cylinder and hollow sphere with different temperatures at the two free surfaces, and for heating hollow cylinders and spheres from outside to inside, as well as in the opposite sense. These cases will be treated in a later paper on the subject.

**ACKNOWLEDGMENT**

The author is indebted to Dr. J. W. Calkin and to Dr. R. Oldenburger for some mathematical advice.

**BIBLIOGRAPHY**


**Discussion**

E. S. Davis. The influence of nonuniform development of heat on the temperature distribution in bodies of simple shape is of rather great engineering interest.

The author has ably covered the case where the temperature coefficient of conductivity is linear. However, there are many materials whose coefficients are not linear. Let us consider the case where the coefficient $\frac{d\varepsilon}{dt}$ may be expressed by the form

$$1 + a\vartheta + B\vartheta^2$$

The basic equation governing the case of the flat plate is still

$$\frac{d^2\theta}{dx^2} = \frac{q'''}{k}$$

If we represent

$$\frac{q'''}{k} = m (1 + a\vartheta + B\vartheta^2)$$

then

$$\frac{d\vartheta}{dx} = m (1 + a\vartheta + B\vartheta^2)$$

Representing the temperature at the median line by $T_m$, at the surface by $T_s$, with $z$ at the surface = 0, and $x = s$ at the median, we may express the boundary conditions as $\theta = (T - T_m) = 0$, and $\frac{d\theta}{dx} = 0$ at $z = s$, and $\theta = (T_s - T_m)$ at $x = 0$.

Multiplying Equation [68] by $d\vartheta$ and integrating

$$\int_0^s \frac{d\theta}{dx} \frac{d\vartheta}{dx} \frac{d\theta}{dx} = \int_0^s m (1 + a\vartheta + B\vartheta^2) d\theta$$

then integrating again

$$\int_0^x \frac{d\theta}{dx} = \sqrt{2m} \int_s^x \frac{d\vartheta}{dx}$$

where $\mu$ and $\nu$ are roots of

$$\frac{B\vartheta^3}{3} + \frac{a\vartheta}{2} + 1 = 0$$

or

$$\frac{(\mu, \nu)}{4} = -\frac{3a}{4B}$$

Here, $m$ represents one of the Legendre, or more precisely, the Jacobi functions.

This solution shows the proper procedure to take in case the thermal conductivity varies greatly from a linear function. The same type of analysis may be applied to the case of the cylinder and the sphere, if the necessity arises.

R. H. Norris. The author has made a useful contribution in reassuring us that no serious error is introduced by the customary neglect of the effect of nonuniformity of heat generation on the calculated temperature rise of electrical coils.

His analysis provides a convenient basis for a discussion of the ratio of the temperature rise, calculated by his new "exact" method, to the rise calculated by the common, less accurate, method which assumes uniformly distributed heat generation. This ratio is of most practical significance when it is based on the same total heat dissipation for both methods. Since its value on this basis, however, does not seem directly evident from the author’s curves, it is discussed more explicitly in the following paragraphs:

The total heat dissipation is proportional to the slope, at the coil surface, of the curve of temperature distribution, as plotted, for example, in Fig. 1 of the paper. The slopes of the highest and lowest curves shown in Fig. 1, differ by a ratio of 1.27 to 1, for the same maximum temperature.

For the extreme case here considered, this means (for reasons explained later) that a 27 per cent error would be introduced in the calculated maximum temperature rise, if the customary approximate method were used, instead of the “exact” method when assuming the same total heat dissipation for both methods. (This extreme case is the one for which the parameter $a$ has the value $a/2$.)

In the range of practical interest, however, the corresponding error becomes practically negligible. It is only 5 per cent for a case more unfavorable than any likely to be encountered in practice, namely, the case in which the temperature rise from

---


5 "General Engineering Laboratory, General Electric Company, Schenectady, N. Y. Jun. A.S.M.E."
surface to center is assumed 100 C, and the temperature coefficient \( \varepsilon \), assumed as high as 0.00364 (corresponding to surface temperature of 40 C). The value of the parameter \( \varepsilon \), corresponding to these conditions, can be shown to be 0.748.

For the general case, a convenient expression can be derived, from the author's results, to express the ratio of the temperature rise, by the "exact" method to the rise by the customary approximate methods. This ratio, for a given total heat generation, is given by the equation:

\[
\frac{(\theta)_{\text{exact}}}{(\theta)_{\text{approx}}} = 2 \frac{1 - \cos \varepsilon}{\sin \varepsilon}
\]

The figures of 27 per cent and 5 per cent discrepancy previously mentioned, were obtained from this expression, since it gives 1.27 and 1.05 for \( \varepsilon = \pi/2 \), and 0.748, respectively. This expression is obtainable by differentiation, with respect to \( \xi \), of the author's Equation [16], and evaluation of the resulting expression for the value \( \xi = 1 \). The values thereby obtained correspond to the relative temperature gradients at the surface, and hence to the relative heat dissipation for a given maximum temperature or, conversely, to the relative maximum temperature for a given heat dissipation.

As regards effective, or average, thermal conductivity of an array of round electrical conductors embedded in homogeneous insulation, data have been made available by R. Richter. It would seem that these data might be better than the author's approximations, for evaluation of the thermal conductivity for the test sample. These data of Richter are also available in a book by A. D. Moore.

Rufus Oldenburger. This paper fills a long existing need for an accurate understanding of the temperatures in electrical coils, which understanding will prove of value in preventing failures caused by the overheating of conductors. The author emphasizes a technique, so important in modern engineering, of treating problems from different branches of engineering by one method applicable to all of these problems, because the underlying mathematical laws are identical.

That the solution of the differential equation for the temperature difference \( \theta \) yields, in general, positive values of \( \theta \) for only certain values of \( x \) in the range \( 0 < x < \infty \) is somewhat surprising since a value of \( \theta \) has physical meaning only if it is positive or zero. For the simplicity of the analysis let us restrict ourselves to the author's solution of the infinitely wide plane plate. Analogous remarks will hold for the other coils treated. With the differential Equation [7] of the paper, and the boundary conditions Equations [12], [13], the solution Equation [14] is obtained, which can be shown by mathematical methods to be the unique solution for the type of functions considered. Since the differential Equation [7] is fundamental, and has been thoroughly verified by theory and experiment, the nonnegativity of \( \theta \) in Equation [14] can be due only to the choice of the boundary conditions. The value of \( \theta \) is negative, for example, when

\[ \cos \varepsilon \phi < \cos \phi, \cos \phi > 0, 0 < \phi \leq 1 \]

These inequality relations can be satisfied for a proper choice of \( \varepsilon \) and \( \phi \) in the ranges.

One of the boundary conditions, namely, Equation [13], is the mathematical equivalent of the property that the surface of the coil is maintained at a constant temperature. Clearly, Equation [13] is indispensable to a description of the physical situation, and cannot be discarded or modified. This brings us to Equation [12], which states that, at the median plane of the plate, where \( \xi = 0, (x = 0) \), the temperature attains a critical value, which, when the solution Equation [14] is obtained, turns out to be the maximum value of \( \theta \). For \( \theta \) to take on its maximum at \( \xi = 0 \), it is not, however, necessary to have

\[ \frac{d\theta}{dx} = 0 \quad \frac{d\theta}{d\xi} = 0 \]

It seems quite likely that the derivative with respect to \( x \) of the solution for \( \theta \) in the problem in question does not vanish at \( x = 0 \), at least for the cases where Equation [14] yields negative values of \( \theta \).

If one allows \( x \) to be negative, the condition Equation [12] is not needed to obtain the author's solution Equation [14]. If we write \( \theta = \Theta (x) \), the symmetry assumption of the author's paper makes Equation [12] superfluous since \( \Theta (-x) = \Theta (x) \) implies that \( N = 0 \), which is all that one obtains from Equation [12]. If \( x \) is not allowed to take on negative values, some condition like Equation [12] is needed to determine \( N \). The author's development is, however, valid for negative as well as positive values of \( x \).

Since by physical considerations, the solution \( \theta \) of Equation [7] and its derivatives should be continuous for \( 0 \leq x \leq 0 \), the solution of Equation [7] will in any case be given by Equation [11]. Also, by Equation [13] we have

\[ M \cos \varepsilon + N \sin \varepsilon = m/n \]

If \( \cos \varepsilon \neq 0 \), solving for \( M \) we have

\[ \theta = \frac{m}{n} \left( \frac{\cos \xi \phi - 1}{\cos \phi} \right) + N \left( \sin \xi \phi - \cos \xi \phi \tan \phi \right) \]

From physical considerations (See Table 2 of the paper) it appears that \( N \) should be chosen so that for \( x \geq 0 \) (i.e., \( \geq 0 \)) we have \( \theta > 0 \), and \( \theta \) decreasing uniformly from its value at \( x = 0 \) to its value at \( x = x \) is \( \theta > 0 \), that is, in the range \( x \geq 0 \), we must have \( d\theta/dx < 0 \). We remark that Equation [7], and \( \theta > 0 \), \( n > 0 \) imply that

\[ d^2\theta/dx^2 < 0 \]

whence the \((x, \theta)\) curve must be concave down over the range

\[ 0 < x < 0 \]

The relations \( \theta > 0 \) and \( d\theta/dx < 0 \) are equivalent to the pair of inequalities

\[ N \left( \cos \xi \phi + \sin \xi \phi \tan \phi \right) > \frac{m}{n} \left( 1 - \cos \xi \phi \right) \]

\[ N \left( \cos \xi \phi - \sin \xi \phi \tan \phi \right) < \frac{m}{n} \cos \xi \phi \]

which are linear in \( N \). For each value of \( N \) which satisfies the foregoing inequalities, the solution obtained for \( \theta \) in terms of \( N \) is one of the desired type. A mathematical study should be made of these inequalities in \( N \) to determine whether or not a solution \( N \) can always be found, and if so, among the possible solutions which value of \( N \) fits the actual physical situation. The value \( N = 0 \), obtained by the author in the cases he considered, satisfies the inequalities mentioned, and is uniquely determined by the property that it yields for \( d\theta/dx \) the smallest possible numeri-
It is gratifying that the present paper has been discussed from such different viewpoints. The author was pleased that even a mathematician such as Mr. Oldenburger found certain results of the paper to be somewhat surprising. In his analysis, the fact that finite temperatures are possible only for rather restricted values of the thickness of the coil is interpreted from the "physical situation," i.e., from the boundary conditions. He further states that, from a mathematical viewpoint, the practical ease of the coil is only a special ease of a wider mathematical complex covered by the differential equations used in the paper. The author, used to looking at a problem from a physical rather than from a mathematical viewpoint sees the solution of the puzzle in the fact that at any place of the coil an increase of temperature causes an increase of the electrical resistance, and an increase of the electrical resistance causes an increase of temperature if the electrical current is not changed. Hence only the relative strength of these two effects is decisive for the question whether the temperature increase is stopped at a finite temperature or never.

Mr. Davis deals with the case of nonlinear dependence of $q''/k$ on the temperature and gives a solution for a dependence of second order. On a former occasion the author has mentioned that for the relation

$$\frac{d\psi}{dx} = -\frac{q''}{k} = \psi(\theta)$$

where $\psi$ is an arbitrary function of $\theta$, the general solution is

$$x = \int \frac{d\theta}{\sqrt{C_1 + 2f\psi(\theta)d\theta}} + C_2$$

with the constants of integration $C_1$ and $C_2$. This corresponds to Davis' Equation 72. However, from a practical standpoint it seems to be preferable to replace the nonlinear function $\psi$ by two or more linear sections as an approximation and to apply to these the simpler solution for linear distribution as given in the paper.

The author had an opportunity to see Mr. Tramontini's unpublished manuscript. In fact, Equation [31] has been derived there for the case of constant thermal conductivity. Certainly the writer will not have expected any considerable temperature difference in the cross section of the fine platinum wire under consideration except for excessively strong heating.

Concerning Mr. Paschkis' discussion the author refers to the second sentence of Part 1 of his paper. According to that statement he is confident that Rogowski and Vieweg's coil with 1416 windings in a cross section of 3 sq in. can be considered as a quasi-homogeneous heat source and conductor and that application of the theory to this and similar cases gives satisfactory results. Therefore the author would be glad if Mr. Paschkis were right with his carefully formulated judgment that the method seems to be more fit for chemical reactions than for electrical coils. However, the author agrees with Mr. Paschkis in so far as a more exact experimental check of the theory than just by the tests of Rogowski and Vieweg would be desirable.

The author learned from Mr. Norris' discussion that Richter and Moore have determined the equivalent thermal conductivity using a graphical method for drawing the flow lines in the insulation. According to their curves the ratio of equivalent thermal conductivity to the conductivity of the insulation material of Rogowski and Vieweg's coil would be 3.6, instead of 5.7 as found by the writer's less exact method. This agreement is very satisfactory.
Another item discussed by Mr. Norris is the case of a fixed heat dissipation which is essential for the precalculation of a coil. There is another practical case, namely, to find the maximum temperature of a given coil for a given amperage. Measurements of the electrical resistance of the coil, cold and under load, yield the mean temperature excess $\theta_m$. The maximum temperature excess $\theta_m = \theta_m / \phi$. For the case of the plane plate, the old theory gave $\phi = 2/3$, throughout. The new theory gives values of $\phi$ between $2/3$ and $2/\pi$.

For the case dealt with by Norris, Equation [24] yields $\theta_m = 66$°C. Then, according to the old theory, $\theta = 1.5(99) = 99$°C instead of 100°C, according to the new theory. The difference in this procedure is so small because here $(d\theta/d\xi)_{\xi = 1}$ is not the same for both theories. Applying the old theory, $d\theta/d\xi = -2\theta = -2(99) = -198$°C; applying the new theory, $d\theta/d\xi = -\sigma \cdot \sin \sigma \over 1 - \cos \sigma = -1.91(100) = -191$°C. This shows that, using the old theory for calculating $\theta$ from $\theta_m$ a 3½ per cent too steep temperature gradient at the surface has been supposed; however, Fig. 1 shows that with a steeper gradient at $\xi = 1$ and parabolic temperature distribution the same point as in the new theory may be reached at $\xi = 0$, that is $\theta_m$ may come out all right. Assuming a fixed value of $\theta_m$ namely, the one found by measurement, the temperature-distribution curves for the old and new theory in a $\theta,\xi$-diagram will intersect between $\xi = 0$ and $\xi = 1$.

It is seen that engineers neglecting in their calculations the influence of nonuniformity of heat generation in a coil, had just luck; they scarcely will have foreseen that the two errors involved, in using the old theory, namely, too steep temperature drop at $\xi = 1$, and parabolic instead of cosinoidal temperature distribution, might cancel one another almost entirely. However, it is comforting that this happens, as Mr. Norris remarks at the beginning of his discussion.