Relativity of Inertial Forces

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Within the framework of the general theory of relativity, the realizability of Mach's ideal on relativity of inertial forces concerning rotation is examined in a modified Thirring shell model. In the model are taken into account the Lorentz contraction and the associated density-modulation of the shell. The conservation law for energy-momentum tensor is properly treated. Calculations are carried out up to second order of the weak field approximation and up to second order of the angular velocity of the shell. Affirmative results are obtained for Mach's ideal.

§ 1. Introduction

As is well known, Einstein\(^7\) got inspiration in developing the general theory of relativity from Mach's critical remark\(^9\) against the concept of absolute space which had been conceived by Newton in connection with his rotating bucket experiment.\(^8\) According to Mach, since every motion is relative, the rotating bucket experiment by Newton shows only the fact that, while the rotation of the water relative to the wall of the bucket gives rise to no appreciable centrifugal force, the rotation of the water relative to the enormous amount of masses such as the earth and the rest of celestial bodies gives rise to the noticeable centrifugal force, and so nobody can say anything whatever qualitative or quantitative about what would happen in the rotating bucket experiment with more and more massive and thicker wall.

Thirring\(^6\) is the first that showed on the basis of the general theory of relativity that a rotating hollow sphere generates inside it the gravitational forces which resemble the inertial forces in a rotating frame of reference; he examined, by making use of Einstein's weak field approximation method,\(^9\) the gravitational effect of a uniformly rotating, infinitely thin and uniformly mass-distributed large spherical shell on a slowly moving test particle in the vicinity of the center of the shell, and obtained as the equation of motion of the test particle the following one:

\[
(\ddot{x}, \ddot{y}, \ddot{z}) = 2\omega (\varepsilon/3) (\dot{y}, -\dot{x}, 0) + \omega^2 (\varepsilon/15) (x, y, -2z) \quad (T)
\]

with the dimensionless constant \(\varepsilon\)

\[
\varepsilon = (M\kappa\epsilon^b) (2\pi\alpha)^{-1}, \quad (1.1)
\]

which is determined from the radius \(a\), rest mass \(M\) and angular velocity \(-\omega\).
of the shell, the light velocity $c$ and Einstein’s gravitational constant $\kappa$.

In comparison with the familiar expression

$$2\omega(\dot{y}, -\dot{x}, 0) + \omega^2(x, y, 0)$$

(1·2)

for Coriolis’ and the centrifugal accelerations in a rotating frame of reference with angular velocity $\omega$ with respect to the so-called fixed-stars system, however, Thirring’s result is unsatisfactory in two respects from Mach’s viewpoint. Firstly, while the first term is just of Coriolis type, the second one is not a genuine centrifugal acceleration because of the appearance of the nonvanishing component along the rotation axis ($z$-axis). Secondly, the ratio of coefficients of these two terms does not coincide with that of the familiar expression.

Recently Okamura, Ohta, Kimura and Hiida have investigated the same problem again in the Thirring model, but now up to the post-post-Newtonian order of approximation with their own method. Their result reads as

$$(\dot{x}, \dot{y}, \dot{z}) = 2\omega (\varepsilon/3) (\dot{y}, -\dot{x}, 0) + \omega^2 (\varepsilon/3)^2 (x, y, 0)$$

$$+ \omega^2 (\varepsilon/20) \{1 - (3737/5040)\varepsilon\} (x, y, -2z). \quad \text{(OOKH)}$$

Again unfortunately there exists no possibility of making their result concordant with the familiar expression by whatever numerical choice of $\varepsilon$.

Even aside from the unsatisfactory features in their results, both the works have, in common, two shortcomings on the side of principle. The one is that the conservation laws are not fulfilled because of the assumed incoherence of the material of the shell. The other is that the material of the shell can get super-light velocities when the product of radius and angular velocity of the shell is larger than $c$ because of the assumed neglect of the Lorentz contraction due to rotation.

Already in 1955 Bass and Pirani remedied the former shortcoming of Thirring’s assumptions, of which they had been communicated from Lanczos, by introducing an additional term representing elastic stresses into the energy-momentum tensor. Moreover they assumed that the density of the material of the shell varies with the latitude according to

$$\rho = \rho_0 (1 + N \omega^2 a^2 \sin^2 \theta),$$

where $\rho_0$ and $N$ are constants, and showed that the choice $N = -1$ can annihilate both the radial and the axial ‘centrifugal’ forces ($\sim \varepsilon$), leaving the ‘Coriolis’ force intact, although their calculations are, as well, confined up to first order of the weak field approximation.

On the other hand, Pietronero showed in his calculation up to second order of the weak field approximation that an infinitely long and infinitely thin cylindrical distribution of matter in rotation gives rise to the ‘Coriolis’ ($\sim \varepsilon$) and the ‘centrifugal’ forces ($\sim \varepsilon^5$) with their satisfactory ratio and does not give rise to any
pseudo-centrifugal forces. In his calculation, however, the conservation laws are not fulfilled because of the assumed incoherence of the material, the situation of which manifests itself in the fact that his solutions for $\xi_{\mu\nu}$ are not divergenceless in contradistinction with his assumption as well as in Thirring's work. Superlight velocities can appear there also. Besides his model is rather unrealistic.

For getting rid of the latter shortcoming in the original Thirring model, the author took into account the Lorentz contraction of the shell due to rotation in a recent short note.\textsuperscript{9} Therein was introduced also a function for describing the density-modulation of the shell associated with the contraction. The report contains, however, the former shortcoming, namely, the problem of the conservation laws is overlooked.\textsuperscript{9} Besides the calculations are confined to first order of the weak field approximation.

The purpose of this article is now to proceed the calculations along the line of Ref. 9) up to second order of $\kappa$ with the proper treatment of the conservation laws and thereby to examine with what choice of the density-modulation function Mach's ideal is realizable.

\section{2. Basic formulae up to second order of $\kappa$}

The fundamental equations assumed in this article are Einstein's gravitational field equation without the cosmological term

$$R_{\mu\nu} - (1/2) R g_{\mu\nu} = -\kappa T_{\mu\nu}$$

(2.1)

and the geodesic line equation which describes the motion of a test particle in the gravitational field. Moreover de Donder's coordinate condition is used,

$$\left(\sqrt{-g} g^{\sigma\nu}\right)_{,\sigma} = 0.$$  \hfill (2.2)

Prior to the specification in later sections for the energy-momentum tensor $T^{\mu\nu}$ which should satisfy the conservation law

$$T^{\mu\nu}_{\ ;\nu} = 0,$$  \hfill (2.3)

let us put formally $T^{\mu\nu}$ and $g_{\mu\nu}$ into power series with respect to the gravitational constant $\kappa$,

$$T^{\mu\nu} = T^{(0)} + T^{(1)} + O(\kappa^2),$$  \hfill (2.4)

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + O(\kappa^2),$$  \hfill (2.5)

where $\eta_{\mu\nu}$ is Minkowski's metric with the signature $(-1, 1, 1, 1)$. After the corre-

\textsuperscript{9} At the Meeting on the general theory of relativity held at the Research Institute for Fundamental Physics, Nov. 10-12, 1976, Professor R. Utiyama of Osaka University kindly pointed it out and informed me of Bass-Pirani's work.\textsuperscript{10} I thank him for his remarks.
responding power series expansions of various quantities, one can safely use \( \eta^{\mu\nu} \) (\( = \eta_{\mu\nu} \)) and \( \eta_{\mu\nu} \) for raising and lowering suffices.

In terms of those quantities which are defined by

\[
\begin{align*}
(1) \quad Z_{\mu\nu} &= h_{\mu\nu} - (1/2) \ h \ \eta_{\mu\nu}, \\
(2) \quad Z_{\mu\nu} &= h_{\mu\nu} - (1/2) \ h \ \eta_{\mu\nu} - \eta_{\rho\sigma} \ Z^\rho_{\mu} Z^\sigma_{\nu} + (1/2) \ h \ \eta_{\mu\nu} \\
& \quad - (1/8) \ (Z)^{\alpha\beta}_{\eta_{\mu\nu}} + (1/4) \ Z^{\alpha\beta}_{\eta_{\mu\nu}} \eta_{\alpha\beta}, \\
\end{align*}
\]

(2.6)

(2.7)

with the abbreviation of suffices like

\[
(1) \quad h = h^{\alpha\beta} \eta_{\alpha\beta},
\]

(2.8)

Eq. (2.2) can be written, up to second order of \( \kappa \), as follows:

\[
\begin{align*}
(1) \quad Z^{\mu}_{\mu\nu} &= 0, \\
(2) \quad Z^{\mu}_{\nu\nu} &= 0.
\end{align*}
\]

(2.9)

(2.10)

Hence Eq. (2.1) reduces to

\[
\begin{align*}
(1) \quad \Box Z_{\mu\nu} &= -2(0) \ \kappa \ T_{\mu\nu}, \\
(2) \quad \Box Z_{\mu\nu} &= 2 \kappa Z^{(0) \mu}_{\nu \tau} T_{\tau \rho} + S_{\mu\nu},
\end{align*}
\]

(2.11)

(2.12)

where \( S_{\mu\nu} \) is given, in terms of \( Z_{\mu\nu} \), by

\[
\begin{align*}
S_{\mu\nu} &= - \eta^{\rho\varphi} Z_{\rho\varphi,\nu} Z_{\mu,\varphi} - (1/2) \ (Z)^{\alpha\beta}_{\eta_{\mu\nu}} \eta_{\rho\varphi,\nu} + (1/4) \ Z_{\mu\nu} \\
& \quad + Z_{\rho\varphi,\alpha} Z^{\rho\varphi}_{\alpha\beta} Z_{\mu,\varphi,\beta} + \eta_{\rho\varphi,\alpha} \ Z_{\mu,\alpha,\beta} Z^{\rho\varphi}_{\varphi,\beta} - \eta_{\rho\varphi,\alpha} \ Z_{\mu,\alpha,\beta} Z^{\rho\varphi}_{\varphi,\beta} + Z^{\rho\varphi}_{\rho\varphi,\beta} \eta_{\mu,\varphi,\beta} \\
& \quad - \eta_{\mu\nu} \ [ (1/8) \ \eta^{\rho\varphi} Z_{\rho\varphi,\nu} Z_{\mu,\varphi} + (1/2) \ (Z)^{\alpha\beta}_{\eta_{\mu\nu}} \eta_{\rho\varphi,\nu} - (1/4) \ \eta^{\rho\varphi}_{\eta_{\mu\nu}} \eta_{\alpha\beta,\mu} \eta_{\alpha\beta,\nu} ].
\end{align*}
\]

(2.13)

Equation (2.3) reduces to

\[
\begin{align*}
(0) \quad T^{\mu}_{\nu\varphi} &= 0, \\
(1) \quad T_{\mu,\nu} &= - \eta^{\rho\varphi}_{\rho\varphi,\nu} T^{\varphi}_{\mu} + (1/2) \ (Z)^{\alpha\beta}_{\eta_{\mu\nu}} T^{\alpha\beta}_{\mu} + Z_{\rho\varphi,\beta} T^{\rho\varphi}_{\mu} - (1/4) \ Z_{\mu,\nu} T^{(0) \mu}_{\nu}.
\end{align*}
\]

(2.14)

(2.15)

From the geodesic line equation it follows that, when the gravitational field is static, a slowly moving test particle with the three-velocity \( v^\rho \) has the acceleration

\[
A^\rho = A^\rho + A_1 + O(\kappa^2, v^2)
\]

(2.16)

in the approximation up to second order of \( \kappa \) and up to first order of \( v^\rho \), where

\[
(1) \quad A^\rho = (c^2/2) h_{0\alpha,\mu} - c \ (h_{0\alpha,\mu} - h_{0\mu,\alpha}) v^\mu,
\]

(2.17)
\[ A_t = \left( \frac{c^2}{2} \right) \left( \frac{c}{2} \right) \left( \frac{h_{0,t}}{h_{0,t}} - h_{0,t} \right) - \left( \frac{c}{2} \right) \left( \frac{h''_0}{h_0} \right) \left( \frac{c}{2} \right) \left( \frac{h_{0,t}}{h_{0,t}} - h_{0,t} \right) \]

\[ -c \left( \frac{c}{2} \right) \left( \frac{c}{2} \right) \left( \frac{h_{0,t}}{h_{0,t}} + h_{0,t} \right) \left( \frac{h''_0}{h_0} \right) \left( \frac{c}{2} \right) \left( \frac{h_{0,t}}{h_{0,t}} + h_{0,t} \right) \left( \frac{c}{2} \right) \left( \frac{h_{0,t}}{h_{0,t}} + h_{0,t} \right) v^t. \] (2.18)

**§ 3. Specification of \( T^\rho_\sigma \)**

Suppose that an infinitely thin material shell is rotating with constant angular velocity \( \omega \) around the \( z \)-axis in a frame of reference \( K \). We assume that in the special case \( \omega = 0 \) the shell, in the \( K \), has the shape of a spherical shell with radius \( a \), the whole mass \( M \) and the uniform mass density \( M/(4\pi a^2) \). In the general case \( \omega \neq 0 \), however, the shape of the shell in the \( K \) is, because of the Lorentz contraction, no longer spherical but of such a shape that the coordinates \((x, y, z)\) of any point on the shell obey the relation (see Appendix C)

\[ x^2 + y^2 = (a^2 - z^2) \left( 1 + \left( a^2 - z^2 \right) \omega^2 \right)^{-1}. \quad (z^2 \leq a^2) \] (3.1)

In fact the Lorentz contraction guarantees that the velocity

\[ u' = (\omega y, -\omega x, 0) \] (3.2)

of any point on the shell never exceeds the light velocity \( c \).

Introduction of angles \( \theta \) and \( \phi \) through

\[ x = R \cos \phi, \quad y = R \sin \phi, \quad z = a \cos \theta \] (3.3)

leads to the following expression for the radius \( R \) of a cross section of the shell perpendicular to the \( z \)-axis:

\[ R = a \left( 1 + \alpha \sin^2 \theta \right)^{-1/2} \sin \theta, \] (3.4)

where \( \alpha \) is the dimensionless constant

\[ \alpha = \left( \omega / c \right)^2. \] (3.5)

Hence the area of an infinitesimal surface element of the shell in the \( K \) is

\[ dS = R d\phi \cdot \left( a^2 \sin^2 \theta + (\partial R / \partial \theta)^2 \right)^{1/2} d\theta. \] (3.6)

Moreover, since the shell is not spherical in the \( K \), the proper mass density per unit area of the shell in the \( K \), which we denote by \( \sigma(\theta, \alpha) \), is no longer uniform though it is still axially symmetric. The whole rest mass of the shell, however, should be equal to \( M \) independently from \( \alpha \) or \( \omega \),

\[ M = \int \sigma dS = \int \sigma_0 dS_0. \] (3.7)

Here \( \sigma_0 \) and \( dS_0 \) are respectively the mass density and the surface element of the shell in the special case \( \omega = 0 \):
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\[ \sigma_s = M/(4\pi a^2), \quad dS_s = a^3 \sin \theta d\theta d\phi. \]  \hspace{1cm} (3.8)

In comparison with

\[ \sigma_\theta dS_\theta = (M/4\pi) \sin \theta d\theta d\phi, \]

let us assume

\[ \sigma dS = (M/4\pi) \{1 + \alpha f(\cos^2 \theta, \alpha)\} \sin \theta d\theta d\phi, \]  \hspace{1cm} (3.9)

where the yet unspecified function \( f(\cos^2 \theta, \alpha) \) represents the degree of modulation of mass distribution of the shell arising from the Lorentz contraction due to rotation. Equation (3.7) imposes on the function \( f \) the \( \alpha \)-independent condition

\[ \int_{\beta=1}^{1} f(\beta^2, \alpha) d\beta = 0. \]  \hspace{1cm} (3.10)

The kinetic energy-momentum tensor of the shell is given by

\[ \rho(\theta, \alpha) U^\nu U^\nu, \]  \hspace{1cm} (3.11)

where

\[ \rho(\theta, \alpha) = \{(1 - (u/c)^2)^{1/2} \sigma(\theta, \alpha) \} \]  \hspace{1cm} (3.12)

and

\[ U^\nu = \{(1 - (u/c)^2)^{1/2}(c, u, u, u^\nu)\}. \]  \hspace{1cm} (3.13)

This tensor, however, does not fulfill by itself the conservation law. Hence, according to Bass-Pirani, let us introduce an additional term \( E^{(0)}_{\mu
u} \) representing elastic stresses and put

\[ T^{(0)}_{\mu
u} = \rho U^\nu U^\mu + E^{(0)}_{\mu
u}. \]  \hspace{1cm} (3.14)

Bass-Pirani’s four conditions for \( E^{(0)}_{\mu
u} \) are

(i) it is symmetric: \( E^{(0)}_{\mu\nu} = E^{(0)}_{\nu\mu} \), \hspace{1cm} (3.15)

(ii) it has no component normal to the shell:

\[ E^{(0)}_{\mu\nu} n_\nu = 0, \]  \hspace{1cm} (3.16)

where \( n_\nu \) is a 4-normal to the world tube of the shell,

(iii) it has no component in the direction of motion of the shell:

\[ E^{(0)}_{\mu\nu} U_\nu = 0, \]  \hspace{1cm} (3.17)

(iv) its physical components, in a local reference frame in which the relevant point of the shell is instantaneously at rest, are bounded:
where $e^{(a)}_\mu$ denote $c^{-1}U_\mu$ and any three unit vectors orthogonal to $U_\mu$ and to each other.

Under these conditions, Eq. (2.14) can be uniquely solved. The details of the calculation are given in Appendix A. The result in the approximation up to first order of $\alpha$ is as follows:

$$
T^{(0)}_\mu = \frac{\varepsilon}{2\kappa a} \left[ 1 + \alpha \left\{ \sin^2 \theta + \frac{3}{2} \sin^2 \phi \cos^2 \theta + f(\cos^2 \theta, 0) \right\} \right],
$$

$$
T^{(0)}_\theta = \frac{\varepsilon}{2\kappa a} \alpha \frac{1}{r^2} \sin \theta \sin \phi,
$$

$$
T^{(0)}_\phi = -\frac{\varepsilon}{2\kappa a} \alpha \frac{1}{r^2} \sin \theta \cos \phi,
$$

others zero,

where $\varepsilon$ is the same dimensionless constant as in Eq. (1.1).

\section{First order effect}

Since the material distribution in the model is stationary, Eq. (2.11) reduces to

$$
\Delta \zeta^{(1)}_{\mu\nu} = -\frac{\varepsilon}{2\kappa a} T^{(0)}_{\mu\nu},
$$

and its solution which vanishes at spatial infinity is given by

$$
\zeta^{(1)}_{\mu\nu}(x, y, z) = \frac{\kappa}{2\pi} \int D^{-1} T^{(0)}_{\mu\nu} dS,
$$

where

$$
D = [ (x - R \cos \phi)^2 + (y - R \sin \phi)^2 + (z - a \cos \Theta)^2 ]^{1/2}.
$$

Let us introduce, for any off-shell point, the coordinate $(s, \Theta, \Phi)$ by

$$
x = s(1 + \alpha \sin^2 \Theta)^{-1/2} \sin \Theta \cos \Phi,
$$

$$
y = s(1 + \alpha \sin^2 \Theta)^{-1/2} \sin \Theta \sin \Phi,
$$

$$
z = s \cos \Theta.
$$

According to whether $s$ is smaller or larger than $a$, the point lies inside or outside the shell. After lengthy calculations we get, in the approximation up to first order of $\alpha$, the inner solution

$$
\zeta^{(1)}_{\mu\nu}^{(inner)} = \varepsilon \left[ 1 + \alpha \left\{ \frac{3}{5} + \frac{f^{(3)}}{5} - \frac{19}{105} \right\} a^{-2} s^2 P_2(\cos \Theta)
$$

$$
+ \sum_{n=2}^{m} \frac{f^{(2n)}}{4n+1} a^{-2n} s^n P_{2n}(\cos \Theta) \right]\right],
$$
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\( Z_{tt}(\text{inner}) = -\frac{1}{3} \varepsilon \alpha \sigma^2 a^{-1}s \sin \Theta \sin \Phi, \) \hspace{1cm} (4.5)

\( Z_{tt}(\text{outer}) = \frac{1}{3} \varepsilon \alpha \sigma^2 a^{-1}s \sin \Theta \cos \Phi, \)

and the outer solution

\( Z_{tt}(\text{outer}) = \varepsilon a^{-1} \left[ 1 + \alpha \left( \frac{3}{5} + \frac{1}{5} a^2 s^{-2} + \frac{f^{(2)}}{5} a^2 s^{-2} - \frac{8}{21} P_2(\cos \Theta) \right) + \frac{4}{35} \left( 1 - a^2 s^{-2} \right) P_2(\cos \Theta) + \sum_{n=3}^{\infty} \frac{f^{(2n)}}{4n+1} a^{2n-4n} P_{2n}(\cos \Theta) \right], \)

\( Z_{tt}(\text{outer}) = -\frac{1}{3} \varepsilon \alpha \sigma^2 a^2 s^{-2} \sin \Theta \sin \Phi, \) \hspace{1cm} (4.6)

\( Z_{tt}(\text{outer}) = \frac{1}{3} \varepsilon \alpha \sigma^2 a^2 s^{-2} \sin \Theta \cos \Phi, \)

where the \( f^{(2n)} \)'s are the expansion coefficients of \( f(\cos^2 \Theta, 0) \) into Legendre series of even order,

\[ f(\cos^2 \Theta, 0) = \sum_{n=0}^{\infty} f^{(2n)} P_{2n}(\cos \Theta), \]

\hspace{1cm} (4.7)

\[ f^{(2n)} = \frac{4n+1}{2} \int_{-1}^{1} f(\beta^2, 0) P_{2n}(\beta) d\beta. \] \hspace{1cm} (4.8)

Equation (3.10) implies that

\[ f^{(0)} = 0. \] \hspace{1cm} (4.9)

Moreover, because of the last line of Eq. (3.19) we have

other components of \( Z_{\mu \nu} = 0. \) \hspace{1cm} (4.10)

The \( Z_{\mu \nu} \text{ (inner)} \) and \( Z_{\mu \nu} \text{ (outer)} \) coincide with each other in the limit \( s \to a, \) and hence

\[ Z_{\mu \nu} \text{ (on shell)} = \lim_{s \to a} Z_{\mu \nu} \text{ (inner)} = \lim_{s \to a} Z_{\mu \nu} \text{ (outer)}. \] \hspace{1cm} (4.11)

The \( Z_{\mu \nu, \rho} \text{ (on shell)} \), however, should be calculated directly from

\[ Z_{\mu \nu, \rho} \text{ (on shell)} = \frac{\kappa}{2\pi} \int (D^{-1})_{\rho \sigma} (\text{on shell}) T_{\rho \sigma} dS. \] \hspace{1cm} (4.12)

The calculations show that it coincides with the simple average of the limit values of \( Z_{\mu \nu, \rho} \text{ (inner)} \) and \( Z_{\mu \nu, \rho} \text{ (outer)}, \)
\[ Z_{\mu, \rho}^{(1)} \text{(on shell)} = \frac{1}{2} \lim_{\varepsilon \to 0} \left[ \lim_{\varepsilon \to 0} Z_{\mu, \rho}^{(1)} \text{(inner)} + \lim_{\varepsilon \to 0} Z_{\mu, \rho}^{(1)} \text{(outer)} \right]. \] (4.13)

Near the center of the shell, where quantities of order higher than the second on \( x, y \) and \( z \) may be neglected, the expressions of Eq. (4.15) reduce to
\[ Z_{\phi}^{(1)} \text{(near the center)} = \frac{\varepsilon}{2} \alpha \left\{ \frac{3}{5} \frac{-1}{5} \left( f^{(c)} - \frac{19}{21} \right) \frac{x^2 + y^2 + z^2}{a^2} \right\}, \]
\[ Z_{a}^{(1)} \text{(near the center)} = -\frac{1}{3} \varepsilon \alpha^{1/2} y a^{-1}, \]
\[ Z_{\phi}^{(1)} \text{(near the center)} = \frac{1}{3} \varepsilon \alpha^{1/2} x a^{-1}. \] (4.14)

Hence, from Eqs. (2.17), (4.10) and (4.14), it follows that
\[ A^{(1)} = \frac{2}{3} \varepsilon \omega (v_y, -v_x, 0) - \frac{1}{20} \varepsilon \omega^2 \left( f^{(c)} - \frac{19}{21} \right) (x, y, -2z). \] (4.15)

§ 5. Determination of \( T^{\mu \nu} \)

Inserting the obtained values of the \( T^{\mu \nu} \) and the \( Z_{\mu, \rho}^{(1)} \text{(on shell)} \) into Eq. (2.15), we get the equation
\[ T^{\mu \nu} = W^\mu, \] (5.1)

where
\[ W^\mu = 0, \]
\[ W^1 = -\frac{\varepsilon^2}{16 \varepsilon a^2} \sin \theta \cos \phi (1 + \alpha B), \]
\[ W^2 = -\frac{\varepsilon^2}{16 \varepsilon a^2} \sin \sin \phi (1 + \alpha B), \] (5.2)
\[ W^3 = -\frac{\varepsilon^2}{16 \varepsilon a^2} \cos \theta (1 + \alpha C), \]

and
\[ B = \frac{89}{21} + \frac{37}{21} P_1(\cos \theta) - \frac{36}{35} P_4(\cos \theta) + \sum_{n=1}^{\infty} \frac{2}{n+1} f^{(2n)} P_{2n+1}(\cos \theta), \]
\[ C = -\frac{106}{105} + \frac{58}{21} P_1(\cos \theta) - \frac{36}{35} P_4(\cos \theta) + \sum_{n=1}^{\infty} \frac{2(n+1)}{4n+1} f^{(2n)} \sec \theta P_{2n+1}(\cos \theta). \] (5.3)

Now let us impose Bass-Pirani's four conditions on the \( T^{\mu \nu} \) also,
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\( T^\nu_{\,\mu} = T^\nu_{\,\mu} \), \hspace{1cm} (5.4)

\( T^\nu_{\,\mu} n_\nu = 0 \), \hspace{1cm} (5.5)

\( T^\nu_{\,\mu} U_\nu = 0 \), \hspace{1cm} (5.6)

\( T^\nu_{\,\mu} d_\nu^{(\mu)} < A \). \hspace{1cm} (5.7)

These conditions together with Eq. (5.1) determine the \( T^\nu_{\,\mu} \) uniquely. The details of calculations are given in Appendix B. The result in the approximation up to first order of \( \alpha \) is as follows:

\[ T^0_0 = \frac{\dot{\varepsilon}^2}{32\kappa a} \cos^2 \theta \sin^2 \phi \left( 1 + \alpha (2A - 3 \sin^2 \theta) \right) \]

\[ T^0_1 = \frac{\dot{\varepsilon}^2}{32\kappa a} \sin \theta \cos \phi \left( 1 + \alpha (2B - \sin^2 \theta) \right), \]

\[ T^1_1 = \frac{\dot{\varepsilon}^2}{32\kappa a} \left[ \cos^2 \theta \cos^2 \phi \left( 1 + \alpha (2A - 3 \sin^2 \theta) \right) + \sin^2 \phi \left( 1 + \alpha (2B - \sin^2 \theta) \right) \right], \]

\[ T^1_2 = \frac{\dot{\varepsilon}^2}{32\kappa a} \sin \theta \cos \phi \left[ \cos^2 \theta \left( 1 + \alpha \left( 2A - \frac{3}{2} \sin^2 \theta \right) \right) - \left( 1 + \alpha (2B - \sin^2 \theta) \right) \right], \]

\[ T^1_3 = -\frac{\dot{\varepsilon}^2}{32\kappa a} \sin \theta \cos \phi \left( 1 + \alpha \left( 2A - 3 \sin^2 \theta \right) \right), \]

\[ T^2_1 = \frac{\dot{\varepsilon}^2}{32\kappa a} \left[ \cos^2 \theta \sin^2 \phi \left( 1 + \alpha \left( 2A - \frac{3}{2} \sin^2 \theta \right) \right) + \cos^2 \phi \left( 1 + \alpha (2B - \sin^2 \theta) \right) \right], \]

\[ T^2_2 = -\frac{\dot{\varepsilon}^2}{32\kappa a} \sin \theta \cos \phi \left( 1 + \alpha \left( 2A - \frac{3}{2} \sin^2 \theta \right) \right), \]

\[ T^2_3 = \frac{\dot{\varepsilon}^2}{32\kappa a} \sin \theta \cos \theta \left( 1 + 2\alpha \right), \]

where \( A \) and \( B \) are those Legendre series of even order which are given in Eqs. (B.16) and (B.20).

\section*{§ 6. Second order effect}

In the stationary case, Eq. (2.12) reduces to

\[ \Delta x_{\mu} = 2\kappa^2 \left( T^\mu_{\,\mu} - 2\kappa T_{\mu\nu} + S^\mu_{\,\mu} \right), \]

(6.1)
and its solution which vanishes at spatial infinity is given by the sum of a surface integral and a volume integral

\[
I_{\nu}\nu = -\frac{e}{2\pi} \int D^{-1}(y T_{\nu\nu} - T_{\nu\nu}) dS - \frac{1}{4\pi} \int \tau^{-1} S_{\nu\nu}(x', y', z') dV'.
\]

(6.2)

with the \( D \) given in Eq. (4.3) and with

\[
\tau = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}.
\]

(6.3)

The volume element \( dV \) in terms of the coordinates \( s, \Theta, \Phi \) defined by Eq. (4.4) is, up to first order of \( \alpha \),

\[
dV = (1 + \alpha (\sin^2 \Theta - 2 \sin \Theta)) \cdot s \cdot \sin \Theta \cdot ds d\Theta d\Phi.
\]

(6.4)

The integrations can be carried out by expanding \( D^{-1} \) and \( \tau^{-1} \) into power series with respect to \( x, y \) and \( z \), and by using the \( I_{\nu\nu} \) (on shell), \( I_{\nu\nu}, I_{\nu\nu}, I_{\nu\nu,\nu} \) (inner) and \( I_{\nu\nu} \) (outer) which have been obtained in the preceding sections. Since our concern lies in the values of \( I_{\nu\nu} \) in the vicinity of the center of the shell, all the terms higher than second order with respect to \( x, y \) and \( z \) may be neglected. The result after very lengthy calculations in the approximation up to first order of \( \alpha \) is as follows: near the center

\[
I_{\nu\nu} = \varepsilon^\lambda \left\{ \frac{9}{16} + \alpha \left[ \frac{71}{90} + \frac{1}{54} \left( \frac{x^2 + y^2 + z^2}{a^2} \right) + \left( \frac{4073}{37800} - \frac{41}{400} \right) \left( \frac{x^2 + y^2 - 2z^2}{a^2} \right) \right] \right\},
\]

(2)

\[
I_{\nu\nu} = -\frac{3}{47} \varepsilon^\lambda \frac{a^{-1} y}{a^2}, \quad I_{\nu\nu} = \frac{47}{144} \varepsilon^\lambda \frac{a^{-1} x}{a^2}, \quad I_{\nu\nu} = 0,
\]

(2)

\[
I_{\nu\nu} = \varepsilon^\lambda \left\{ \frac{1}{154} + \alpha \left\{ \left( \frac{5}{504} + \frac{1}{27} \left( \frac{x^2 + y^2 + z^2}{a^2} \right) + \left( \frac{1}{280} \right) \left( \frac{x^2 - y^2}{a^2} \right) \right) \right\},
\]

(2)

\[
I_{\nu\nu} = \varepsilon^\lambda \left( \frac{1}{140} \right) \frac{x y}{a^2}, \quad I_{\nu\nu} = \varepsilon^\lambda \left( \frac{59}{2520} - \frac{1}{280} \right) \frac{x z}{a^2},
\]

(2)

\[
I_{\nu\nu} = \varepsilon^\lambda \left\{ \frac{1}{504} + \alpha \left\{ \left( \frac{5}{27} \left( \frac{x^2 + y^2 + z^2}{a^2} \right) + \left( \frac{1}{280} \right) \left( \frac{x^2 - y^2}{a^2} \right) \right) \right\},
\]

(2)

\[
I_{\nu\nu} = \varepsilon^\lambda \left( \frac{59}{2520} - \frac{1}{280} \right) \frac{y z}{a^2},
\]

(2)

\[
I_{\nu\nu} = \varepsilon^\lambda \left\{ \frac{169}{2520} - \frac{1}{27} \left( \frac{x^2 + y^2 + z^2}{a^2} \right) + \left( \frac{1}{560} \right) \left( \frac{x^2 + y^2 - 2z^2}{a^2} \right) \right\}.
\]

(6.5)
It is confirmed that the obtained \( \hat{z} \) fulfills Eq. (2.10).

Thus it follows from Eq. (2.18) that

\[
\mathbf{A} = -\frac{1}{72} \omega \varepsilon^z(x, -\hat{y}, 0) + \left( \frac{49}{800} + \frac{17}{400} f^{(2)} \right) \omega \varepsilon^z(x, y, 0) \\
+ \left( \frac{259}{3600} - \frac{17}{200} f^{(2)} \right) \omega \varepsilon^z(0, 0, z). \tag{6.6}
\]

§ 7. Conclusion

In the above we have calculated, up to second order of the weak field approximation, the gravitational effects of the material shell, which is rotating with constant angular velocity \(-\omega\) around an axis (z-axis) through its center, to a test particle moving with a low velocity \(v\) near the center of the shell. In the calculations we have taken into account the Lorentz contraction of the shell due to rotation. Also the associated change in the material distribution of the shell has been taken into account by introducing the modulation function \(f(\cos^2\theta, 0)\) for the mass density. Furthermore the conservation law of the energy-momentum tensor has been treated properly according to Bass-Pirani’s method. The calculations show that the net acceleration of the test particle is, according to Eqs. (2.16), (4.15) and (6.6), now expressed as

\[
\mathbf{A} = \mathbf{A}^{(1)} + \mathbf{A}^{(2)} \\
= -2k_1\omega \times v - k_2\omega \times (\omega \times r) + k_3(0, 0, z) \tag{7.1}
\]

with

\[
k_1 = \frac{1}{3} \varepsilon - \frac{1}{144} \varepsilon^z,
\]

\[
k_2 = \frac{1}{200} \left( \frac{19}{21} - f^{(2)} \right) \varepsilon + \frac{1}{400} \left( \frac{49}{2} + 17f^{(2)} \right) \varepsilon^z,
\]

\[
k_3 = \frac{1}{10} \left( f^{(2)} - \frac{19}{21} \right) \varepsilon + \frac{1}{200} \left( \frac{259}{18} - 17f^{(2)} \right) \varepsilon^z. \tag{7.2}
\]

Now from the above result we see that, if the value of the \(f^{(2)}\), namely, the first non-vanishing coefficient in the Legendre series expansion of the \(f(\cos^2\theta, 0)\), is chosen to be

\[
f^{(2)} = \left( \frac{17}{20} \varepsilon - \frac{19}{20} \right) \left( \frac{17}{20} \varepsilon - 1 \right)^{-1}, \tag{7.3}
\]

then the coefficient \(k_3\) vanishes and no pseudo-centrifugal acceleration appears in the \(\mathbf{A}\); there remain only the ‘Coriolis’ and the ‘genuine centrifugal’ acceleration with
the coefficients

\[ k_1 = \frac{1}{3} \varepsilon - \frac{1}{144} \varepsilon^2, \]

\[ k_2 = \frac{7}{72} \varepsilon^2. \]  

(7.4)

Furthermore if the scale of the shell is such that

\[ \varepsilon = \frac{M \kappa c^2}{2\pi a} = \frac{16}{5}, \]  

(7.5)

then the ratio of \( k_1 \) and \( k_2 \) reduces to unity and the \( A \) now takes the form

\[ A = -k \{2 \omega \times v + \omega \times (\omega \times r)\} \]  

(7.6)

with the factor

\[ k = \frac{224}{225}. \]  

(7.7)

This numerical value of \( k \) is not exactly equal to but slightly different from unity. Considering, however, the limited nature of the weak field approximation and the oversimplification taken here for our universe by replacement with an infinitely thin shell, the result of our model calculations is, on the whole, rather satisfactory enough to exemplify the realizability of Einstein's idea that the empirical fact of the very existence of Newton's inertial systems may be explained within the framework of the general theory of relativity by the material distribution of our real universe in conformity with Mach's original thought.

**Appendix A**

The unit normal \( n_\sigma \) at a point \((x, y, z)\) on the shell is

\[ n_\sigma = N(0, (1 + \alpha \sin^2 \theta)^{1/2} \sin \theta \cos \phi, (1 + \alpha \sin^2 \theta)^{1/2} \sin \theta \sin \phi, 0) \]  

(A.1)

with

\[ N = \{\cos^2 \theta + (1 + \alpha \sin^2 \theta)^2 \sin^2 \theta\}^{-1/2}. \]

Let us transform the coordinates from \((ct, x, y, z)\) into \((ct, q, \theta, \phi)\) by

\[ x = Q \cos \phi, \quad y = Q \sin \phi, \quad z = q \cos \theta \]  

(A.2)

with

\[ Q = q (1 + \alpha q^{-1} \sin^2 \theta)^{-1/2} \sin \theta, \]  

(A.3)

and restrict ourselves to the approximation up to first order of \( \alpha \). Then in the curvilinear coordinate system we have
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\[ n_s = (0, 1, 0, 0). \]  
(A.4)

Namely the direction of the \( q \)-coordinate coincides with that of the normal of the shell. The covariant components of the metric are

\[ g_{00} = -1, \quad g_{11} = 1, \quad g_{22} = -(3/2) \alpha a \sin^2 \theta \cos \theta, \]
\[ g_{33} = a^2 (1 - 3 \alpha a \sin^2 \theta \cos^2 \theta), \quad g_{55} = a^2 \sin^2 \theta (1 - \alpha a \sin^2 \theta), \]
other components zero.  
(A.5)

The conservation equation for the \( T^{(0)}_{\mu \nu} \) now takes the form:

\[ \Gamma^{(0)}_{\nu \sigma} + \Gamma^{(0)}_{\nu \sigma} \Gamma^{(0)}_{\mu \nu} + \Gamma^{(0)}_{\mu \nu} \Gamma^{(0)}_{\nu \sigma} = 0. \]  
(A.6)

On the shell \( (q = a) \), \( \Gamma^{(0)}_{\nu \sigma} \) have the following values:

\[ \Gamma^{(0)}_{12} = (3/2) \alpha \sin^3 \theta \cos \theta, \quad \Gamma^{(0)}_{13} = -a \left[ 1 + \alpha \left( 3 \sin^2 \theta - \frac{3}{2} \sin^4 \theta \right) \right], \]
\[ \Gamma^{(0)}_{33} = -a \sin^2 \theta \left[ 1 + \alpha \left( \sin^2 \theta - \frac{3}{2} \sin^4 \theta \right) \right], \]
\[ \Gamma^{(0)}_{23} = -\alpha \sin \theta \cos \theta \left( 3 - \frac{3}{2} \sin^2 \theta \right), \]
\[ \Gamma^{(0)}_{33} = -\sin \theta \cos \theta \left[ 1 + \alpha \left( \sin^2 \theta - \frac{3}{2} \sin^4 \theta \right) \right], \quad \Gamma^{(0)}_{33} = \cot \theta (1 - \alpha \sin^2 \theta), \]
and others zero.  
(A.7)

Since the four-velocity \( U^\nu \) of a material particle of the shell now takes the value

\[ U^\nu = (c (1 + \alpha \sin^2 \theta)^{-1/2}, 0, 0, -\omega (1 + \alpha \sin^2 \theta)^{1/2}), \]  
(A.8)

the kinetic energy-momentum tensor \( \rho U^\nu U^\mu \) has a simple form.

Now if follows from Eqs. (A.4), (A.5) and Eqs. (3.16), (3.17) that

\[ E^{(0)}_{\mu 1} = 0, \]
\[ E^{(0)}_{\mu 3} = -\alpha^{1/2} a \sin^2 \theta E^{(0)}_{\mu 3} + O(\alpha^{3/2}). \]  
(A.9)

(A.10)

Since our model is stationary and axially symmetric, all the derivatives with respect to \( t \) and \( \phi \) drop out from Eq. (A.6). Moreover, by virtue of Eqs. (A.8) and (A.9), the derivatives with respect to \( q \) do not appear at all in Eq. (A.6). Thus Eq. (A.6) reduces to

\[ \frac{d E^{(0)}_{\mu 2}}{d \theta} + \Gamma^{(0)}_{13} E^{(0)}_{\mu 2} = 0, \]  
(A.11)

\[ \Gamma^{(0)}_{33} (E^{(0)}_{\mu 3} + \rho U^3 U^\mu) + \Gamma^{(0)}_{33} E^{(0)}_{\mu 3} = 0, \]  
(A.12)
\[
\frac{dE^{(0)}_{32}}{d\theta} + \Gamma^{(0)}_{32} (E^{(0)}_{33} + \rho U^3 U^3) + (\Gamma^{(0)}_{22} + \Gamma^{(0)}_{10}) E^{(0)}_{22} = 0, \tag{A.13}
\]
\[
\frac{dE^{(0)}_{23}}{d\theta} + (2 \Gamma^{(0)}_{32} + \Gamma^{(0)}_{12}) E^{(0)}_{33} = 0. \tag{A.14}
\]

A general solution of Eq. (A.11) is
\[
E^{(0)}_{32} = k \csc \theta \left(1 + \alpha \left(2 \sin^2 \theta - \frac{3}{2} \sin^4 \theta\right)\right), \tag{A.15}
\]
where \(k\) is an integration constant. In order for the \(E^{(0)}_{32}\) to satisfy Bass-Pirani's fourth condition the \(k\) must be zero, hence
\[
E^{(0)}_{32} = 0. \tag{A.16}
\]

Then it follows from Eq. (A.10) that
\[
E^{(0)}_{23} = 0 \tag{A.17}
\]
with which Eq. (A.14) also is satisfied. Next, elimination of \(E^{(0)}_{33}\) from Eq. (A.12) and (A.13) leads to
\[
\frac{dE^{(0)}_{32}}{d\theta} + \left[\Gamma^{(0)}_{22} + \Gamma^{(0)}_{12} - \Gamma^{(0)}_{32} \Gamma^{(0)}_{33} (\Gamma^{(0)}_{33})^{-1}\right] E^{(0)}_{32} = 0, \tag{A.18}
\]
whose general solution is
\[
E^{(0)}_{22} = k' \csc^2 \theta \left\{1 + \alpha \left(2 \sin^2 \theta - \frac{3}{2} \sin^4 \theta\right)\right\}, \tag{A.19}
\]
with an integration constant \(k'\). Again by virtue of the Bass-Pirani fourth condition the \(k'\) should be zero, hence
\[
E^{(0)}_{22} = 0, \tag{A.20}
\]
which together with Eq. (A.12) brings out
\[
E^{(0)}_{33} = -\rho U^3 U^3 = -\varepsilon \alpha (2\kappa \alpha)^{-1}. \tag{A.21}
\]
Thus the \(T^{(0)}_{rr}\) in the curvilinear coordinate system are, in the approximation up to first order of \(\alpha\), as follows:
\[
T^{(0)}_{10} = \frac{\varepsilon}{2\kappa \alpha} \left[1 + \alpha \left(\sin^2 \theta + \frac{3}{2} \sin^4 \theta \cos^2 \theta + f (\cos^2 \theta, 0)\right)\right],
\]
\[
T^{(0)}_{33} = -\alpha^{1/2} \varepsilon (2\kappa \alpha)^{-1}, \tag{A.22}
\]
other \(T^{(0)}_{rr} = 0\).
Finally from this, by transforming back to the original coordinate system \((ct, x, y, z)\), we arrive at the expression of Eq. (3·19).

Appendix B

Let us work again in the same coordinate system \((ct, q, \theta, \phi)\) as in Appendix A. By virtue of Eqs. (A·4) and (A·5), then, the Eqs. (5·5) and (5·6) respectively reduce to

\[
\overset{(1)}{T}^{01} = 0
\]  

(B·1)

and

\[
\overset{(1)}{T}^{03} = -a\alpha^{1/2}\sin^{1/2}\theta\overset{(1)}{T}^{03}.
\]  

(B·2)

Moreover Eq. (5·1) reduces to

\[
\frac{dT^{02}}{d\theta} + \Gamma^{02}_{02} T^{02} = 0,
\]  

(B·3)

\[
\Gamma^{02}_{03} T^{03} + \Gamma^{02}_{02} T^{02} = -\frac{\varepsilon^2}{16\kappa a^2} [1 + a (B \sin^2 \theta + C \cos^2 \theta)],
\]  

(B·4)

\[
\frac{dT^{02}}{d\theta} + \Gamma^{02}_{03} T^{03} + (\Gamma^{02}_{02} + \Gamma^{02}_{01}) T^{02} = -\frac{\varepsilon^2 a}{16\kappa a^2} \sin \theta \cos \theta (B - C),
\]  

(B·5)

\[
\frac{dT^{03}}{d\theta} + (2\Gamma^{03}_{02} + \Gamma^{03}_{01}) T^{03} = 0,
\]  

(B·6)

where \(B\) and \(C\) are the functions given in Eq. (5·3). In just the same way as the \(E^{02}\) of Eq. (A·11), we see from Eq. (B·3) and the condition (iv)' that

\[
\overset{(1)}{T}^{02} = 0,
\]  

(B·7)

and consequently from Eq. (B·2) that

\[
\overset{(1)}{T}^{03} = 0
\]  

(B·8)

with which Eq. (B·6) also is satisfied. Elimination of \(T^{03}\) from Eqs. (B·4) and (B·5) leads to

\[
\frac{dT^{02}}{d\theta} + 2 \cot \theta (1 + \alpha (3 \sin^2 \theta - 2 \sin^2 \theta)) \overset{(1)}{T}^{02} = \frac{\varepsilon^2}{16\kappa a^2} \cot \theta [1 + \alpha C].
\]  

(B·9)

Now put

\[
\overset{(1)}{T}^{02} = \frac{\varepsilon^2}{16\kappa a^2} \left(\frac{1}{2} + X + \alpha \mathcal{L} + O(\alpha^3)\right).
\]  

(B·10)
Then Eq. (B·4) reduces to

$$\frac{dX}{d\theta} + 2X \cot \theta = 0,$$

(B·11)

$$\frac{d\mathcal{A}}{d\theta} + 2\mathcal{A} \cot \theta = \cot \theta [C + (1 + 2X) (2 \sin^2 \theta - 3 \sin^4 \theta)].$$

(B·12)

Then solution of Eq. (B·11) which conforms with the condition (iv)' is again

$$X = 0,$$

(B·13)

hence Eq. (B·12) reduces to

$$\sin \theta \frac{d\mathcal{A}}{d\theta} + 2\mathcal{A} \cos \theta = \sum_{n=0}^{\infty} b^{(2n)}P_{2n+1}(\cos \theta)$$

(B·14)

with the coefficients

$$b^{(0)} = \frac{22}{105}, \quad b^{(2)} = \frac{22}{15} + \frac{6}{5} f^{(2)}, \quad b^{(4)} = -\frac{20}{21} + \frac{10}{9} f^{(4)},$$

$$b^{(2n)} = \frac{2(2n + 1)}{4n + 1} f^{(2n)} \quad \text{for} \quad n \geq 3.$$  

(B·15)

The solution of Eq. (B·14), which conforms with the condition (iv)' is obtained as the following Legendre series:

$$\mathcal{A} = \sum_{n=0}^{\infty} g^{(2n)}P_{2n}(\cos \theta),$$

(B·16)

where

$$g^{(0)} = \frac{41}{210} + \frac{1}{10} f^{(2)} + \sum_{m=2}^{\infty} \frac{f^{(2m)}}{(m + 1)(4m + 1)},$$

$$g^{(2)} = \frac{19}{42} + \frac{1}{2} f^{(2)} + 5 \sum_{m=2}^{\infty} \frac{f^{(2m)}}{(m + 1)(4m + 1)},$$

$$g^{(4)} = -\frac{2}{7} + 9 \sum_{m=2}^{\infty} \frac{f^{(2m)}}{(m + 1)(4m + 1)},$$

$$g^{(2n)} = (4n + 1) \sum_{m=n}^{\infty} \frac{f^{(2m)}}{(m + 1)(4m + 1)} \quad \text{for} \quad n \geq 3.$$  

(B·17)

Thus the $^{(1)}T^{12}$ is written as

$$^{(1)}T^{12} = \frac{\varepsilon^2}{32\pi a^3} (1 + 2\mathcal{A})\mathcal{A}.$$  

(B·18)

Accordingly, by virtue of Eq. (B·4), we get
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\[
\mathcal{T}^{\alpha\beta} = \frac{\varepsilon^2}{32\kappa a^2} \csc^2 \theta (1 + 2\alpha \mathcal{B}),
\]

where

\[
\mathcal{B} = \sum_{n=0}^{\infty} j^{(2n)} P_{2n}(\cos \theta)
\]

and

\[
\begin{align*}
  j^{(0)} &= \frac{1697}{630} - \frac{1}{10} f^{(2)} - \sum_{m=2}^{\infty} \frac{f^{(2m)}}{(m+1)(4m+1)}, \\
  j^{(2)} &= \frac{1649}{630} + \frac{7}{10} f^{(2)} - \frac{5}{9} \frac{f^{(2m)}}{(m+1)(4m+1)}, \\
  j^{(4)} &= \frac{2}{7} + \frac{10}{9} f^{(4)} - \sum_{m=1}^{\infty} \frac{f^{(2m)}}{(m+1)(4m+1)}, \\
  j^{(2n+2)} &= \frac{2(2n+1)}{4n+1} f^{(2n)} - (4n+1) \sum_{m=0}^{\infty} \frac{f^{(2m)}}{(m+1)(4m+1)} \quad \text{for} \quad n \geq 3.
\end{align*}
\]

Therefore Eq. (B.2) gives

\[
\mathcal{T}^{\alpha\beta} = -\frac{\varepsilon^2 \alpha^2}{32\kappa a^2}, \quad \mathcal{T}^{\alpha\theta} = \frac{\varepsilon^2 \alpha}{32\kappa a} \sin^2 \theta.
\]

Thus all components of \( \mathcal{T}^{\alpha\beta} \) in the curvilinear coordinate system \((ct, q, \theta, \phi)\) have been obtained.

By transforming these back to the original coordinate system \((ct, x, y, z)\), we arrive at the expression of Eq. (5.8).

**Appendix C**

The rotating shell can be regarded as a collection of an infinite number of rotating rings with the common angular velocity \(-\omega\) around the \(z\)-axis. Let us consider the ring at height \(z\). The diameter \(2R\) of this ring in the \(K\) is one half of the sum, over the ring, of the projected length \(dl = |\sin \phi| ds\) of an infinitesimal segment \(ds\) of the ring onto a certain radial direction in the \(K\),

\[
2R = \frac{1}{2} \int |\sin \phi| ds,
\]

where \(\phi\) is the angle between the above stated direction and the normal of the segment. By the Lorentz contraction, however, the length of the segment is reduced by the factor \(\{1 - (v/c)^2\}^{1/2}\) compared with the corresponding length \(\sqrt{a^2 - z^2} \; d\phi\) in the case \(\omega = 0\),

\[
ds = \sqrt{1 - (v/c)^2} \sqrt{a^2 - z^2} \; d\phi.
\]
Hence we get
\[ R = \frac{1}{4} \sqrt{1 - \left(\frac{v}{c}\right)^2 \sqrt{a^2 - z^2}} \int_0^{2\pi} |\sin \phi| d\phi = \sqrt{1 - \left(\frac{v}{c}\right)^2 \sqrt{a^2 - z^2}}, \] (C·1)

which means that the radius of the rotating ring in the K is reduced by the Lorentz factor compared with the corresponding radius \( \sqrt{a^2 - z^2} \) of the ring in the case \( \omega = 0 \).

Now in terms of the coordinates of a particle of the ring we have the relations
\[ R = \sqrt{x^2 + y^2}, \quad v_x = \omega y, \quad v_y = -\omega x, \quad v_z = 0. \] (C·2)

Thus we finally get Eq. (3.1) as the equation for the rotating shell in the K.

References

1) A. Einstein, Berl. Ber. (1914), 1030.
2) E. Mach, Die Mechanik in ihrer Entwicklung, historischkritisch dargestellt (1883), Kap. II Nr. 6.
4) H. Thirring, Phys. Z. 19 (1918), 33; 22 (1921), 29.