Testing a palaeomagnetic study for the averaging of secular variation

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Summary. It is shown that the interval estimate for $K$ given by Cox (1969) is inappropriate as a test for determining whether secular variation has been averaged out in a particular study. The appropriate test is presented in two formulations, one for use with $\chi^2$ tables and the other for use with $F$ tables.

The well known secular variation models are investigated and it is shown that Model A due to Irving & Ward (1964) is, statistically, the most efficient predictor. Finally, for the 83 palaeomagnetic results selected by Brock (1971), it is shown that the most efficient predictor for the secular variation $K$ is $k' = 18.1 \left(1 + 3 \sin^2 \lambda \right)$, where $\lambda$ is the palaeolatitude, and the uncertainty in this predicted value is discussed.

1 Introduction

In the analysis of palaeomagnetic data it is frequently of interest to determine whether secular variation has been averaged out in a particular study. This problem has usually been approached by determining the value of the Fisher precision parameter, $\kappa$, predicted by a model of palaeosecular variation (e.g. Irving & Ward 1964; Creer 1962; Cox 1970) and then checking whether this lies within the limits, given by Cox (1969), of $\kappa$ for a study. This test is inappropriate in that Cox's (1969) test gives the interval estimate for $\kappa$ deduced from a (statistical) sample; it does not test the hypothesis that the observed estimate of $\kappa$ could have been achieved by random sampling from a population with a prescribed value of $\kappa$.

It is the intention here to set out the appropriate statistical test. Since this requires consideration of the secular variation models the well known models are discussed with regard to their ability as predictors for comparisons with observed estimates of $\kappa$.

2 Testing against a prescribed value of $\kappa$

Most statistical analyses of palaeomagnetic data assume the distribution of Fisher (1953). Watson (1956) has shown for this distribution that over a wide range of practical cases (see McFadden (1980) for a discussion of the range of validity) the statistic $2\kappa(N - R)$ is $\chi^2$ distributed with $2(N - 1)$ degrees of freedom, i.e.

$$2\kappa(N - R) = \chi^2_{2(N - 1)}$$  \hspace{1cm} (1)

where $N$ is the number of unit vectors observed and $R$ the length of their vector sum.
From equation (1) it may be shown (McFadden 1980) that \(1/k\) is an unbiased estimate for \((1/K)\) when
\[
k = \frac{N - 1}{N - R}.
\]

With this definition
\[
\frac{K}{k} = \frac{2(N - 1)}{\chi^2(N - 1)},
\]
the relation used by Cox (1969) to determine the interval estimate for \(k\). Here, however, it is required to test the null hypothesis that \(\kappa_S = \kappa\), where \(\kappa_S\) is the precision of the population from which the sample is drawn and the prescribed precision is \(\kappa\). There are then three alternative hypotheses and these are discussed below.

2.1 ALTERNATIVE HYPOTHESIS \(\kappa_S \neq \kappa\)

If the observed value of \(k\) exceeds \(k_u^*\), where
\[
k_u^* = \frac{2\kappa(N - 1)}{\chi^2[2(N - 1); (1 - \alpha/2)]},
\]
or is less than \(k_l^*\), where
\[
k_l^* = \frac{2\kappa(N - 1)}{\chi^2[2(N - 1); (\alpha/2)]},
\]
then reject the hypothesis \(\kappa_S = \kappa\) with probability \((1 - \alpha)\) and accept the alternative \(\kappa_S \neq \kappa\). Here \(\chi^2[n; p]\) is the value of \(\chi^2\) with \(n\) degrees of freedom which will be exceeded with a probability \(p\). The ratios given by this alternative are the inverses of those given by Cox (1969).

2.2 ALTERNATIVE HYPOTHESIS \(\kappa_S > \kappa\)

If the observed value of \(k\) exceeds \(k_u\), where
\[
k_u = \frac{2\kappa(N - 1)}{\chi^2[2(N - 1); (1 - \alpha)]},
\]
then reject the hypothesis \(\kappa_S = \kappa\) with probability \((1 - \alpha)\) and accept the alternative \(\kappa_S > \kappa\).

2.3 ALTERNATIVE HYPOTHESIS \(\kappa_S < \kappa\)

If the observed value of \(k\) is less than \(k_l\), where
\[
k_l = \frac{2\kappa(N - 1)}{\chi^2[2(N - 1); \alpha]},
\]
then reject the hypothesis \(\kappa_S = \kappa\) with probability \((1 - \alpha)\) and accept the alternative \(\kappa_S < \kappa\).

In each case if the hypothesis is not rejected then it is accepted that \(\kappa_S = \kappa\).

2.4 CHOICE OF ALTERNATIVE

In a given study if the observed value of \(k\) is larger than the prescribed value of \(\kappa\) then the question requiring an answer is usually 'did the samples acquire their magnetization over a
period long enough to average out secular variation?’. This requires the one-tailed alternative hypothesis that $\kappa_S > \kappa$. Conversely, if the observed value of $k$ is less than the prescribed value of $\kappa$ then the question requiring an answer is usually ‘did something happen which caused a dispersion in excess of that produced by secular variation alone?’. This requires the one-tailed alternative hypothesis that $\kappa_S < \kappa$. Thus one of the two one-tailed tests is generally the correct test, rather than the two-tailed test.

2.5 FORMULATION AS AN $F$-TEST

From equation (3)

$$\frac{\kappa}{k} = \frac{\chi^2(N-1)}{2(N-1)} = F'[2(N - 1); \infty]$$

(8)

where $F[n; m]$ is a random variate $F$ distributed with $n$ and $m$ degrees of freedom respectively. Similarly

$$\frac{k}{\kappa} = \frac{2(N-1)}{\chi^2(N-1)} = F[\infty; 2(N - 1)].$$

(9)

Hence if $CF[p; n; m]$ is the value of $F[n; m]$ which will be exceeded with probability $p$ then the tests may be formulated as follows.

2.5.1 Alternative hypothesis $\kappa_S \neq \kappa$

If

$$\frac{\kappa}{k} > CF[(1 - \alpha/2): 2(N - 1); \infty]$$

(10)

or if

$$\frac{k}{\kappa} > CF[(1 - \alpha/2): \infty; 2(N - 1)]$$

(11)

then reject the hypothesis $\kappa_S = \kappa$ with probability $(1 - \alpha)$ and accept the alternative that $\kappa_S \neq \kappa$.

2.5.2 Alternative hypothesis $\kappa_S > \kappa$

If

$$\frac{\kappa}{k} > CF[(1 - \alpha): \infty; 2(N - 1)]$$

(12)

then reject the hypothesis $\kappa_S = \kappa$ with probability $(1 - \alpha)$ and accept the alternative that $\kappa_S > \kappa$.

2.5.3 Alternative hypothesis $\kappa_S < \kappa$

If

$$\frac{k}{\kappa} > CF[(1 - \alpha): 2(N - 1); \infty]$$

(13)

then reject the hypothesis $\kappa_S = \kappa$ with probability $(1 - \alpha)$ and accept the alternative that $\kappa_S < \kappa$. 
Equations (10), (11), (12) and (13) are identical to equations (4), (5), (6) and (7). However, with the $F$ formulation the required ratio is available immediately from $F$ tables whereas the value obtained from $\chi^2$ tables must be substituted into the appropriate formula to obtain the ratio.

From equations (12) and (13) it is evident that the test used by Gough, Opdyke & McElhinny (1964) is identical to the two one-tailed tests given here. Thus the difference, noted by Cox (1969), between these ratios and the ratios given by Cox (1969) is to be expected since they give the answer to distinctly different questions.

3 Angular standard deviation

The most widely used measure of palaeosecular variation is the angular standard deviation, which is the semi-angle of a cone about the true mean direction within which 63 per cent of the population lies. If the population is Fisherian then for $\kappa > 3$ this angle, $\theta_{63}$, is given by

$$\cos \theta_{63} = 1 + \frac{\ln 0.37}{\kappa}. \quad (14)$$

When $\theta_{63}$ is small the solution is closely approximated by

$$\theta_{63} = \frac{81}{\kappa^{1/2}} \quad (15)$$

with $\theta_{63}$ in degrees.

Generally the statistic $s$, given by

$$s = \frac{81}{\kappa^{1/2}} \quad (16)$$

with $\kappa$ defined as in equation (2), is used as an estimate of $\theta_{63}$. Unfortunately this estimate is biased, as is now shown. From equation (1), using $E[z]$ to denote the expectation value of $z$,

$$E[\sqrt{2\kappa(N-R)}] = E[\sqrt{\chi^2_{2}(N-1)}] \quad (17)$$

and it may easily be shown that

$$E[\sqrt{\chi^2_{2}}] = \frac{2^{1/2}\Gamma[(f+1)/2]}{\Gamma[f/2]} \quad (18)$$

where $\Gamma[z]$ is the gamma function of $z$. Thus from equations (17) and (18)

$$E\left(\frac{\Gamma[N-1]^{\sqrt{N-R}}}{\Gamma[N-\frac{1}{2}]}\right) = (1/\kappa)^{1/2}. \quad (19)$$

Hence if

$$k^* = \left(\frac{\Gamma[N-\frac{1}{2}]}{\Gamma[N-1]}\right)^2 \frac{1}{N-R} \quad (20)$$

then

$$E(s^*) = E\left[\frac{81}{(k^*)^{1/2}}\right] = 81/(\kappa)^{1/2} \quad (21)$$
and $s^*$ is an unbiased estimate of $\theta_{63}$. A good approximation for $k^*$ is given by

$$k^* = \frac{N - 1\%}{N - R}. \quad (22)$$

The error in this approximation is less than 1 per cent for $N$ greater than 2 and less than 0.1 per cent for $N$ greater than 6. Now $k$ is always greater than $k^*$ and hence $s$ is always less than $s^*$. Thus $s$ is a biased estimate for $\theta_{63}$, the bias being towards a value which is too small. In consequence a curve fitted to experimental values of $s$ will predict a value of $\kappa$ which is too high.

Evidently this bias may be overcome by fitting the secular variation model to experimental values of $s^*$. However, the most common estimate for $\kappa$ is $k$, so this would be inconvenient. The best way to overcome the bias is simply to fit the secular variation model to experimental values of $s^*$ since

$$E(s^2) = \frac{81^2}{\kappa} = \frac{81^2}{k} = \theta_{63}. \quad (23)$$

Further, the distribution of $k$ for a given $\kappa$ is a tabulated function whereas the distribution of $k^*$ is not. This is useful when considering the uncertainty of the predicted value of $\kappa$.

4 Analysis of secular variation models

The three models which will be examined are Model A due to Irving & Ward (1964), Model B due to Creer, Irving & Nairn (1959) and Model D due to Cox (1970). Model C due to Cox (1962) will not be considered since it has been superseded by Model D.

Brock (1971) used 83 selected palaeomagnetic results in his experimental study of palaeo-secular variation. Those same results are used in this analysis. Brock's (1971) analysis was aimed at discerning the causes of secular variation whereas the analysis here is aimed at discriminating between the models on the basis of their ability to predict a value for $\kappa$ both accurately and precisely.

Model A considers a geocentric axial dipole field of fixed dipole moment with secular variation caused by a component of fixed magnitude but random direction which perturbs the dipole field. The predicted latitude variation is given by

$$s = s_e(1 + 3 \sin^2 \lambda)^{-1/2} \quad (24)$$

where $s_e$ is the value of $s$ at the equator and $\lambda$ is the latitude. Brock (1971) performed a fit to this model by calculating the value of $s_e$ for each datum and then averaging to obtain the fitted value of $s_e$. As previously discussed this leads to a biased estimate for $\kappa$ and is thus a poor fitting procedure for a predictive model. However, from this procedure, Brock (1971) was able to show that $s_e$ is not dependent upon $\lambda$ and thus suggested that $s_e$ is a useful parameter for describing secular variation.

Model B postulates a wobble of the main dipole, the wobble distribution being Fisherian. No appeal is made to non-dipole components and the predicted latitude variation is given by

$$s = S_D \left( \frac{5 - 2 \sin^2 \lambda}{2(1 + 3 \sin^2 \lambda)} \right)^{1/2} \quad (25)$$

where $S_D$ is the angular standard deviation of the dipole wobble. Using a fitting process exactly as for Model A, Brock (1971) found that $S_D$ is latitude dependent. It is therefore
P. L. McFadden concluded that $S_D$ is not a useful parameter for predicting secular variation and this model is not considered any further.

Model D allows variations caused by a non-dipole field and by dipole wobble. The predicted latitude variation is given by

$$s^2 = AX + BY$$

where

$$X = (1 + 3 \sin^2 \lambda)^{-1}$$

describes the variations due to the non-dipole field and

$$Y = \frac{5 + 3 \sin^2 \lambda}{5(1 + 3 \sin^2 \lambda)^2}$$

describes the variation due to dipole wobble.

Evidently the latitude variation predicted by Model A may be written as

$$s^2 = CX$$

It is then simply a matter of performing least squares analyses to determine the coefficients $A$, $B$ and $C$, from which unbiased estimates of $\theta_{23}^2$ may be obtained.

4.1 LEAST SQUARES ANALYSIS OF MODEL A

Using the 83 palaeomagnetic results selected by Brock (1971) the coefficient $C$ was found, from a least squares fit, to be 362.6, an unbiased estimate for the variance being 445.2 (82 degrees of freedom). The root mean square deviation of the data from the fitted curve is 134.6.

The scatter is large and in order to determine whether the criteria for a least squares analysis are met the residuals have been examined. These residuals, normalized to the root mean square deviation, are plotted in Fig. 1. It may be seen that the data are slightly heteroscedastic (i.e. the variance of the data is not constant but a function of $X$), the variance tending to increase with increasing $X$ (i.e. decreasing $\lambda$). Further, it is apparent that the model tends to overestimate $s^2$ for $X$ in the approximate range of 0.83–1.00 (i.e. for $\lambda$

Figure 1. Plot of the residual deviations, normalized to the root mean square deviation, against $X$ for the least squares fit to Model A.
from 0 to 15°) and to underestimate $s^2$ for $X$ in the approximate range 0.55–0.75 (i.e. for $\lambda$ from 20 to 30°). However the discrepancy between the observed deviation distribution and a perfectly homoscedastic (i.e. constant variance), normal distribution is not gross. Hence the model may be considered as reasonably acceptable.

4.2 LEAST SQUARES ANALYSIS MODEL D

Using the same 83 results $A$ was found to be 621.3 (unbiased estimate of variance = 13 389) and $B$ was found to be $-305.6$ (unbiased estimate of variance = 18 101), with now 81 degrees of freedom. The root mean square deviation of the data from the fitted curve is 130.5. The analysis indicates serious drawbacks to using Model D as a predictive model.

As $\lambda$ decreases monotonically from 90 to 0° both $X$ and $Y$ increase monotonically, $X$ increasing from 0.25 to 1.00 and $Y$ increasing from 0.10 to 1.00. Thus it is to be expected that these two explanatory variables should be highly correlated. For the data set used the correlation coefficient of $Y$ on $X$ is 0.9924. When extreme collinearity of explanatory variables occurs the estimates determined for the coefficients are still unbiased but the variances of the estimates are frequently inflated, sometimes grossly so. Comparison of the variance estimates for $A$ and $B$ with the variance estimate for $C$ shows this to be the case in this instance. With such extreme collinearity between $X$ and $Y$ it is to be expected that $(A + B)$ should be approximately equal to $C$, which is so. Thus, as a predictive model, Model D is virtually the same model as Model A except that when calculating the variance of the predicted value the covariance of the parameters $A$ and $B$ must be considered as well as their variances.

The residuals for the Model D fit are plotted in Fig. 2. A rough envelope has been drawn around the data in both Figs 1 and 2 to show the close similarity between the residual patterns, indicating again that Model D is little different from Model A. Assuming a normal distribution for the predicted value the maximum z-score for difference in the values predicted by the two models (this maximum occurs at $\lambda = 0$) is 1.06. When sampling from a normal distribution there is a 29 per cent chance of obtaining a z-score in excess of this and there is therefore no indication that the two models predict different values.

It is interesting to note that Brock's (1971) fit for $s$ with $A^{1/2} = 18.6$ ($A = 345.96$) and $B^{1/2} = 3.2$ ($B = 10.24$) gives a root mean square deviation of 135.0. The fact that the root

![Figure 2. Plot of the residual deviations, normalized to the root mean square deviation, against $X$ for the least squares fit to Model D.](https://academic.oup.com/gji/article-abstract/61/1/183/602077)
mean square deviation increases by so small an amount above the minimum for such large changes in the coefficients $A$ and $B$ is a further indication of the statistical inefficiency of Model D.

However, the most important feature is that the coefficient $B$ is negative. This implies a correlation between the dipole wobble and the non-dipole components which causes the dipole wobble variation to interfere destructively with the variation from the non-dipole components. The theoretical and physical basis of Model D quite definitely denies this possibility. Least squares analysis with the additional constraint of physical realizability (i.e. $B > 0$) gives the solution $B = 0$. This is, of course, simply Model A.

4.3 CHOICE OF MODEL

Model B has already been rejected on the basis of the analysis given by Brock (1971). The choice is therefore between Models A and D. On the basis of the analysis presented in the previous section Model D is rejected in favour of Model A as a predictive model. The reasons are summarized below.

1. Models A and D do not predict values significantly different from each other and Model A is far more efficient (statistically) than Model D.
2. A least squares fit of the data to Model D gives coefficients which are physically unrealistic.
3. A least squares fit to Model D with the additional constraint that the solution be physically realistic leads immediately to Model A.

5 Uncertainty in the predicted value of $\kappa$

In considering the uncertainty in a predicted value of $\kappa$ it is useful, for purpose of comparison with an observed value of $\kappa$, to determine the number of readings from a Fisher population which would give the same uncertainty in the observed value of $\kappa$. The observed value of $s^2$ obtained by taking $N$ readings from a Fisher population is, from equations (3) and (16), given by

$$s^2 = c\chi^2_f$$

(30)

where $c$ is a constant and $f = 2(N - 1)$. Assuming the least squares analysis to be valid the predicted value of $s^2$ will be normally distributed. Since the $\chi^2$ distribution is asymptotically normal it may be assumed (for a reasonably well defined fit) that the predicted value of $s^2$ is $\chi^2$ distribution with mean $\mu = cf$ and variance $\sigma^2 = 2c^2f$. Thus the equivalent number of readings, $N_e$, for the same uncertainty in the prediction of $s^2$ is given by

$$N_e = 1 + \frac{\mu^2}{\sigma^2}$$

(31)

Since neither $\mu$ nor $\sigma^2$ are known their estimates from the least squares fit must be used. For Model A, $\mu$ at a given point may be estimated by $\hat{s}^2(= CX)$ at that point and $\sigma^2$ by $X^2\text{var}(C)$, where var$(C)$ is the unbiased estimate of the variance of $C$. Thus from equation (31) the estimate of $N_e$ is given by

$$N_e = 1 + \frac{C^2}{\text{var}(C)}$$

(32)

For the data used here $C = 362.6$ and var$(C) = 445.2$ and so the estimated value of $N_e$ is 295. Thus in most cases the value predicted by Model A may be considered as well
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defined. However, if the number of readings, \( N \), in a sample is large (i.e. becoming comparable with 295) then the predicted value should be considered as an observed value of \( k \) from 295 observations and the two \( k \) values compared by the method given by Watson (1956).

6 Conclusions

The test presented by Cox (1969) gives the interval estimate for \( k \) deduced from a sample; it does not test the hypothesis that the observed value of \( k \) could have been obtained by random sampling from a population with a prescribed value of \( k \). The appropriate test has been formulated here in a manner that either \( \chi^2 \) of \( F \) tables may be used to determine the critical values. In the event of a one-tailed test being chosen (as will usually be the case) the test reduces to that used by Gough et al. (1964).

It has been shown that in order to determine the prescribed value of \( k \), data should not be fit to a secular variation model of the angular standard deviation, \( s = 81/k^{1/2} \), but rather to the square of this parameter, i.e. \( s^2 = 6561/k \). An analysis of the common theoretical models indicates that the most efficient model as a predictor for \( k \) is Model A, in the form

\[ s^2 = CX \]

where

\[ X = (1 + 3 \sin^2 \lambda)^{-1}. \]

A least squares fit using the 83 palaeomagnetic results selected by Brock (1971) gives \( C = 362.6 \) with an unbiased estimate of variance of 445.2. The distribution of the predicted \( s^2 \) indicates that it may be considered as an observed value obtained from 295 readings. Thus it is concluded that the most efficient predictor, \( k' \), for \( k \) is given by

\[ k' = 18.1 (1 + 3 \sin^2 \lambda) \]

and is equivalent to an observed value of \( k \) obtained from 295 readings.

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References

