rical configuration of Bleuler's equipment on the measured characteristics of valve, such as torque and discharge coefficients, cannot be decided on the basis of existing experimental results. According to Bleuler, "the form of the cavity due to cavitation shows that outside the safety limit the distribution of pressure on the suction side must be practically constant" and that "it is the pressure on the delivery side only which primarily influences the magnitude of the torque."

In closing, it should be emphasized that the intention of the writer's paper was, first, to give a mathematical meaning to the analysis of butterfly valves through the application of the admittedly idealized and somewhat restricted free-streamline theory, and, second, to compare wherever possible the results thus obtained with those obtained experimentally. Application of the free-streamline solution results to butterfly valves and comparison with experiments were made on the basis of clearly stated hypotheses without resorting to "various stratagems."

Experiments which will be conducted in the future will undoubtedly shed more light on the extrapolation of free-streamline solution results to three-dimensional flows, on the effects of back pressure, valve configuration, and other variables on the development of cavitation. Already at hand is convincing evidence of a surprisingly close correspondence between these two classes of flows, at least insofar as certain bulk characteristics are concerned. The significance of the various restrictions and the degree of confidence in the results obtained can be determined only by comparing the calculated results with those obtained from observations of real flows.

## The Integration of Some Expressions Arising in Beam Theory[1]

**L. Y. BAHAR.**[2] The author is to be commended for his contribution to a lively and recently much discussed subject. It seems to the writer, however, that the integration by parts indicated in this paper is of limited practical value. The integral

$$\int_0^l \{x - a\}\{x - c\} dx$$

with $a > c$ written in terms of "Macauley brackets," readily reduces to

$$\int_a^l (x - a)(x - c) dx$$

where the brackets have been replaced by parentheses in the ordinary sense. This expression can now be integrated easily. Integration by parts can be retained if desired, since this makes the resulting expression vanish at the lower limit.

### Author's Closure

I wish to thank Mr. Bahar for his interest and comment. In the second example of my Note there are, after integration, seven terms to be evaluated. If the Macauley brackets are retained, all terms are evaluated for the same limits. If the Macauley brackets are dropped, the limits are, for the first term 0 and $l$, for the next two terms $c$ and $l$, and for the remaining four terms $a$ and $l$. Thus the retention of the Macauley brackets makes the computation simpler and less liable to errors. The advantage may be marginal, but it appeared worthwhile to draw attention to it.

[1] By A. Ormerod, published in the December, 1961, issue of the Journal of Applied Mechanics, vol. 28, Trans. ASME, vol. 83, Series E, p. 633.

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## Dynamics of Nonholonomic Systems[1]

**T. P. MITCHELL.**[2] The method of treating nonholonomic problems which the author proposes is not new, but is an application of the principle of virtual work in which the presence of linear constraint equations is handled by direct algebraic elimination rather than by the use of Lagrange multipliers. This can be seen, in the author's notation, as follows:

A virtual displacement of the $i$th mass is

$$\delta \mathbf{p}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{p}_i}{\partial q_j} \delta q_j + \frac{\partial \mathbf{p}_i}{\partial t} \delta t \tag{1}$$

The nonholonomic constraint equations are

$$\sum_{r=1}^{n} A_{rs}\delta q_r + B_s \delta t = 0; \quad s = 1 \ldots m \tag{2}$$

D'Alembert's generalization of the principle of virtual work states that

$$\sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{F}_i')\cdot \delta \mathbf{p}_i = 0 \tag{3}$$

for virtual displacements compatible with the instantaneous constraints. For such displacements, one can solve equation (2) for, say, the final $m$ of the $\delta q_r$ to find

$$\delta q_{\bar{n}+s} = \sum_{r=1}^{\bar{n}} \delta q_r C_{r,\,\bar{n}+s}; \quad s = 1 \ldots m$$

and, from equation (1),

$$\delta \mathbf{p}_i = \sum_{r=1}^{\bar{n}} \left( \frac{\partial \mathbf{p}_i}{\partial q_r} + \sum_{s=1}^{m} \frac{\partial \mathbf{p}_i}{\partial q_{\bar{n}+s}} C_{r,\,\bar{n}+s} \right) \delta q_r \tag{4}$$

where $\bar{n} = n - m$. The coefficient of $\delta q_r$ in (4) is the quantity which the author denotes by $\mathbf{v}_{,q_r}{}^{p_i}$. On substituting from equation (4) in equation (3) one obtains

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{r=1}^{\bar{n}} \mathbf{v}_{,q_r}{}^{p_i}\delta q_r + \sum_{i=1}^{N} \mathbf{F}_i' \cdot \sum_{r=1}^{\bar{n}} \mathbf{v}_{,q_r}{}^{p_i}\delta q_r = 0 \tag{5}$$

However, in equation (5) the $\delta q_r$ are independent, and so

$$K_{qr} + K_{qr}' = 0; \quad r = 1 \ldots \bar{n} \tag{6}$$

which is the author's equation (19) and his principal result.

If the forces of constraint are such that they do no work in a virtual displacement, $\delta \mathbf{p}_i$, which is compatible with the instantaneous constraints, then these forces do not appear in equation (6). This is the content of the author's development immediately following and including his equation (20).

The approach to dynamics problem, under discussion, is unsatisfactory for two reasons: (*a*) It is not a true variational principle, and (*b*) it does not treat the generalized coordinates $q_i$ symmetrically. These drawbacks are eliminated by proceeding from equation (3) to Hamilton's principle and by using Lagrange multipliers to incorporate equations (2).

**HERMAN SCHAEFER.**[3] The author cites Whittaker's "Treatise on the Analytical Dynamics of Particles and Rigid Bodies" (Dover Publications, 1937). This famous work dates nearly unchanged from 1904. As to nonholonomic systems, it is unfortunately out

[1] By T. R. Kane, published in the December, 1961, issue of the Journal of Applied Mechanics, vol. 28, Trans. ASME, vol. 83, Series E, pp. 574–578.

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