A Duality between Blocked Ising Models

Yasuhiro Kasai

Department of Applied Physics, Faculty of Engineering
Osaka University, Suita, Osaka 565

(Received March 30, 1978)

For planar lattices, there exists a dual relation between extended (blocked) Ising models in which interactions are expressed by face elements (blocks) instead of bonds. The total interaction for a block contains all even spin products among spins on the face (g-polygon); for \( g = 2 \) or 3, it contains only two-spin product, for \( g = 4 \) it contains crossed bond interactions and four-spin product, and so on. Formally, there exists a self-dual lattice for every \( g \). The triangular lattice is self-dual for \( g = 3 \) and the self-dual lattice for \( g \geq 4 \) has non integer coordinate number (a rational number) which suggests a universality for the coordination number.

§ 1. Introduction

Exact properties for generalized Ising models which are extended in range of interaction and in many-body spin interaction are interesting. The duality of the Ising model makes it possible to find the exact critical temperature for the square lattice without knowing the partition function. Wegner extended the duality for generalized Ising models in lattice dimensionality and in many-spin interaction; every Ising model has the dual lattice of it. Merlini and Gruber discussed the group theoretical aspect of the duality and found a compact rule to find the dual lattice. Up to now, the discussion for the duality has been based on the expansions, high and low temperature, of the partition function with respect to every single interaction term. In this paper, we treat a different type of expansion, with respect to blocked interaction terms. Thus a new type of duality can be obtained which is a natural extension of the duality introduced by Kramers, Wannier and Onsager.

The restriction on the interaction parameter is found to be natural when we treat an extended Ising model transformed from the dilute ferromagnetic model with blocked components, which can be related to the site problem, in the annealed system. On the other hand, the duality is topological property of lattices. We treat also the duality from the concept of connectedness of graphs on the basis of Euler’s theorem. The treatment leads to the extended coordination number (rational number) of the lattices. This suggests a universal property of the Ising models for the coordination number.

§ 2. Blocked lattices

We start from a planar Ising lattice. Each spin variable, \( s_i \) at the \( i \)-th vertex,
can take $\pm 1$. The interaction among spins (vertices) on a face is introduced. If there exist $g$ spins on the face; $s_1, s_2, \cdots, s_g$, we can construct symmetric sums of even number products of the spin variables:

$$\sum_j(t(2)) = \sum_{i_1, i_2} s_{i_1} s_{i_2}, \quad \text{(two spin sum)}$$
$$\sum_j(t(4)) = \sum_{i_1, i_2, i_3, i_4} s_{i_1} s_{i_2} s_{i_3} s_{i_4}, \quad \text{(four spin sum)}$$
$$\vdots$$
$$\sum_j(t(2[g/2])) = \sum_{i_1, i_2, \cdots, i_{g/2}} s_{i_1} s_{i_2} \cdots s_{i_{g/2}}, \quad (2.1)$$

where $\sum_j(t(m))$ denotes the symmetric sum for $m$ spin product on the $j$-th face, and $[x]$ is the Gaussian integer for $x$. We assign the interaction $H_j$ to the $j$-th face:

$$H_j = -J \{ \sum_j(t(2)) + \cdots + \sum_j(t(2[g/2])) \} = -J \cdot B_j, \quad (2.2)$$

where

$$B_j = \frac{1}{2} \left\{ \prod_{k=1}^g (1 + s_k) + \prod_{k=1}^g (1 - s_k) \right\} / 2 - 1 \quad (2.3)$$

$$= \begin{cases} 2^{g-1} - 1 & \text{if all spins are the same value} \\ -1 & \text{otherwise,} \end{cases}$$

and we call $B_j$ the block spin variable for the $j$-th face, or the block. The restriction of the same factor $J$ for the all $\sum_j(t(k))$'s in (2.2) seems to be too strong and unnatural. This restriction is originally found from an annealed dilute ferromagnet with general ferromagnetic interaction (Szyoz model; see an equivalence between a random Ising ferromagnet and a regular Ising model) at 0K. For simplicity, we treat lattices with the same $g$ for every block. The Hamiltonian $H$ can be written by

$$H = \sum_{\text{all blocks}} H_j, \quad (2.4)$$

and the partition function $Z(K)$ is given by

$$Z(Y) = \sum_{\{i_1, i_2\}} e^{-\beta H}, \quad (2.5)$$

where $K = \beta J, \beta = 1/kT, k$ and $T$ are the Boltzmann constant and the absolute temperature respectively. In Fig. 1, the blocks are expressed by those faces with circles in their centers. If two blocks contact with a bond, we insert an digon face, namely a double bond (see Fig. 1, every (b) in cases $g=3, 4, 6$). Then, every block marked by the circle is surrounded by non-block faces. We call those non-block faces the complementary faces. For $g=2$ or 3, the interaction $H_j$ contains only two-spin interactions. For $g \geq 4$, the $H_j$ contains crossed bond interactions and
many = (4, 6, ⋯) spin interactions for which the corresponding Ising model has not been solved yet.

Fig. 1. Blocked lattices with \( g \)-polygons. The circles indicate the blocks. An extended coordination number \( z \) is given for each lattice (see the last part of § 4). See Fig. 3 for the duality.

§ 3. High temperature expansion

The Boltzmann factor in the partition function (2.5) can be written using (2.3) as

\[
\exp[KB_j] = a + b \cdot B_j = a (1 + rB_j),
\]

where

\[
a = \{\exp[K(2^{g-1} - 1)] + (2^{g-1} - 1) e^{-K}/2^{g-1},
\]
\[
b = \{\exp[K(2^{g-1} - 1)] - e^{-K}/2^{g-1},
\]
\[
r = \{\exp[2^{g-1}K] - 1\} / \{\exp[2^{g-1}K] + 2^{g-1} - 1\}.
\]

Then, the partition function \( Z(K) \) can be expanded as follows:

\[
Z(K) = \sum_{\{t_i = \pm 1\}} \prod_j \exp[KB_j] = a^N \sum_{\{t_i = \pm 1\}} \prod_j (1 + r \cdot B_j)
\]
\[
= a^N \sum_{\{t_i = \pm 1\}} \left\{ \sum_{(f \in \Theta)} r^{N(f)} \prod_{(j \in f)} B_j \right\},
\]

(3.3)
where $N$ is the total number of the blocks, and the graph $\theta$ denotes a definite configuration which is a sub-set selected from all blocks (there exist $2^N$ different $\theta$'s and $M(\theta)$ is the total number of blocks in $\theta$. When $g=2$, the expansion (3·3) is well known, i.e., the expansion by the closed graphs. For $g\geq 3$, we must extend the expansion theorem. As the first step, we consider a cluster (a connected component of the graph) composed of block spin variables, $B_i, B_2, \ldots, B_n$. We perform the partial spin sum over the connected spins (the total number $m$) in the cluster:

$$
\sum_{\{\theta_i=\pm 1; i=1,\ldots, m\}} \prod_{j \in \text{cluster}} B_j = 2^m \cdot d,
$$

(3·4)

where $d$ is the number (we call it the degeneracy of the cluster) of different terms which do not vanish when we take the spin sum $\sum_{\{\theta\}}$. More precisely, the

Fig 2. Degeneracies $d$ for some clusters. The non-vanishing sub-configurations are shown for each cluster. The number supplemented to the sub-configuration indicates the number of symmetric sub-configurations.
degeneracy $d$ is the number of non-vanishing (for the spin sum) sub-configuration, each of which is a set of elements taking one element (which corresponds to a term of spin products) from each block in the cluster; the sub-configuration is an extension of the closed polygon for $g = 2$. For $g = 2$, the degeneracy is always one. For $g \geq 3$, the degeneracy depends on the cluster (see Fig. 2). For a given configuration $\theta$, the total degeneracy $D(\theta)$ is the product of degeneracies for all clusters in $\theta$:

$$D(\theta) = \prod \limits_j d_j(\theta),$$  \hspace{1cm} (3.5)

where $d_j(\theta)$ denotes the degeneracy for the $j$-th cluster in $\theta$. The partition function (3.3) can be written by

$$Z(K) = a^N 2^V \sum \limits_{\forall \theta} D(\theta) r^{N(\theta)},$$ \hspace{1cm} (3.6)

where $V$ is the total number of the spins (vertices). To discuss the duality, we require also low temperature expansion for $Z(K)$. We treat it in the context of the duality.

§ 4. Duality

We define a duality of a block with general $g$ in a blocked lattice. The dual block for a given block is the block whose vertices (spins) are set in every complementary face around the original block (see Fig. 3). The dual-blocked lattice for a blocked lattice (original) can be constructed using the corresponding Fig. 3 for each block. If the complementary face is common for two or more original blocks (solid line), the vertices (crossed vertices) of the dual blocks (broken line) on the complementary face must be looked as one vertex. This extended duality is also reciprocal, i.e., the dual lattice of a dual lattice is the original lattice. We can define the partition function $Z^* (K^*)$ for the dual blocked lattice with the interaction parameter $K^*$:

$$Z^* (K^*) = \sum \limits_{(\eta_i = \pm 1)} \prod \exp [K^*B_j^*],$$ \hspace{1cm} (4.1)

where $s_i^*$ and $B_j^*$ are the $i$-th spin variable and the $j$-th block interaction respec-

---

Fig. 3. Duality for a block with $g$. The solid line element and the broken
line element are dual to each other. Examples of the dual blocked lattices
are shown in Fig. 1.
A Duality between Blocked Ising Models

963

tively for the dual-blocked lattice. We prepare another type of expansion, low temperature, for the partition function $Z^*(K^*)$. Since every block spin variable, $B_j^*$, can take value $2^{s-1} - 1$ or $-1$, we define a configuration $\theta^*$ which is the set of blocks corresponding to the block spin value $-1$, for a total spin configuration. Since some different spin configurations may correspond to the same configuration $\theta^*$, we consider a degeneracy $D^*(\theta^*)$ (the number of the spin configurations) for $\theta^*$, where the degeneracy does not count the totally inverse spin configurations. The partition function $Z^*(K^*)$ can be expanded as follows:

$$Z^*(K^*) = 2 \exp[K^*(2^{s-1} - 1)N] \{ \sum_{\{\theta^*\}} D^*(\theta^*) \exp[-2^{s-1}K^*M(\theta^*)] \},$$

(4.2)

where the factor 2 in the r.h.s. denotes the degeneracy for the totally inverse spin configurations, $N$ is the number of blocks for the dual-blocked lattice and is the same for the original lattice, and $M(\theta^*)$ is the number of blocks with $B_j = -1$. In the Appendix, we will prove the following equality between $D(\theta)$ in (3.5) and $D^*(\theta^*)$ in (4.2):

$$D(\theta) = D^*(\theta^*).$$

(4.3)

If we set

$$r = \{ \exp[2^{s-1}K] - 1 \} / \{ \exp[2^{s-1}K] + 2^{s-1} - 1 \} = \exp[-2^{s-1}K^*]$$

(4.4)

or

$$\{ \exp[2^{s-1}K] - 1 \} \{ \exp[2^{s-1}K^*] - 1 \} = 2^{s-1},$$

(4.4')

then the following equality is valid from (3.6) and (4.2):

$$Z(K) / \{ a^{sV^*} \} = Z^*(K^*) / \{ 2 \exp[(2^{s-1} - 1)K^*N] \}$$

or

$$Z(K) / \{ e^{-NK}(\exp[2^{s-1}K] - 1)^{N/2V^{1/2}} \} = Z^*(K^*) / \{ e^{-NK^*}(\exp[2^{s-1}K^*] - 1)^{N/2V^{1/2}} \},$$

(4.5)

where $V^*$ is the number of vertices (spins) for the dual blocked lattice. We can prove a symmetric relation between the mutually dual-blocked lattices, using (4.4'), (4.5) and their derivatives with respect to $K$ and $K^*$:

$$p + p^* = 1,$$

(4.6)

where

$$p = 2^{-s+1}(1 - \exp[-2^{s-1}K]) (1 + E),$$

$$p^* = 2^{-s+1}(1 - \exp[-2^{s-1}K^*]) (1 + E^*),$$

(4.7)

$$E = (1/N) \partial \ln Z(K) / \partial K$$

and

$$E^* = (1/N) \partial \ln Z^*(K^*) / \partial K^*.$$
This type of relation (4.6) for $g=2$ was obtained by Syozi\(^5\) between the critical concentrations $p_c$ and $p_c^*$ of dilute ferromagnets (a mixture of ferromagnetic bonds and non-magnetic bonds) defined on the dual lattices to each other. The relation (4.6) itself suggests an extension of the dual relation between the critical concentrations to the dilute ferromagnets with general $g$-blocks.

Here we propose a topological relation for the dual-blocked lattices. From Euler’s theorem for graph components (sites, bonds and faces), we obtain a relation between the dual-blocked lattices with the coordination numbers $z$ and $z^*$ for $g$-blocks:

$$[g/(g-1)](1/z + 1/z^*) = 1,$$

(4.8)

where the coordination number denotes the number of $g$-blocks radiated from each site. The relation (4.8) is similar to (4.6), if we put

$$p_b = [g/(g-1)]/z \quad \text{and} \quad p_b^* = [g/(g-1)]/z^*,$$

(4.9)

in fact, this type of estimation ($g=2$) for the critical concentration was given by Brout\(^6\) in the quenched system.

\section{Self-duality}

The self-duality proposed by Kramers and Wannier for the square lattice can be extended for general $g$. From (4.4'), the critical temperature $T_c (= J/[Kc])$ for the self-dual lattice can be obtained by

$$\exp[2\pi^2 Kc] = 1 + 2^{8-2^{1/2}}.$$  

(5.1)

For the case $g=2$, the critical temperature agrees with that of the square lattice which is self-dual in the meaning given in §4. For $g=3$, it agrees with that of the triangular lattice which is also self-dual. This self-duality is a new result.

From (4.8), we can guess more explicitly the self-dual lattice by the coordination number $z$:

$$z = 2g/(g-1),$$

(5.2)

since $z = z^*$. This gives consistent results in $g=2$ and 3. For $g=4$, the formula (5.2) suggests a strange lattice with $z=8/3$.

Here we try to extend lattices to ones with rational number $z (= p/q; p, q$ positive integers$)$. We demand for simplicity to lattices that every lattice site is equivalent, that $g$-polygons radiated from each site make equal angles successively and that the complementary faces are equivalent regular polygons ($\omega$-polygons) to each other. Those demands are satisfied by the self-dual lattices for $g=2$ and 3 and such lattices as the triangular lattice and honey comb lattice for $g=2$. The complementary $\omega$-polygon is related with coordination number $z$ by
A Duality between Blocked Ising Models

\[ w = g z / [(g - 1) z - g], \]

(5.3)

and the interior angle \( \alpha \) for the \( w \)-polygon is given by

\[ \alpha = (w - 2) \pi / w. \]

(5.4)

We can make a lattice by the following procedure for \( z (= p / q) \). Put a site at first on a plane. Connect a vertex of the first \( g \)-polygon in the plane to the site. Connect a vertex of the next \( g \)-polygon to the first site and make the interior angle of the complementary face be \( \alpha \) (see Fig. 4), and so on. The radiation angle \( \beta \) which is defined as the angle between successive \( g \)-polygons is \( \alpha \) plus an interior angle of the \( g \)-polygon. The \((p+1)\)-th \( g \)-polygon for the procedure just overlaps on the first polygon, although the total of the radiation angles for the procedure is \( 2\pi q \). Therefore we must consider many-(\( q \))-fold structure around each site just like Riemann’s plane. Further we must extend the complementary \( w \)-polygon (interior angle \( \alpha \)) for rational number \( w (= p' / q' \)). Figure a circle. Put successively points on the circle whose successive angles from the center are \( \pi - \alpha \) respectively. The \((p+1)\)-th point overlaps on the first point, although the total of the successive angles from the center is \( 2\pi q' \). Connect all those points successively by chords. The \( w \)-polygon has also \( q' \)-fold structure. For example, the area \( S \) of the \( w \)-polygon is given by

\[ S = wa^2 / [4 \tan (\pi / w)], \]

(5.5)

where \( a \) is the length of each edge. Now we can construct the whole lattice. In the previous stage, we constructed around the first site. New sites are generated as the vertices of the connected \( g \)-polygons to the first site. We repeat the same procedure as for the first site to construct the \( q \)-fold structure at every new site, then we have new sites further and so on. If some sites overlap, those sites

---

**Fig. 4.** The successive construction of a lattice with rational coordination number \( z \) for \( g=4 \), around the first site \( P \). The next Fig. 5 is an example of the completed successive construction around \( P \) for \( \alpha = \pi / 4 \).

**Fig. 5.** (a) The lattice structure around each site (white circle) for \( z=8/3 \) and \( g=4 \). The numbers indicate the order of the successive construction shown in Fig. 4, and the black circles are new sites generated by the construction. (b) The complementary faces (\( w \)-polygon) for \( w=8/3 \).
are defined to be connected and looked as a single site. The complementary faces which are \( \omega \)-polygons related by (5·3) are constructed consistently through the whole procedure. It is noted that the distribution of the lattice points becomes dense for the infinite lattice. However, the limit lattice is well defined for every site and the upper side and the lower side for the \( g \)-polygons can be distinguished locally; this property certifies the existence of duality. In Fig. 5, the procedure is exemplified for \( z = 8/3 \).

Potts\(^7\) pointed out an interesting property of the magnetization \( M \) for a lattice with interaction \( K \) and the dual one with \( K^* \):

\[
M = (1 - X)^{1/8},
\]

where

\[
X = \sinh^2 2K^*/\sinh^2 2K
\]

\[
= \frac{p^2}{[(p^2 - 1)^2 + 2(p^{8r-1} - 1)z^{r-1}]^{1/2}}
\]

\[
/\left[ (p^2 - 1)^2 + 2(p^{8r-1} - 1)z^{r-1} \right]^{1/2}
\]

and \( p = e^{2K} \). For the partition function, a similar simple property has not been found yet. The blocked Ising model can be related to the eight vertex model.\(^8\) Therefore the duality is also satisfied by the eight vertex models with specific vertex weights \( \{e_i; i = 1, \ldots, 8\}; e_1 = -7J, e_i = J \) for other \( i \).

Acknowledgements

The author would like to thank Professor I. Syozi for suggesting this problem, and Mr. H. Nakanishi at H. E. Stanley’s Laboratory for new informations and useful discussions.

Appendix

There exists a correspondence between a configuration \( \theta \) for a blocked lattice and \( \theta^* \) for the dual-blocked lattice. Configuration \( \theta \) may be decomposed into clusters, \( \{C_i\} \). The correspondence between \( \theta \) and \( \theta^* \) can be proved for each cluster. In Fig. 6, we define an active element in a block for a sign configuration of the complementary faces, where the sign configuration is consistent with the block spin variable \( B^*_j \) (see § 4) for the dual block to be \( -1 \). We call the sign configuration the permissive sign configuration for the block. The total active elements for the whole blocks in a cluster make up a non-vanishing sub-configuration, since the total wavy boundary (see Fig. 6) separates the complementary faces into some parts in each of which the complementary faces has the same sign. Inversely, for a non-vanishing sub-configuration, we can determine the permissive sign configuration except the whole change of signs, so as to be the same sign for each parts separated by the wavy boundary. Thus we get the one to one correspondence
A Duality between Blocked Ising Models

between a permissive sign configuration and a non-vanishing sub-configuration for each cluster $C_i$:

$$d_i(\theta) = d_i^*(\theta^*), \quad (A.1)$$

and we get

$$D(\theta) = \prod_i d_i(\theta) = \prod_i d_i^*(\theta^*) = D^*(\theta^*). \quad (A.2)$$

Fig. 6. Correspondence between an active element of a block and a sign configuration of the complementary faces around the block. The active element is expressed by a wavy line for two-spin product or a shaded area for four- or more-spin product. The wavy lines for the shaded area show the boundaries when the plus signs are considered to be outside.

References


2) H. A. Kramers and G. H. Wannier, Phys. Rev. 60 (1941), 252.  


Y. Kasai, to be published.

