On an Origin of the Spin-Orbit and the Spin-Spin Coupling Forces in the Relativistic Quark Model

Yoshimasa TAKAYA

Meteorological College, Kashiwa, Chiba 277

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We show a general method of adding the spin-dependent terms to the potential term in the relativistic two-particle problems in order to keep the relativistic invariance of the theory. We derive the most general form of the mass operator of the relativistic two-particle problems so far as the particle-creations are neglected.

§ 1. Introduction

The harmonic oscillating quark model is known to be valid for the understanding of the hadron phenomenology. Recently various authors try to deal with this model in the framework of the special relativity. However, so far, all these investigations are incomplete. The first group of them neglects the spin variable of the constituent quarks. Therefore they must add the spin-dependent terms in an arbitrary manner, if they want to understand the actual hadronic mass spectra. The second group of them, though they take account of the effects of the spin, uses the free spinors for the bounded quarks. Accordingly the relativistic covariance is not complete in their theory.

In this paper, we treat the problem in a fully relativistically invariant manner and prove that the harmonic oscillator potential must be followed by certain spin-dependent terms which are interpreted to be spin-orbit coupling potential and spin-spin coupling potential. Further we derive the most general form of the mass operator of the relativistic two-particle problems in which the particle-creations are neglected.

§ 2. Relativistic quantum mechanics in the point form

In 1945, Dirac showed three types of mechanics; the instant form, the front form and the point form. Natural extension of the Newtonian mechanics is led to the instant form in which the initial conditions are given on the hyper-plane of constant time. Six of the ten generators of the Poincaré group (three space-rotations and three space-translations) are written in the free form. The remaining four generators (three space-time-rotations and one time-translation) have interaction terms. In the front form, the initial conditions are given on the light cone.
Finally in the point form, which we treat in this paper, the initial conditions are
given on the hyperbola and six generators (three space-rotations and three space-
time-rotations) have the free form. The generators of three space-translations and
one time-translation have the interaction terms. In this section, we explain how
to construct the relativistic mechanics of two particles in the point form.

First we define some kinematical quantities of the two-particle system:

\[ \begin{align*}
\mathbf{p}_1; & \text{ free momentum of the particle 1,} \\
\mathbf{p}_2; & \text{ free momentum of the particle 2,} \\
m_1; & \text{ mass of the particle 1,} \\
m_2; & \text{ mass of the particle 2,} \\
\omega_1 = \sqrt{\mathbf{p}_1^2 + m_1^2}; & \text{ free energy of the particle 1,} \\
\omega_2 = \sqrt{\mathbf{p}_2^2 + m_2^2}; & \text{ free energy of the particle 2,} \\
s_1; & \text{ spin matrix of the particle 1,} \\
s_2; & \text{ spin matrix of the particle 2,} \\
M = \sqrt{(\omega_1 + \omega_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2}; & \text{ free mass of} \\
& \text{ the total system.}
\end{align*} \]

By using these quantities, the ten generators of the Poincaré group are written
in the following way\(^6\) when the interaction terms are neglected:

\[ \begin{align*}
\mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2 \quad \text{(space-translation)}, \\
H &= \omega_1 + \omega_2 \quad \text{(time-translation)}, \\
\mathbf{N} &= i\omega_1 \frac{\partial}{\partial \mathbf{p}_1} + i\omega_2 \frac{\partial}{\partial \mathbf{p}_2} + \frac{\mathbf{p}_1 \times s_1}{\omega_1 + m_1} + \frac{\mathbf{p}_2 \times s_2}{\omega_2 + m_2} \quad \text{(space-time-rotation)}, \\
\mathbf{J} &= -i\mathbf{p}_1 \times \frac{\partial}{\partial \mathbf{p}_2} - i\mathbf{p}_2 \times \frac{\partial}{\partial \mathbf{p}_1} + s_1 + s_2 \quad \text{(space-rotation).}
\end{align*} \]

They satisfy the usual commutation relations:

\[ \begin{align*}
[P_1, P_2] &= 0, \\
[N_1, P_2] &= 0, \\
[N_1, N_2] &= -i\epsilon_{ijk}J_k, \\
[N_1, J_2] &= i\epsilon_{ijk}N_k, \\
[J_1, J_2] &= i\epsilon_{ijk}J_k, \\
[N_1, H] &= iP_1.
\end{align*} \] (2.3)

Now we introduce another set of variables instead of those given in Eq. (2.1):
\[ G = \frac{p_1 + p_2}{M} \] (velocity vector of the center-of-mass),
\[ G_0 = \sqrt{\frac{\omega_1 + \omega_2}{M}} = \sqrt{G^2 + 1}, \]
\[ k = \frac{1}{2} (p_1 - p_2) - \frac{(p_1 - p_2 \cdot G)}{2G_0(1 + G_0)} G - \frac{G(m_1^2 - m_2^2)}{2M}, \]
\[ q_1 = \sqrt{m_1^2 + k^2} \] (energy of the particle 1 in the center-of-mass)
and
\[ q_2 = \sqrt{m_2^2 + k^2} \] (energy of the particle 2 in the center-of-mass).

\[(G_0, G)\] forms the four vector under the Lorentz transformations. By using these quantities, those defined in Eq. (2.1) are rewritten as follows:
\[ \omega_1 = q_1 G_0 + (G \cdot k), \]
\[ \omega_2 = q_2 G_0 - (G \cdot k), \]
\[ p_1 = k + \left( q_1 + \frac{(G \cdot k)}{G_0 + 1} \right) G, \]
\[ \text{and} \]
\[ p_2 = -k + \left( q_2 - \frac{(G \cdot k)}{G_0 + 1} \right) G. \]

Further we obtain
\[ |k|^2 = k^2 = \frac{(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)}{4M^2}. \]

We can completely describe the orbital motions of the two particles by these variables \( G, k \) instead of \( p_1, p_2 \). The generators of the Poincaré group (without interactions) is rewritten as follows:
\[ P = MG, \quad J = -iG \times \frac{\partial}{\partial G} - ik \times \frac{\partial}{\partial k} + s_1 + s_2, \]
\[ N = iG \frac{\partial}{\partial G} + \frac{G \times L}{G_0 + 1} + N^{(\text{spin})}, \]
\[ N^{(\text{spin})} = \frac{k \times s_1}{G_0 q_1 + (G \cdot k) + m_1} - \frac{k \times s_2}{G_0 q_2 - (G \cdot k) + m_2} + \frac{q_1 + (G \cdot k)/(G_0 + 1)(G \times s_1)}{G_0 q_1 + (G \cdot k) + m_1} + \frac{q_2 - (G \cdot k)/(G_0 + 1)(G \times s_2)}{G_0 q_2 - (G \cdot k) + m_2}. \]
where

\[ L = -i \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}}. \quad (2.8) \]

\( L \) is interpreted as the inner angular momentum operator. We see from Eq. (2.7) that these operators can change the direction of the vector \( \mathbf{k} \) but cannot change its length \( |\mathbf{k}| = \bar{k} \) which is, via Eq. (2.6), uniquely related to the mass of the total system which is the Casimir-operator of the Poincaré group.

We assume that the interaction terms are written in terms of the relative coordinate of two interacting particles in the center-of-mass frame which will be written as \( i (\partial/\partial \mathbf{k}) \), for this operator satisfies

\[ \left[ i \frac{\partial}{\partial k_i}, k_j \right] = i \delta_{ij}. \quad (i, j = 1, 2, 3) \]

This operator can change the length of the vector \( \mathbf{k} \) and if it is contained in the mass operator, its eigenstates will be the superposition of the various free mass states.

Now we construct the generators of the Poincaré group for the interacting two-particle system in the following way: Six generators (three space-rotations and three space-time-rotations) are as the same as those in Eq. (2.7), namely, they have the free form; the remaining four generators (three space-translations and one time-translation) are given by

\[ P_\mu = G_\mu \mathcal{M}, \quad (2.9) \]

where \( \mathcal{M} \) is the mass operator which contains the interaction terms constructed by \( i (\partial/\partial \mathbf{k}) \). In order that these generators satisfy the same commutation relations as in Eq. (2.3), it is necessary and sufficient that

\[ [G_\mu, \mathcal{M}] = 0, \quad (\mu = 0, 1, 2, 3) \quad (2.10) \]

\[ [J_i, \mathcal{M}] = 0, \quad (i = 1, 2, 3) \quad (2.11) \]

\[ [N_i, \mathcal{M}] = 0. \quad (i = 1, 2, 3) \quad (2.12) \]

Equation (2.10) requires that \( \mathcal{M} \) must not contain the derivative operators with respect to the vector \( \mathbf{G} \), and Eq. (2.11) shows that \( \mathcal{M} \) must be scalar under the spacial rotations. In the next section, we investigate the consequences of Eq. (2.12).

§ 3. Covariant vector operators connected with the inner motion

When each particle has no intrinsic spin, the derivative operator \( i (\partial/\partial \mathbf{k}) \) is transformed in the following way:
\[ \left[ N_t, i \frac{\partial}{\partial k_j} \right] = i \kappa_{ijk} i \frac{\partial}{\partial k_i}, \quad \left[ J_t, i \frac{\partial}{\partial k_j} \right] = i \epsilon_{ijk} i \frac{\partial}{\partial k_3}, \quad (3.1) \]

where
\[ \kappa_{ijk} = \frac{\delta_{ij} G_k - \delta_{ik} G_j}{G_0 + 1}. \quad (3.2) \]

Besides in this case, \( L \) and \( k \) satisfy the same type of the commutation relations:
\[ [N_t, L_j] = i \kappa_{ij} L_k, \quad [J_t, L_j] = i \epsilon_{ij} L_k, \quad (3.3) \]
\[ [N_t, k_j] = i \kappa_{ij} k_k, \quad [J_t, k_j] = i \epsilon_{ij} k_k. \quad (3.4) \]

It is noticeable that the “structure constant” \( \kappa_{ijk} \) depends only on the motions of the whole system and independent of the inner motions which are described by the variable \( k \).

Now we define a new technical term “covariant vector \( V \) connected with the inner motion” (abbreviated to C.V.I.) which means that they obey the next commutation relations:
\[ (1) \quad [J_t, V_j] = i \kappa_{ij} V_k, \quad (3.5) \]
\[ (2) \quad [N_t, V_j] = i \kappa_{ij} V_k \quad (3.6) \]

and
\[ (3) \quad [G_t, V_j] = 0. \quad (3.7) \]

When each particle or one of the two particles has its intrinsic spin, \( i (\partial / \partial k) \) and \( L \) obviously do not belong to the C.V.I. By adding suitable spin dependent terms to these operators, we get the following set of C.I.V.’s:
\[ S = L + \sum_{a=1}^{2} \left\{ \frac{q_a + G_0 s_a}{A_a} \frac{1}{(k \cdot s_a)} - \frac{(-1)^{a-1}}{A_a} \frac{1}{A_a(G_0 + 1)} \right\}, \quad (3.8) \]
\[ \xi = i \frac{\partial}{\partial k} + \sum_{a=1}^{2} \left\{ \frac{1}{A_a a_a} (k \times s_a) - \frac{(-1)^{a-1} m_a}{A_a} (G \cdot s_a) + \frac{1}{A_a(G_0 + 1)} (k \times G) (G \cdot s_a) \right\} \]
\[ + \sum_{a=1}^{2} \left\{ -k \times s_a + \frac{(-1)^{a-1}}{A_a} \frac{1}{A_a(G_0 + 1)} (k \times G)(k \cdot s_a) + \frac{q_a - m_a}{A_a(G_0 + 1)} (k \times G)(G \cdot s_a) \right\} \chi_a(k^2), \quad (3.9) \]

where
\[ A_a = G_0 a_a + (-1)^{a-1} (G \cdot k) + m_a \quad (3.10) \]

and \( \chi_a(k^2) \) is an arbitrary function of \( k^2 \). It is easily seen that \( k \) again belongs to C.V.I.

In addition to Eqs. \( (3.5) \sim (3.7) \), \( S, \xi \) and \( k \) satisfy the next commutation relations one another:
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\[ [S_i, S_j] = i \epsilon_{ijk} S_k, \]
\[ [S_i, \xi_j] = i \epsilon_{ijk} \xi_k, \]
\[ [S_i, k_j] = i \epsilon_{ijk} k_k, \]
\[ [\xi_i, k_j] = i \delta_{ij}, \]

and

\[ [k_i, k_j] = 0. \]

Commutation relations among the components of \( \xi \) are complicated but this has a simple form when two particles have the same mass and the arbitrary function \( \chi_x \) is chosen to be 0, namely,

\[ [\xi_i, \xi_j] = i \epsilon_{ijk} (-S_k - (k \times \xi)_k) / q^2, \]

where

\[ q^2 = m_i^2 + k^2 = m_j^2 + k^2. \]

The operator \( S \) is interpreted to be the spin operator of the relativistic two-particle system because \( S \) can be expressed in another form:

\[ S = J - X \times P, \]

where \( P \) and \( X \) are the total momentum and the Newton-Wigner position operator\(^{21,0}\) for the relativistic two-particle system respectively. \( \xi \) can be named to be a covariant relative position operator in the center-of-mass frame.

From the commutation relations (3.5), (3.6) and (3.7), it is easily proved that the scalar products of any pair of these C.V.I.'s commute with \( J, N \) and \( G \), but the scalar triple product of \( \xi, S \) and \( k \) does not commute with \( N \). (Conclusion A)

The wave function depends on the \( k, G \) and on the spin variables of each particle. The mass operator acts on the \( k \) and the spin variables but not on \( G \) because of the commutation relation (2.10). There are two kinds of operations which work on the variable \( k \); the operations to change the directions of the vector \( k \) but not change the length of \( k \), which are performed by \( L \); to change the length of the vector \( k \), which are done by \( i \partial / \partial k \). Therefore \( L \) and \( i \partial / \partial k \) exhaust all the operations on the variable \( k \). Then we conclude that \( S, \xi \) and \( k \) cover all the operations on the wave function which can be contained in the mass operator. (Conclusion B)

The conclusions A and B are combined to prove the third conclusion C in which we say that the mass operator must be the function which depends only on the scalar products of the pair of the C.V.I.'s, namely,

\[ M = M (\xi^2, (\xi \cdot S), (\xi \cdot k), S^2, k^2, (S \cdot k)). \]

Equation (3.13) gives the most general form of the mass operator for the relati-
viciss two-particle system with intrinsic spin, so far as the particle-creations and
destructions are neglected.

§ 4. Spin-orbit and spin-spin coupling forces
in a Hooke-type potential

If the interaction between the two particles occurs through a Hooke-type potential, the simplest form of the mass operator is

$$\mathcal{M} = (q_1 + q_2)^2 + \frac{g}{2} (\xi \cdot \xi), \quad (4.1)$$

where $g$ is the coupling constant.

The mass operator is invariant under the space-time rotations, therefore we can solve the eigenvalue problem

$$\mathcal{M} \Psi = m^2 \Psi \quad (4.2)$$

in the center-of-mass frame ($G = 0$) without loss of generality. In this frame

$$\xi = i \frac{\partial}{\partial k} + \sum_a \left[ \frac{1}{q_a(q_a + m_a)} - \chi_a(k^2) \right] (k \times s_a) \quad (4.3)$$

and

$$S = L + s_1 + s_2. \quad (4.4)$$

The mass operator $\mathcal{M}$ becomes rather simple:

$$\mathcal{M} = (q_1 + q_2)^2 + \frac{g}{2} \left\{ \frac{\partial^2}{\partial k^2} - \frac{2}{k} \frac{\partial}{\partial k} + 2 \left[ \sum_a F_a s_a \cdot L \right] + \left[ k \times \sum_a F_a s_a \right] + \frac{L^2}{k^2} \right\}, \quad (4.5)$$

where

$$F_a = \frac{1}{q_a(q_a + m_a)} - \chi_a(k^2) \quad \text{and} \quad k = |k|. \quad (4.5)$$

The spin-spin coupling and spin-orbit coupling terms are intimately related to the harmonic oscillator potential. When two particles have the same mass and the functions $\chi_s(k^2)$ and $\chi_s(k^2)$ are chosen to be equal, Eq. (4.5) is more simply written as

$$\mathcal{M}^2 = 4k^2 + 4m^2 + \frac{g}{2} \left\{ \frac{\partial^2}{\partial k^2} - \frac{2}{k} \frac{\partial}{\partial k} + \frac{L^2}{k^2} \\
+ 2 \left[ \frac{1}{\sqrt{k^2 + m^2} (\sqrt{k^2 + m^2} + m)} - \frac{\chi(k^2)}{\sqrt{k^2 + m^2} (\sqrt{k^2 + m^2} + m)} \right] \mathbf{S} \cdot \mathbf{L} \\
+ (k^2 \mathbf{S}^2 - (k \cdot \mathbf{S})^2) \left\{ \frac{1}{\sqrt{k^2 + m^2} (\sqrt{k^2 + m^2} + m)} - \frac{\chi(k^2)}{\sqrt{k^2 + m^2} (\sqrt{k^2 + m^2} + m)} \right\} \right\}, \quad (4.6)$$
where \( m = m_1 = m_2 \) and \( s = s_1 + s_2 \). The complete set of observables of this system which includes the mass operator itself is \( \mathcal{H}, S^\alpha, \sigma^\alpha, \sigma^\beta, S, \) and \( \mathcal{P} \), where \( \mathcal{P} \) is the parity operator.

\section{Conclusions and discussion}

1) The main conclusion of this article is that in the relativistic two-particle problems, the potential \( U(r) \) must be attended by certain type of spin dependent terms which are given by the rule,

\[ U(r) \rightarrow U(\sqrt{\xi^2}) \quad (5.1) \]

in order that theory satisfies the relativistic invariance.

2) The second conclusion is that the most general form of the mass operator of the relativistic two-particle problem is given by Eq. (3.13) so far as the particle creations are neglected.

3) We have used the irreducible representations of the Poincaré group to describe the particles with spin in which we cannot discriminate between the particle and the anti-particle. Further it is not clear how to add the minimal interactions to the mass operator because we have used the irreducible representations of the Poincaré group sacrificing the manifest covariance. These defects will be improved elsewhere.

4) The mass operator in Eq. (4.6), when it is applied to the \( q\bar{q} \) system, will be able to reproduce the existing data of the meson mass spectra by choosing suitably the arbitrary function \( \chi(q^2) \) in Eq. (3.9). A full discussion about this will be given by our next issue.

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