_{\mu} \left\{ \frac{p_{\mu} - e}{c} \mathcal{A}_{\mu}(x_i) + \frac{e^2}{2c^2} \partial_{\mu} \mathcal{D}(x_{ii} - x_i) + \frac{e^2}{c^n} n_{\mu} n_{\nu} \partial_{\nu} \mathcal{D}(x_{ii} - x_i) \right\} \right. \]
\[ + m \gamma^2 \] \[ \Phi = 0 \tag{19a} \]

neglecting the static self-energy term \( \lim_{x \rightarrow x_i} \{ \}. \) We have thereby obtained the
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equations of motion for $\Phi$ which no longer contain the electromagnetic field variables involved in the supplementary condition. The additional term thus introduced must correspond evidently to the covariant generalization of the Coulomb interaction between charged fermions. To exhibit this property somewhat more clearly, we choose such arbitrary time-like unit vector $n_{\mu}$ as normal to a certain space-like plane surface containing two space-like points $x_i$ and $x_{II}$, then the following relations hold:

\[
(\partial_{\mu} + n_{\mu} \partial_{\nu}) \mathcal{D}(x_i - x_{II}) = 0,
\]

\[
n_{\mu} \partial_{\nu} \mathcal{D}(x_i - x_{II}) = \frac{1}{4\pi} \left[ (x_i^\mu - x_{II}^\mu)^2 \right]^{-\frac{3}{2}}.
\]

These relations enable the term containing $\delta$-function in the equation (19a) to be simplified, yielding

\[
-\frac{i e^4}{8\pi} n_{\mu} \left[ (x_i^\mu - x_{II}^\mu)^2 \right]^{-\frac{3}{2}} = V_i.
\]

The procedure of Heisenberg-Pauli theory corresponds to the special choice $n_{\mu} = (0, 0, 0, i)$. In this case $\gamma_i^4 V_i$ becomes a half of the usual three dimensional Coulomb potential;

\[
\gamma_i^4 V_i = -\frac{e^2}{8\pi r}, \quad r = |r_i - r_{II}|.
\]

The corresponding procedure is applied to Eq. (16b) and we get

\[
\gamma_{II}^4 V_{II} = -\frac{e^2}{8\pi r}.
\]

Thus, when the time-points of two particles are taken to be the same, Eq. (9a) is entirely equivalent to the equation of Heisenberg-Pauli. As for Eq. (9b), its meaning will be clarified when the transverse potential $\mathfrak{H}_{\mu}$ is eliminated in the next subsection.

(b) Möller interaction

We show, in this subsection, the general method to derive the interaction potential, whose relativity-theoretical interpretation can be furnished more clearly, when we take the procedure of the canonical transformation in the many-time theory, than when the perturbation-theoretical one is taken.

In the previous subsection we derived the following equations:

\[
[-i\hbar \partial_{\nu} + i\gamma_i^4 \gamma^4_{\mu} P_{\mu} + \gamma_i^4 m_{\nu} c^2 + \gamma_i^4 V_i - i\gamma_i^4 \gamma^4_{\mu} \mathfrak{H}_{\mu}(x_i)] \Phi = 0,
\]

\[
[-i\hbar \partial_{\nu} + i\gamma_{II}^4 \gamma^4_{\mu} P_{\mu} + \gamma_{II}^4 m_{\nu} c^2 + \gamma_{II}^4 V_{II} - i\gamma_{II}^4 \gamma^4_{\mu} \mathfrak{H}_{\mu}(x_{II})] \Phi = 0.
\]

These equations are given in the Schrödinger representation. To derive the interaction potential by the method of the canonical transformation, we perform
a unitary transformation which changes the Schrödinger representation into the interaction representation. Namely, we transform the Schrödinger function \( \Phi(x_n, x_{II}) \) into \( \tilde{\Phi}(x_n, x_{II}) \) of the interaction representation by means of a unitary operator \( U(x_n^0, x_{II}^0) \):

\[
\Phi = U \tilde{\Phi} = U_I(x_n^0) U_{II}(x_{II}^0) \tilde{\Phi},
\]

where

\[
U_I = \exp \left\{ -\frac{i}{\hbar c} H_I x_n^0 \right\}, \quad U_{II} = \exp \left\{ -\frac{i}{\hbar c} H_{II} x_{II}^0 \right\},
\]

and

\[
H_I^0 = i\gamma_{kI}^0 \rho_I^0 + \gamma_{I}^0 m c^2, \quad H_{II}^0 = i\gamma_{kII}^{II} \rho_{II}^0 + \gamma_{II}^{II} m c^2.
\]

Then the transformed equations become

\[
[-i\gamma_{\mu} \partial_t - i\gamma_{\mu} U_{\mu} (x_I) + \gamma_{I} \tilde{V}_I] \Phi = 0,
\]

\[
[-i\gamma_{\mu} \partial_t - i\gamma_{\mu} U_{\mu} (x_{II}) + \gamma_{II} \tilde{V}_{II}] \Phi = 0,
\]

where the symbol \( \sim \) is used to denote the transformed quantities, and \( \tilde{\gamma}, \tilde{\mathcal{A}}(x) \) are defined by

\[
\tilde{\gamma}_\mu^I = U_I^{-1} \gamma_\mu^I U_{II}, \quad \tilde{\mathcal{A}}_\mu(x_I) = U_I^{-1} \mathcal{A}_\mu(x_I) U_I = \mathcal{A}_\mu(\tilde{\gamma}_\mu^I, x_I),
\]

\[
\tilde{\gamma}_\mu^{II} = U_{II}^{-1} \gamma_\mu^{II} U_I, \quad \tilde{\mathcal{A}}_\mu(x_{II}) = U_{II}^{-1} \mathcal{A}_\mu(x_{II}) U_I = \mathcal{A}_\mu(\tilde{\gamma}_\mu^{II}, x_{II}).
\]

Now, the transformed \( \tilde{\gamma} \)'s are time-dependent—operators in the Heisenberg representation—but the commutation relations between these operators retain the original form, that is:

\[
| \tilde{\gamma}_\mu^I(x_I), \tilde{\gamma}_\nu^{II}(x_{II}) \rangle = | \tilde{\gamma}_\nu^{II}(x_{II}), \tilde{\gamma}_\mu^I(x_I) \rangle = 2\delta_{\mu\nu},
\]

\[
[\tilde{\gamma}_\mu^I(x_I), \tilde{\gamma}_\nu^{II}(x_{II})] = 0, \quad \text{for all } \mu, \nu.
\]

And it is easily shown that the time-dependence of \( \gamma \)'s is determined by the following equations:

\[
i\hbar c \frac{d\tilde{\gamma}_\mu^I}{dx_I^0} = [\tilde{\gamma}_\mu^I, \tilde{H}_I^0], \quad i\hbar c \frac{d\tilde{\gamma}_\mu^{II}}{dx_{II}^0} = 0,
\]

where

\[
\tilde{H}_I^0 = U_I^{-1} H_I^0 U_I = \hbar c \gamma_{kI}^0 \frac{\partial}{\partial x_I^0} + \gamma_I^0 m c^2,
\]

and the corresponding relations for \( \gamma^{II} \)'s can be obtained by interchanging suffixes \( I \) and \( II \) in (29). On the other hand, the transformed \( \tilde{\mathcal{A}}_\mu(x) \) satisfies evidently
the following commutation relation:

\[
\begin{align*}
[\mathcal{H}_\mu(x_I), \mathcal{H}_\nu(x_{II})] &= [\mathcal{H}_\mu(\mathbf{r}_I, x_I^0), \mathcal{H}_\nu(\mathbf{r}_{II}, x_{II}^0)] \\
&= i\hbar c (\delta_{\mu\nu} + n_{\mu}n_{\nu}) \mathcal{D}(\mathbf{r}_I - \mathbf{r}_{II}, x_I^0 - x_{II}^0) \\
&\quad - i\hbar c (\delta_{\mu\nu} + n_{\mu}n_{\nu} \delta_{\mu}'(\delta_{\mu}' + n_{\nu} \delta_{\nu}')) \mathcal{D}(\mathbf{r}_I - \mathbf{r}_{II}, x_I^0 - x_{II}^0).
\end{align*}
\]  

(30)

Now, our task is to solve the Eqs. (26a) and (26b) simultaneously under the conditions of (28), (29) and (30). This can be carried out quite analogously to the case of Tomonaga-Schwinger theory, though the equations to be solved in our case are simultaneous ones. Thus we perform the following canonical transformation:

\[
\bar{\phi} = \exp[-iS] \phi = \exp[-i(S_I + S_{II})] \phi
\]  

(31)

where

\[
\begin{align*}
S_I &= -\frac{ie}{\hbar c} \int_{-\infty}^{\infty} \mathcal{H}_I(\mathbf{r}_I, x_I^0) \, dx_I^0, \\
S_{II} &= -\frac{ie}{\hbar c} \int_{-\infty}^{\infty} \mathcal{H}_{II}(\mathbf{r}_{II}, x_{II}^0) \, dx_{II}^0.
\end{align*}
\]  

(32)

Here, since \(S_I\) and \(S_{II}\) are not commutable, we define the operator \(\exp[-iS]\) to be such as shown in (15). The transformed equations of motion for \(\bar{\phi}\) become

\[
\begin{align*}
\{ -i\hbar c \partial_{r_I} + \frac{e}{2} [S_I, \mathcal{H}_I(\mathbf{r}_I, x_I^0)] + \gamma_I \bar{V}_I \} \bar{\phi} &= 0, \\
\{ -i\hbar c \partial_{r_{II}} + \frac{e}{2} [S_{II}, \mathcal{H}_{II}(\mathbf{r}_{II}, x_{II}^0)] + \gamma_{II} \bar{V}_{II} \} \bar{\phi} &= 0
\end{align*}
\]  

(33a, 33b)

in which we have retained only the second order interactions. We neglect, here-after, the terms \([S_I, \mathcal{H}_I(\mathbf{r}_I, x_I^0)]\) and \([S_{II}, \mathcal{H}_{II}(\mathbf{r}_{II}, x_{II}^0)]\) in Eqs. (33a) and (33b) respectively, which represent the self-energy. On the other hand, the interaction energy of Möller type (excluding Coulomb energy) becomes,

\[
\begin{align*}
\frac{e}{2} [S_{II}, \mathcal{H}_{II}(\mathbf{r}_{II}, x_{II}^0)] &= \mathcal{B}_{II}, \\
\frac{e}{2} [S_I, \mathcal{H}_I(\mathbf{r}_I, x_I^0)] &= \mathcal{B}_I.
\end{align*}
\]  

(34a, 34b)

Although \(\gamma_I's\) and \(\mathcal{B}_I\) are time-dependent, the actual evaluation of this potential can be easily done with the aid of a method proposed by Nambu. Here we describe only the result obtained, leaving the detailed calculations for Appendix at the end of this paper. Expanding (34a) and (34b) in powers of \((v_I/c)(v_{II}/c)\), we have, as is expected, a half of the Breit potential in the order of \((v_I/c)(v_{II}/c)\) respectively:

\[
\begin{align*}
\mathcal{B}_I = \mathcal{B}_{II} &= \frac{e^2}{16\pi} \frac{1}{r} \left\{ \left( \mathcal{K}_I \mathcal{K}_I' \right) \left( \mathcal{K}_{II} \mathcal{K}_{II}' \right) + \left( \mathcal{K}_I \mathcal{K}_{II}' \right) \left( \mathcal{K}_I' \mathcal{K}_{II} \right) \right\} \\
&= \frac{e^2}{16\pi} \frac{1}{r} \left\{ \left( \mathcal{K}_I \mathcal{K}_I' \right) \left( \mathcal{K}_{II} \mathcal{K}_{II}' \right) + \left( \mathcal{K}_I \mathcal{K}_{II}' \right) \left( \mathcal{K}_I' \mathcal{K}_{II} \right) \right\}.
\end{align*}
\]  

(35)
where we assumed the time-points of both the particle \( I \) and \( II \) to be the same. Terms of the order of \((v_I/c)^2(v_{II}/c)^2\) representing the recoil effect, depend, as is natural, on the velocity of particles.

Further detailed discussion about the potential energy, for example about the fourth order potential of charge \( e \), is left over. Our theory differs from Tomonaga-Schwinger’s in the point that in the latter the second quantization is applied to the matter field, but not in the former. It may be interesting to investigate this difference in the potentials of the fourth order or higher with respect to \( e \).

Finally, we return to the Schrödinger representation from the interaction representation by means of the inverse transformation of (23),

\[
\psi = U^{-1} \phi = U^{-1}_I U^{-1}_II \phi.
\]

Then the operators with the symbol \( \sim \) return to those without \( \sim \) and Eqs. (33a) and (33b) become as follows:

\[
[-i\hbar \partial_t + H_I^0 + \gamma_I^I V_I + B_I] \phi = 0, \quad (37a)
\]

\[
[-i\hbar \partial_t^II + H_{II}^0 + \gamma_{II}^II V_{II} + B_{II}] \phi = 0. \quad (37b)
\]

\section*{§ 4. Some properties of equations for the stationary state}

When we denote the potentials derived in section 3 as \( J_I \) and \( J_{II} \), the fundamental equations become:

\[
[-i\hbar \partial_t + H_I^0 + J_I] \phi = 0, \quad (37a')
\]

\[
[-i\hbar \partial_t^II + H_{II}^0 + J_{II}] \phi = 0 \quad (37b')
\]

where, for the time being, we take \( J_I \) and \( J_{II} \) as general potentials which are not restricted to the same time-points of two particles.

To solve these equations generally, it will be desired to separate the motion of particles into the motion of centre of gravity and the relative motion, the former showing the motion of the two-particle system as a whole, that is, the motion of a Bose particle. But, reserving this problem for future occasion, we will discuss here only the stationary state.

In our theory also, \( \phi^* (x_I, x_{II}) \phi (x_I, x_{II}) \) has the meaning of the probability density, then we can take the following condition

\[
\phi^* \phi = \text{constant}, \quad (38)
\]

as the generalized definition for the stationary state. The wave function \( \phi \) which satisfies (38) can generally be given in the form:

\[
\phi (x_I, x_{II}) = \varphi (r_I, r_{II}) \exp [iE (x_I^0, x_{II}^0)], \quad (39)
\]

where \( E \) depends only on \( x_I^0 \) and \( x_{II}^0 \) explicitly. Substituting (39) into (37a’), we have
In order that this equation may hold identically, \( J_i \) should be divided into two parts.

\[
J_i = J_i^0(r_i, r_{1i}, \partial/\partial x_i, \partial/\partial x_{i1}), \tag{40a}
\]

\[
J_i' = J_i'(x_i, x_{1i}, \partial/\partial x_i, \partial/\partial x_{1i}) \tag{40b}
\]

where \( J_i^0 \) is a function only of \( r_i, r_{1i}, \partial/\partial x_i \) and \( \partial/\partial x_{i1} \), while \( J_i' \) depends only on \( x_i, x_{1i}, \partial/\partial x_i \) and \( \partial/\partial x_{1i} \). On the other hand, the potential derived in the previous section can be transformed into the form (40a) through a real transformation, but this is not the case for (40b). Thus we put \( J_i' = 0 \), and only such potentials as take the form (40a) survive. This means that, in the actual calculations, the potentials for the stationary state should be taken as those which correspond to the case where the time-points of the two particles are the same. Accordingly, the Eq. (37a') becomes

\[
-\hbar c \frac{\partial E}{\partial x_i} = \frac{(\psi^*(H_0^0 + J_i^0)\psi)}{(\psi^*, \psi)} \text{const.} = E_i \tag{41}
\]

whence we have

\[
E = -\frac{E_i x_i^0}{\hbar c} + E'(x_{1i}). \tag{42}
\]

The analogous discussion can be made for (37b') and we have

\[
-\hbar c \frac{\partial E'}{\partial x_{1i}} = \frac{dE'}{dx_{1i}} = \frac{(\psi^*(H_0^0 + J_{1i}^0)\psi)}{(\psi^*, \psi)} \text{const.} = E_{1i}. \tag{43}
\]

Thus, for the following solution of the stationary state:

\[
\Phi = \psi(r_i, r_{1i}) \exp \left[ -\frac{i}{\hbar c} (E_i x_i^0 + E_{1i} x_{1i}) \right]. \tag{44}
\]

Eqs. (37a') and (37b') are reduced to

\[
[-E_i + H_0^0 + J_i^0] \psi = 0, \tag{45a}
\]

\[
[-E_{1i} + H_0^0 + J_{1i}^0] \psi = 0. \tag{45b}
\]

From Eqs. (45a) and (45b), we can construct the equations corresponding to Eqs. (9a) and (9b) as follows:

\[
[-E_i - E_{1i} + H_0^0 + J_i^0 + V^0(=J_i^0 + J_{1i}^0)] \psi = 0, \tag{46a}
\]

\[
[-E_i + E_{1i} + H_i^0 - H_{1i}^0] \psi = 0. \tag{46b}
\]

Eq. (46b) can be reduced to

\[
E_i - E_{1i} = \pm \sqrt{c^2 p_i^2 + m_i^2 c^4} = \sqrt{c^2 p_{1i}^2 + m_{1i}^2 c^4} \tag{46b'}
\]
which is identically satisfied and indicates the possibility of the negative energy states for each particle.

The energy-momentum four vector of the centre of gravity is covariantly defined by

\[ P_\mu = p_\mu^I + p_\mu^II. \]

Then the proper mass \( M_0 \) of the centre of gravity is defined, in the centre of gravity system \((\mathcal{P} = 0)\), by

\[ P_\mu^2 = -(E_I + E_{II})^2/c^2 = -M_0^2c^2 \]

and Eq. (46a) in this system becomes

\[ \pm M_0^2 \phi = [\beta^a p_I^a + \beta^a p_{II}^a + c(a^a - c^a) \rho^a + \mathcal{V}^a] \phi \]

with

\[ \rho^a = \rho^a = -\rho^a; \quad a^a = a^a; \quad \beta^a = \beta^a \quad a = 1, 2. \]

We can obtain the mass spectrum of the centre of gravity by solving the above equation. This form of equation had previously been proposed by Kemmer for the deuteron problems.

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**Appendix**

Here we calculate the Möller interaction energy of the following type:

\[ \frac{1}{2} [S_{II}, \overline{\mathcal{M}}_{\mu}(\mathbf{r}_{II}, x^0)] = B_I, \quad \tag{34a} \]

\[ \frac{1}{2} [S_{II}, \mathcal{M}_{\mu}(\mathbf{r}_{II}, x^0)] = B_{II}. \quad \tag{34b} \]

As \( S_I \) and \( S_{II} \) are operators given by (32), (34a) becomes as follows:

\[ B_I = -\frac{i e^2}{2\hbar c} \int_{-\infty}^{x^0} [\mathcal{M}_{\mu}(\mathbf{r}_{II}, x^0)] \overline{\mathcal{M}_{\mu}(\mathbf{r}_{II}, x^0)] dx^0. \quad \tag{A1} \]

The integral \( \int_{-\infty}^{x^0} dx^0 \) can be written as \( \frac{1}{2} \int_{-\infty}^{x^0} [1 + \epsilon(x^0)] dx^0 \) where \( \epsilon(x^0) = \pm 1 \) according as \( x^0 \approx 0 \). We are concerned only with the phenomena in which any real process to emit or absorb the photon does not occur, so the integral \( \int_{-\infty}^{x^0} dx^0 \) can be omitted. Moreover, if the time-points of the two particles \( I \) and \( II \) are the same \( (x^0_I = x^0_{II}, \text{ and } n_{II} = (0, 0, 0, i)) \), (34a) is reduced, on account of (30), to the following:
As the operator $\tilde{r}$'s are time-dependent, the rigorous evaluation of these integrals is not easy. Accordingly, we take up the convenient procedure employed by Nambu, then we have:

$$\int_{-\infty}^{\infty} D(\tilde{r}, x^0 - x_i^0) f(x^0) dx^0 = \frac{1}{4\pi} \frac{d}{dx^0} \left( \frac{1}{\tilde{r}} \right) f(x^0) \bigg|_{x_0 = x_i^0}$$

(A3)

$$\int_{-\infty}^{\infty} \overline{D}(\tilde{r}, x^0 - x_i^0) f(x^0) dx^0 = \frac{1}{4\pi} \left( \frac{1}{\tilde{r}} \right) \left( \frac{d}{dx^0} \right)^2 f(x^0) \bigg|_{x_0 = x_i^0}$$

(A4)

As is easily seen, this is no more than the form expanded in powers of $(v/c)$. Thus we have:

$$H_1 = \sum_{n=1}^{\infty} H_1^{(n)} = \frac{e^2}{8\pi \alpha} \left( \frac{1}{\tilde{r}} \right) + \frac{1}{2} \left( \frac{d}{dx^0} \right)^2 \tilde{r} + \cdots \right) \tilde{r}^{(1)} \tilde{r}^{(2)} \tilde{r}^{(3)} \tilde{r}^{(4)} \tilde{r}^{(5)}$$

K_1 = \sum_{n=1}^{\infty} K_1^{(n)} = -\frac{e^2}{16\pi} \partial_i \partial_j \left( \tilde{r} + \frac{1}{12} \left( \frac{d}{dx^0} \right)^2 \tilde{r} + \cdots \right) \tilde{r}^{(1)} \tilde{r}^{(2)} \tilde{r}^{(3)} \tilde{r}^{(4)} \tilde{r}^{(5)} \right)$$

(A2')

The first term of each equation corresponds to the order of $(v/c)(v/c)$, and $H_1^{(n)} + K_1^{(n)}$ is reduced to (35). For the other terms, on account of (29), we substitute the relations

$$\frac{d}{dx^0} = \frac{i}{\hbar c} [\tilde{H}_i^{(0)}, \cdots], \quad \frac{d}{dx^0} = \frac{i}{\hbar c} [\tilde{H}_i^{(0)}, \cdots]$$

(A5)

in (A2'). If we put the time of the two particles in the integrand of $B_j$ to be the same, Nambu's procedure can be applied to our case without contradicting the integrability condition for Eqs. (33a) and (33b). Moreover, we take into account the law of energy-conservation in the form:

$$\frac{d}{dx_i^0} = -\frac{d}{dx_i^0}$$

and the higher powers of temporal differential operator, we write for symmetry's sake:
Then we can obtain the explicit form of the term $H^{(2)}_{I}+K^{(2)}_{I}$ which corresponds to the order of $(v_{I}/c)^{2}(v_{II}/c)^{2}$ and depends on the velocity of particles. The calculation for the term $B_{II}$ is carried out analogously to the case of $B_{I}$, and the result is obtained by exchanging the symbol $I$ and $II$ in $B_{I}$.

Thus we have the Möller interaction energy of the order of $(v_{I}/c)^{2}(v_{II}/c)^{2}$ which runs as follows:

\[
H^{(2)}_{I}+K^{(2)}_{I}+H^{(2)}_{II}+K^{(2)}_{II}
= \frac{e^{2}}{8\pi} \left[ x_{I} \cdot x_{II} \right] \left\{ 3 \left( \gamma_{I}\gamma_{II}^{*} \right) - \frac{\left( \gamma_{I}\gamma_{II}^{*} \right) \left( \gamma_{I}^{*}\gamma_{II} \right)}{r^{2}} \right\} \\
+ x_{I} \left\{ 3 \left[ \gamma_{I} \left( \gamma_{I}^{*}\gamma_{II}^{*} \right) - \left( \gamma_{I}\gamma_{II}^{*} \right) \left( \gamma_{II}^{*}\gamma_{I} \right) \right] - \frac{1}{r} \left( \gamma_{I}\gamma_{II}^{*} \right) \left( \gamma_{II}^{*}\gamma_{I} \right) \right\} \\
+ x_{II} \left\{ 3 \left( \gamma_{II}^{*}\gamma_{I} \right) - \left( \gamma_{I}\gamma_{II}^{*} \right) \left( \gamma_{II}^{*}\gamma_{I} \right) \right\} \\
- \left\{ \frac{(\gamma_{II}^{*}\gamma_{I})^{2}}{r} - \frac{2}{r^{3}} \left( \gamma_{I}\gamma_{II}^{*} \right) \left( \gamma_{II}^{*}\gamma_{I} \right) \right\}
\]

where

\[
x_{I} = \frac{m_{I}c}{\hbar}, \quad x_{II} = \frac{m_{II}c}{\hbar}.
\]

References

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Note added in proof

In our calculation of the potential in § 3 (b), the negative energy states are neglected. Araki* has recently showed that, in the calculation of the potential, the neglect of negative energy states is not adequate and that, if the negative states are taken into account, much better coincidence with experiment can be obtained in the case of hydrogen atom.

Accordingly, it may be necessary to show that our method adopted in this paper can also be applied to the case when the negative energy states are considered.

For this purpose, it is convenient to carry out the calculation of the potential in the momentum representation in the following.

Bringing the transformations (23), (31) and (38) into one, we can write it as

$$\Theta = \mathbf{S}_{m} \exp \left[ -\frac{i}{\hbar c} \left( H_{\mu} x_{\mu} + H_{\mu} x_{\mu} \right) \right] \times \exp \left[ -i(S_{z} + S_{II}) \right] \exp \left[ -\frac{i}{\hbar c} \left( H_{\mu} x_{\mu} + H_{\mu} x_{\mu} \right) \right] \phi.$$  \hspace{1cm} (N 1)

This transformation can be written in the momentum representation as

$$\mathbf{S} = (p_{m}, p_{\mu}; N \pm 1, \phi_{m}, \phi_{\mu}; N_{k})$$

$$= \int \left( p_{m}, p_{\mu}|r_{I}, r_{II} \right) (N \pm 1, r_{I}, r_{II} | S_{z} S_{II} N_{k}) \times (r_{I}, r_{II}|p_{m}, p_{\mu}) d^{4}r_{I} d^{4}r_{II}.$$  \hspace{1cm} (N 2)

Taking into account the relation

$$\left( r_{I}, r_{II}|p_{m}, p_{\mu} \right) = \left( p_{I}, p_{II} | r_{I}, r_{II} \right) \times \left( p_{I}, p_{II} | r_{I}, r_{II} \right),$$

(N 2) runs as follows:

$$\mathbf{S} = \exp \left[ -i(S_{z} + S_{II}) \right],$$

$$\left( p_{m}, N \pm 1, \phi_{m}, N_{k} \right)$$

$$= \delta \left( p_{m} - p_{I} \pm k \right) \left( \phi_{m} \phi_{I}, \phi_{m} \phi_{I} \right) (N \pm 1, \phi_{m} \phi_{I}, \phi_{m} \phi_{I} | N_{k} / E_{m} - E_{I} = \omega_{k},$$

$$\left( p_{\mu}, N \pm 1, \phi_{\mu}, N_{k} \right)$$

$$= \delta \left( p_{\mu} - p_{II} \pm k \right) \left( \phi_{\mu} \phi_{II}, \phi_{\mu} \phi_{II} \right) (N \pm 1, \phi_{\mu} \phi_{II}, \phi_{\mu} \phi_{II} | N_{k} / E_{\mu} - E_{II} = \omega_{k}. \hspace{1cm} (N 3)$$

As is well known, van Hove's procedure can also be applied to the calculation of the potential when the negative energy states are considered, and thus this is also true in our method. But it may need further investigation whether the consideration of the negative energy states in the calculation of the interaction energy is also necessary in meson field as well as in electromagnetic field.

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