Tides and tidal friction in a hemispherical ocean centred at the equator

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Summary. A numerical model is constructed of the tides in a hemispherical ocean centred at the equator. This is used to study the role of resonances and friction in determining the rms amplitude of the tide and the energy dissipated by the tides.

Without friction, all of the resonances of the ocean may be excited by the tidal forces but when a realistic amount of friction is present most of the resonances merge into a smooth background. Remaining are the ones whose velocity fields best match the tidal forces. These are also important in determining the energy dissipated by the tides.

When an energy decay time of 30 hr is used, the model is in good agreement with the tides of the North Pacific and with Lambeck's estimate of the energy dissipated by the $M_2$ tide. However, the energy dissipated is very dependent on two of the remaining resonances lying within two radians per day of the tidal band.

If the remaining resonances were distributed in a more random fashion an energy dissipation rate of one third of the calculated value is not impossible. If this were also true for the world's oceans, it would put the Gerstenkorn event near the time of formation of the solar system.

1 Introduction

In modelling oceanic tides, one has to compromise between realism and practicality. Large 'realistic' models of the tides take many hours of computer time to run and so are not very practical for investigating the properties of the ocean. The present paper goes to almost the opposite extreme in using a model of a flat bottomed hemispherical ocean to investigate the effect of the resonances of the deep ocean on the tides. The model is also used to study the role of friction. The main drawback of the model is that it uses a rather artificial bottom friction. But the model needs only a few seconds of computer time to run and this allows many different aspects to be investigated.

The problem of determining the tides in an ocean bounded by a complete meridian, that is in a hemispherical ocean centred on the equator, was first solved in detail by Proudman and Doodson (Proudman & Doodson 1936; Doodson 1938). They looked at oceans of different depths but driven at the same frequency, and discussed the changes in the tidal
wave produced. They hoped that their work would give insight into the behaviour of the tides in the Pacific but ran into two important problems. The first was that small changes in depth produced large changes in the solution. Secondly, the maximum height of the forced tide was many times the height of the equilibrium tide. For example in Doodson’s solution for the semi-diurnal tide, the maximum height was up to 26.6 times the height of the equilibrium tide and was on average 10 times the height of the equilibrium tide. In contrast the tide in the actual deep ocean is rarely more than twice the height of the equilibrium tide.

The large amplitudes observed in Doodson’s hemispherical model were apparently due to resonances of the ocean. The natural frequency of each ocean resonance varies rapidly with the depth of the ocean (Longuet-Higgins & Pond 1970), and so the effect of each resonance on a fixed frequency also changes rapidly with the depth of the ocean.

In recent years, experience with models of tides in the actual ocean (Bogdanov & Magarik 1967; Pekeris & Accad 1969; Zahel 1970, 1977; Hendershott 1972) have shown that when a suitable frictional term is introduced, the resulting tidal height is reasonable, although the solution may still be sensitive to the actual depth or boundaries used. Thus in these models the resonances are still important but the friction has somehow tamed them.

It would be useful if these realistic ocean models could be used to learn more about the resonances of the deep ocean and to an extent this has been done, by Platzman (1975, 1978). But unfortunately the large models are very expensive to run and cannot be easily used to see how different factors interact. Therefore in this paper we approach the problem by going back to the computationally simple model of Proudman & Doodson.

However, before doing this there are at least two other reasons why the study of the tidal resonances is interesting. The first of these is concerned with the Gerstenkorn event (Gerstenkorn 1967; Lambeck 1977). At present the tidal wave on the rotating Earth applies a small torque to the Moon, accelerating it in its orbit. By this process angular momentum is transferred from the rotating Earth to the Moon. There is some energy left over in this transfer and so the angular momentum can only be transferred if the extra energy can somehow be dissipated by the tides.

As discussed by Lambeck, the rate of tidal dissipation can be estimated from measurements of ocean currents, from the acceleration of the Moon in its orbit, from the orbits of Earth satellites and from tidal models. At present the latter three methods give values for the dissipation rate of about $4 \times 10^{12}$ W.

This value can be used to extrapolate the Moon’s orbit into the past. Allowing for the fact that the tidal height and the dissipation rate increases when the Moon is nearer the Earth, this shows that the Moon would have been within 10 Earth radii about $1.5 \times 10^9$ yr ago (Munk 1968; Lambeck 1977; Brosche & Sundermann 1978).

At such a small separation the tidal deformation of the Earth and Moon would have been large and the extra tidal dissipation would have been enough to boil off the oceans. These effects might be expected to leave some trace in the geological record but no such traces have been found.

A possible explanation is that the dissipation rate in the present ocean is usually high and that on average, in the oceans of the past, it was much smaller. Thus if the dissipation rate was on average only a third of its present value, the period of closest approach would have occurred $4.5 \times 10^9$ yr ago, at the time of the formation of the solar system.

However, this is speculative, for at present we have no way of knowing how tidal dissipation, in an average ocean of the past, might behave. Thus any general principles which are indicated by studies of the tides would be useful.

A final and more practical reason for studying the properties of tides in the deep ocean concerns the problem of extracting energy from the tides. The usual way planned to do this
is to build a barrage across an estuary like the Severn Estuary or the Bay of Fundy where the tidal range is large.

The building of such a barrage may reduce the tidal range though, and make the project uneconomic. To check this it is usual to develop a model of the tides on the continental shelf near to the barrage. Unfortunately conflicting results can be obtained, depending on the model used (Heaps 1972; Heaps & Greenberg 1974; Greenberg 1977; Garratt & Greenberg 1977).

One reason for this is that the open boundary conditions used in the models are really trying to model the response of the rest of the ocean to the presence of the barrage. The different open boundary conditions used in the models then correspond to different models of the rest of the ocean.

What is needed is a better understanding of the properties of the tides in the deep ocean, to see if a simple but realistic open boundary condition for the shelf models can be obtained. If in the deep ocean only a few resonances affect the tides then it might be possible to devise a simple boundary condition to model the resonances. Alternatively if there are very many overlapping resonances then a boundary condition based on their average effect might be suitable.

With these problems in mind, it was decided to develop a model of the tides in an ocean, which was as realistic as possible and yet which was computationally quick to run, so that a large number of aspects could be investigated. A flat bottomed hemispherical model of the ocean was chosen as being suitable. It includes the curvature of the Earth, the changing coriolis force with latitude and it also includes coastal boundaries. The hemispherical model also has the advantage that it has already been studied by a number of other authors, including Proudman & Doodson (1936), Doodson (1938) and Longuet-Higgins & Pond (1970).

Drawbacks include the flat bottom, the straight coastlines and the north–south symmetry, all of which are rather artificial. The greatest drawback concerns the way tidal friction is introduced. In the real ocean much of the dissipation is believed to take place on resonant continental shelves. However, this cannot be easily modelled and instead a uniform linear bottom friction is used.

The model to be discussed is not going to solve completely any of the problems mentioned above. But the problems act as a rationale for the work and give us something to think about on the way.

2 The model

Using Laplace's tidal equations, we shall study the tides in a hemispherical ocean centred at the equator. The depth of the model ocean is taken to be 4400 m which is typical of a deep ocean like the Pacific (Sverdrup, Johnson & Fleming 1942). The Pacific is also similar to our model in that it is roughly circular in shape, but whereas the area of our model corresponds to approximately 80 per cent of the total area of the deep oceans, the Pacific makes up only 50 per cent.

We shall investigate how the model responds to the forces which correspond to equilibrium tides in the form of $Y_2^{-1}$ and $Y_2^{-2}$ spherical harmonics. These are the spherical harmonics which, when driven at one and two cycles per day, give the diurnal and semidiurnal tides respectively (Munk & Cartwright 1966; Webb 1976b). However, we shall investigate a range of frequencies to see, in particular, how the resonances affect the rms amplitude and the rate of energy dissipation by the tides. For convenience the spherical
harmonic equilibrium tides used to drive the model ocean are normalized so that they have a
rms amplitude of 1 m.

The behaviour of the tides is described to a good approximation by Laplace’s tidal
equations (Lamb 1932). With a linear friction term these are the momentum equation,

\[
\rho \frac{\partial u}{\partial t} = \kappa \rho f \times u + \rho g \nabla \zeta = \rho g \nabla \zeta,
\]

and the continuity equation,

\[
\frac{\partial \zeta}{\partial t} + \nabla \cdot (hu) = 0.
\]

In these equations \( u \) is the horizontal water velocity and is assumed to be uniform with
depth, \( \zeta \) is the tidal height and \( \zeta^e \) the height of the equilibrium tide. \( \rho \) is the density of water,
\( h \) the depth, \( \kappa \) the friction coefficient and \( g \) the acceleration due to gravity. Points within the
ocean are defined by their colatitude \( \theta \) and longitude \( \phi \). The ocean lies between two coast-
lines along longitudes 0 and \( \pi \).

If \( \Omega \) is the angular velocity of the Earth about its axis and \( \hat{r} \) is the unit radial vector,
then the coriolis vector \( f \) in the above equation is given by,

\[ f = 2\Omega \cos \theta \hat{r}. \]

The boundary condition is that there is no flow across the coastline. Thus on longitudes
0 and \( \pi \),

\[ u \cdot \hat{n} = 0, \]

where \( \hat{n} \) is the unit vector normal to the coastline.

**METHOD OF SOLUTION**

The problem can be solved by expanding the velocities and tidal heights in terms of
complete sets of simple functions and obtaining a matrix equation for the coefficients.
Proudman (1916), Proudman & Doodson (1936) showed that a convenient way to do this
is first to write the velocity as a time derivative of a horizontal displacement \( X \),

\[ u = \frac{\partial}{\partial t} X. \]

Using Helmholtz’s theorem (Morse & Feshbach 1953), \( X \) can be written as the sum of the
gradient of a scalar and the curl of a zero divergence vector.

\[ X = \nabla \Phi + \nabla \times A, \quad \nabla \cdot A = 0. \]

Then as \( X \) has no vertical component, \( A \) can be written as,

\[ A = \hat{r} \Psi \]

giving,

\[ X = \nabla \Phi + \nabla \times \hat{r} \Psi, \]

\[ = \nabla \Phi + \nabla \Psi \times \hat{r}. \quad (2.3) \]

\( \Phi \) has the properties of a potential and \( \Psi \) has the properties of a stream function. These two
scalars are now expanded in terms of complete sets of functions satisfying Laplace’
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The potential $\Phi$ is expanded as,

$$\Phi(\theta, \phi) = \sum_{r=1}^{\infty} p_r \phi_r(\theta, \phi), \quad (2.4)$$

where the coefficients $p_r$ have to be determined later and the functions $\phi_r$ satisfy the eigenfunction equation,

$$\nabla^2 \phi_r + \mu_r \phi_r = 0, \quad (2.5)$$

with the boundary condition,

$$\nabla \phi_r \cdot \hat{n} = 0.$$

In a similar manner the stream function $\Psi$ is expanded as,

$$\Psi(\theta, \phi) = \sum_{r=1}^{\infty} p_{-r} \psi_r(\theta, \phi), \quad (2.6)$$

where,

$$\nabla^2 \psi_r + \nu_r \psi_r = 0, \quad (2.7)$$

and on the boundary,

$$\psi_r = 0.$$

The functions $\phi_r$ and $\psi_r$ are normalized so that,

$$\int \phi_r \phi_r \, dA = \int \psi_r \psi_r \, dA = 1,$$

where the integral $A$ is over the area of the ocean.

Substituting these expansions into equations (2.1) and (2.2), the boundary conditions are satisfied exactly and one finds that,

$$\zeta = h \sum_{r=1}^{\infty} p_r \mu_r \phi_r. \quad (2.8)$$

Multiplying equation (2.1) by $\nabla \phi_r$ and $\nabla \psi_r \times \hat{t}$, and integrating over the area of the ocean then gives the equations for the coefficients $p_r$ and $p_{-r}$,

$$\frac{\partial^2 p_r}{\partial t^2} + \frac{\kappa \mu_r}{\rho h} \frac{\partial p_r}{\partial t} - \frac{2\Omega}{\rho h} \sum_{s=-\infty}^{\infty} \beta_{r,s} \frac{\partial p_s}{\partial t} + g h \mu_r p_r = g \phi_r, \quad (2.9)$$

where $r = 1, 2, \ldots, \infty$.

In these equations,

$$\beta_{r,s} = -\int \hat{t} \cos \theta \cdot \nabla \phi_r \times \nabla \phi_s \, dA,$$

$$\beta_{r,-s} = \int \cos \theta \, \nabla \phi_r \cdot \nabla \psi_s \, dA,$$

$$\beta_{-r,s} = -\int \cos \theta \, \nabla \psi_r \cdot \nabla \phi_s \, dA,$$

$$\beta_{-r,-s} = -\int \hat{t} \cos \theta \cdot \nabla \psi_r \times \nabla \psi_s \, dA. \quad (2.10)$$
Equation (2.9) shows that in the absence of rotation, i.e. when $\Omega$ equals zero, the basis functions $\phi_r$ and $\psi_r$ are the normal modes or eigenfunctions of the ocean (Veltkamp 1960). The set of functions $\phi_r$ are then the long gravity wave modes with frequencies $\omega_r$ given by,

$$\omega_r = -i(k/2\rho h) \pm \left[ gh\mu_r - (k/2\rho h)^2 \right]^{1/2},$$

and the functions $\psi_r$ are modes with zero frequency, representing the non-divergent steady currents of the ocean. When the coriolis term is not zero, the modes are mixed by the off diagonal terms $2\Omega\beta_{r,s}$ of equation (2.9). Further details of the basis functions, the terms $\beta_{r,s}$, and the driving terms $\xi_r$ are given in Appendix 1.

The equations are now Fourier transformed in order to study the behaviour of the ocean at a fixed angular velocity $\omega$. The Fourier transform from time to angular velocity is defined so that, for example,

$$p_r(t) = \int_{-\infty}^{+\infty} \exp(-i\omega t)p_r(\omega) \, d\omega,$$

and for convenience $p_r(\omega)$ will usually be written as $p_r$. Note that if $\xi_0(t)$ is the constituent of the tidal wave with angular velocity $\omega_0$, it is a real quantity, so that,

$$\xi_0(t) = \xi_0 \exp(-i\omega_0 t) + \xi_0^* \exp(i\omega_0 t).$$

$\xi_0$ is the complex function we shall calculate and $\xi_0^*$ its complex conjugate.

Then at a fixed value of $\omega$, equation (2.9) becomes,

$$(-\omega^2 - i\omega k/\rho h)p_r - 2i\omega \mu_r^{-1} \sum_{s = -\infty}^{+\infty} \beta_{r,s} p_s + gh\mu_r p_r = g\xi_r,$$

$$(-\omega^2 - i\omega k/\rho h)p_{-r} - 2i\omega \mu_r^{-1} \sum_{s = -\infty}^{+\infty} \beta_{-r,s} p_s = 0.$$

Equation (2.14) represents an infinite set of equations which in practice one has to truncate. The justification for doing this is that the tidal waves in the ocean are long, whereas the terms being neglected represent short wavelength waves.

The functions $\phi_r$ and $\psi_r$ are similar to spherical harmonics and for most of the work reported in this paper the expansions were truncated after the terms of order ten. Truncation leaves a matrix equation for the coefficients $p_r$ and $p_{-r}$ which can be solved using standard techniques.

THE EQUILIBRIUM TIDE

The forces $F$ which drive the tides are usually represented in terms of the equilibrium tide $\xi$,

$$F = \rho g \nabla \xi.$$

$\xi$ can be expanded in terms of the spherical harmonics $Y_n^m(\theta, \phi)$ giving,

$$\xi = \sum_{n = 2}^{\infty} \sum_{m = -n}^{+n} C_n^m(r) Y_n^m(\theta, \phi),$$
where \( C_n^m \) is the complex conjugate of \( C_n^{-m} \). The magnitude of the terms decreases rapidly as \( n \) increases so as a result only those terms with \( n \) equal to 2 need be considered. Each term \( C_n^m \) has a period of approximately \(-m\) cycle day\(^{-1}\) so it is conveniently expanded in the form,

\[
C_n^m(t) = \exp(-im\omega t)S_n^m(t).
\]

Here time is measured in days. The functions \( S_n^m(t) \) depend on the motion of the Sun and the Moon relative to the Earth and so only contain periods of a fortnight or more.

Thus in order to understand the tides, we need to understand the response of the ocean to the forces which correspond to the \( Y_{22} \) and \( Y_{21} \) spherical harmonic equilibrium tides. The tidal bands are near 2 cycle day\(^{-1}\) for \( Y_{22} \) and near 1 cycle day\(^{-1}\) for \( Y_{21} \). But in order to obtain a broad view of how the ocean is behaving, a much wider range of frequencies will be considered.

**Bulk Properties**

Once equation (2.14) has been solved for the coefficients \( p_r \), a number of quantities, describing the overall behaviour of the ocean, can be calculated. Later in this paper we shall use the rms amplitude of the tide and the energy dissipated by the tide. However, first it is convenient to calculate the average potential and kinetic energy of the ocean.

If we use the brackets \( \langle \rangle \) to represent time averaging, then the total potential energy \( P.E. \) of the ocean is given by,

\[
P.E. = \left\langle \frac{\rho g}{2} \int \xi(t)^2 \, dA \right\rangle,
\]

\[
= \frac{\rho g}{2} \int \left\langle \xi \exp(-i\omega t) + \xi^* \exp(i\omega t) \right\rangle^2 \, dA,
\]

where the integration is over the area of the ocean. On carrying out the time averaging this gives,

\[
P.E. = \rho g \int \xi^* \xi \, dA.
\]

Substituting for \( \xi \) from equation (2.8),

\[
P.E. = \rho gh^2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \mu_r^* p_r^* \mu_s p_s \int \phi_r \phi_s \, dA,
\]

giving finally,

\[
P.E. = \rho gh^2 \sum_{r=1}^{\infty} \mu_r^2 p_r^* p_r^*.
\]

The total kinetic energy is,

\[
K.E. = \left\langle \frac{\rho h}{2} \int |u(t)|^2 \, dA \right\rangle,
\]

\[
= \frac{\rho h}{2} \int \left\langle |u \exp(-i\omega t) + u^* \exp(i\omega t)|^2 \right\rangle \, dA,
\]

\[
= \rho h \int u^* \cdot u \, dA.
\]
Now,

\[ u = -i\omega(\nabla \Phi + \nabla \Psi \times \mathbf{\hat{r}}), \]

therefore,

\[ \text{K.E.} = \rho \omega^2 \int (\nabla \Phi^* + \nabla \Psi^* \times \mathbf{\hat{r}}) \cdot (\nabla \Phi + \nabla \Psi \times \mathbf{\hat{r}}) \, dA, \]

\[ = \rho \omega^2 \int (\nabla \Phi^* \cdot \nabla \Phi + \nabla \Psi^* \cdot \nabla \Psi) \, dA. \]

Substituting for \( \Phi \) and \( \Psi \) from equations (2.4) and (2.6), and integrating over the area of the ocean, gives finally,

\[ \text{K.E.} = \rho \omega^2 \sum_{r=1}^{\infty} (\mu_r p_r^* p_r + \nu_r p_{-r}^* p_{-r}). \tag{2.18} \]

The rms amplitude of the tide is,

\[ H_{\text{rms}} = \left( \frac{1}{2\pi R^2} \int \langle \xi(t)^2 \rangle \, dA \right)^{1/2}, \]

and as in equations (2.15) and (2.16), this becomes,

\[ H_{\text{rms}} = \left( \frac{h^2}{2\pi R^2} \sum_{r=1}^{\infty} \mu_r^2 p_r^* p_r \right)^{1/2}. \tag{2.19} \]

If \( W_1 \) is the rate of working by the tides against the frictional force \( -\kappa u \), then,

\[ W_1 = \left\langle \int u(t) \cdot \kappa u(t) \, dA \right\rangle. \tag{2.20} \]

From equations (2.17) and (2.18),

\[ W_2 = 2\kappa \omega^2 \sum_{r=1}^{\infty} (\mu_r p_r^* p_r + \nu_r p_{-r}^* p_{-r}). \tag{2.21} \]

This should equal the work done on the ocean by the tidal forces, \( W_2 \),

\[ W_2 = \left\langle \rho g h \int \nabla \bar{\xi}(t) \cdot u(t) \, dA \right\rangle, \tag{2.22} \]

\[ = \rho g h \int (\nabla \bar{\xi}^* \cdot u + \bar{\xi}^* \cdot u^*) \, dA, \]

\[ = 2\rho g \text{Re} \left[ \int (\nabla (\bar{\xi}^* hu) - \bar{\xi}^* \nabla (hu)) \, dA \right], \]

\[ = 2\rho g \text{Re} \left[ \int \bar{\xi}^* \frac{\partial \xi}{\partial t} \, dA \right]. \]

Thus,

\[ W_2 = 2\rho g \text{Re} \left[ -i\omega \int \bar{\xi}^* \xi \, dA \right]. \]
or

\[ W_2 = 2\rho g \omega \text{Im} \left[ \int \bar{\xi}^* \xi \, dA \right]. \] (2.23)

This equation shows that the work done by the Moon on the ocean is proportional to that component of the tide which is \( \pi/2 \) out of phase with the equilibrium tide.

Substituting for \( \xi \) from equation (2.8) gives,

\[ W_2 = 2\rho gh \omega \text{Im} \left[ \int \bar{\xi}^* \sum_{r=1}^{\infty} p_r \mu_r \phi_r \, dA \right], \]

Then using equation (2.11) one finds that,

\[ W_2 = 2\rho gh \omega \text{Im} \left[ \sum_{r=1}^{\infty} p_r \mu_r \bar{\xi}_r^* \right]. \] (2.24)

3 The resonances

In order to study the effect of resonances on the ocean it is helpful first to write the response of the ocean in terms of the eigenfunctions of the complete tidal equation. If the matrix \( L \) and the vectors \( \zeta \) and \( \bar{\zeta} \) are defined by,

\[ L = \begin{pmatrix} \kappa/\rho h + f \times g \nabla \\ \nabla h \\ 0 \end{pmatrix}, \]

\[ \zeta = \begin{pmatrix} u \\ \xi \end{pmatrix} \quad \text{and} \quad \bar{\zeta} = \begin{pmatrix} 0 \\ \bar{\xi} \end{pmatrix}, \] (3.1)

then the tidal equations (2.1) and (2.2) can be written as,

\[ \left( L + \frac{\partial}{\partial t} \right) \zeta = L \zeta. \] (3.2)

Fourier transformed this becomes,

\[ (L - i\omega) \zeta = L \bar{\zeta}. \] (3.3)

The solution to this equation is (Morse & Feshbach 1953; Webb 1974),

\[ \zeta = -\sum_r \phi_r \frac{\omega_{r-} - \omega_{r+}}{\omega - \omega_r} \gamma_r, \] (3.4)

where \( \gamma_r \) is the overlap integral,

\[ \gamma_r = \int \phi_r^* \bar{\xi} \, dA. \]

\( \phi_r \) are the eigenfunctions of the equation,

\[ (L - i\omega_r) \phi_r = 0, \] (3.5)

with eigenvalues \( \omega_r \). \( \psi_r \) are the corresponding eigenfunctions of the Hermitian adjoint equation,

\[ (L^* + i\omega_r^*) \psi_r = 0. \] (3.6)
The functions $\phi_r$ and $\Psi_r$ form a bi-orthonormal set, i.e.
\[
\int \Psi_r^* \phi_s \, dA = \delta_{r,s},
\]
\[
\sum_r \phi_r(x) \Psi_r^*(x') = \delta(x - x').
\] (3.7)

If the scalar product is defined so that,
\[
\int \xi_1^* \cdot \xi_2 \, dA = \int (u_1^*, L_2) \left( \begin{array}{c} \rho h_1 \\ 0 \\ \rho g \end{array} \right) \left( \begin{array}{c} u_2 \\ \xi_2 \\ 0 \end{array} \right) \, dA,
\]
\[
= \int (\rho u_1^* \cdot u_2 + \rho g \xi_1^* \xi_2) \, dA,
\] (3.8)

then the Hermitian adjoint matrix $L^*$ is similar to $L$, but with the sign of all terms, except the frictional term, reversed (Garrett & Greenberg 1977). In particular if there is no friction $L^*$ equals $-L$ and $\Psi_r$ equals $\phi_r$.

The functions $\phi_r$ are the resonances of the ocean. They come in pairs so that if there is an eigenfunction $\phi_r$ with eigenvalue $\omega_r$, there is also an eigenfunction $\phi_r^*$ with eigenvalue $-\omega_r$. Physically the two eigenfunctions correspond to the same physical resonance of the ocean, mathematically they arise from the slightly artificial use of both positive and negative frequencies.

Equation (3.4) can now be used to obtain expressions for the rms amplitude and other quantities. The total energy is given by,
\[
E = \int \left( \frac{\rho h}{2} u^2 + \frac{\rho g}{2} \xi^2 \right) \, dA,
\]
\[
= \int (\rho u^* \cdot u + \rho g \xi^* \xi) \, dA.
\] (3.9)

Thus from equation (3.8),
\[
E = \int \xi^* \cdot \xi \, dA,
\] (3.10)

and from equation (3.4),
\[
E = \sum_r \sum_s \left( \int \phi_r^* \phi_s \, dA \right) \left( \frac{\omega_r \gamma_r}{\omega - \omega_r} \right)^* \left( \frac{\omega_s \gamma_s}{\omega - \omega_s} \right).
\] (3.11)

Notice that if there is no friction, equation (3.11) for the energy simplifies and becomes,
\[
E = \sum_r \left| \frac{\omega_r \gamma_r}{\omega - \omega_r} \right|^2.
\] (3.12)

In a similar manner one can show that the time averaged potential energy is,
\[
P.E. = \rho g \sum_r \sum_s \left( \int \xi_r^* \xi_s \, dA \right) \left( \frac{\omega_r \gamma_r}{\omega - \omega_r} \right)^* \left( \frac{\omega_s \gamma_s}{\omega - \omega_s} \right).
\] (3.13)
and as in equation (2.19) the rms amplitude is proportional to the square root of this quantity. At zero frequency, the ocean takes on the form of the equilibrium tide, so all its energy is in the form of potential energy and equals the potential energy of the equilibrium tide.

Using equation (3.4), one can also show that $W_1$, (equation 2.20), the rate of working by the tides against friction is given by,

$$W_1 = 2 \kappa \sum_r \sum_s \left( \int u_r^* \cdot u_s \, dA \right) \left( \frac{\omega_r \gamma_r}{\omega - \omega_r} \right)^* \left( \frac{\omega_s \gamma_s}{\omega - \omega_s} \right).$$

(3.14)

Similarly $W_2$ (equation 2.22), the work done by the tidal forces on the ocean, is,

$$W_2 = 2 \omega \Im \left[ -\sum_r \frac{\gamma_r^* \omega_r \gamma_r}{\omega - \omega_r} \right],$$

(3.15)

where,

$$\gamma_r^* = \int \tilde{\Phi}_r \cdot \cdot dA.$$

Although not obvious from equation (3.14), equation (3.15) shows that the power throughput of the ocean is just the imaginary part of a fairly simple analytic function.

4 Solutions with no friction

The numerical model was used first to determine the behaviour of a hemispherical ocean when no friction was present. Forces corresponding to both the $Y_2^{-1}$ and $Y_2^{-2}$ equilibrium tides were used to drive the ocean and its response was calculated for a range of frequencies.

Plots of the rms amplitude of the tide were found to be useful for representing the overall response of the ocean and are shown in Figs 1 and 2. Positive angular velocities correspond to the tide producing body moving from east to west across the sky, negative frequencies correspond to motion from west to east.

Figure 1. The rms amplitude response of the model ocean, without friction, when excited by the force corresponding to the $Y_2^{-1}$ spherical harmonic equilibrium tide with unit rms amplitude.
Calculations were made at an interval of 0.02 cycle day$^{-1}$. At this resolution many of the peaks appear truncated whereas, at the resonant frequencies, the ocean should have an infinite response. Also near the origin some of the Rossby waves are missed because they are so closely packed together.

At zero frequency, the apparent position of the tide producing body in the sky remains stationary and so the ocean should take on the form of the equilibrium tide; that is seen to be the case.

The resonances shown in Figs 1 and 2 are broadest and most widely separated near 1 and 2 cycle day$^{-1}$. They lie symmetrically about the origin because each physical resonance may be excited by the tide producing body travelling eastwards or westwards across the sky at the resonant frequency. However, the widths of the resonances are not symmetric about the origin, because the amount by which each resonance is excited does depend on the direction in which the tide producing body travels.

The model ocean is symmetric about the equator, and so the resonances fall into two classes, one symmetric and the other anti-symmetric about the equator (Longuet-Higgins & Pond 1970). $Y_2^{-1}$ being itself anti-symmetric excites only the anti-symmetric resonances (Fig. 1) and $Y_2^{-2}$ being symmetric excites only the symmetric resonances (Fig. 2). In a real irregular ocean, each equilibrium tide would excite every resonance.

**THE DISTRIBUTION OF RESONANCES**

The resonance frequencies of a frictionless hemispherical ocean centred at the equator have been calculated previously by Longuet-Higgins & Pond (1970), who also studied the effect of varying rotation, either by directly changing the Earth's rotation rate or by changing the depth of the ocean.

As the effect of rotation is reduced, resonances below 3.4 rad day$^{-1}$ are reduced in frequency and resemble non-divergent Rossby waves, eventually becoming the steady currents described by the stream functions $\psi_r$ of Section 2.

Resonances with frequencies above 3.4 rad day$^{-1}$ tend to the long gravity waves described by the potential functions $\phi_r$. Rotation resolves the degeneracy of resonances of the same order $n$ but for a realistic rotation rate the splitting is not large and so gives the groups of nearby resonances seen in Figs 1 and 2. The effect of rotation on the resonance frequencies, is discussed further in Appendix 2.
If the distribution of resonances in an ocean is not greatly affected by rotation, an estimate of the density of eigenvalues can be obtained from that for the density of eigenvalues for Laplace's equation. For a region of arbitrary shape, this is given approximately by (Morse & Feshbach 1953),

$$n(\omega) = \left( \frac{A \omega}{2 \pi c} + \frac{L}{4 \pi c} \right)^{-1}. \quad (4.1)$$

$A$ is the area of the ocean, $L$ its perimeter and it has been assumed that $\omega = ck$,

$$\omega = ck, \quad (4.2)$$

where $k$ is the characteristic wavenumber of the eigenfunction and $c$ is the speed of long gravity waves in the ocean.

Near 10 rad day$^{-1}$, equation (4.1) gives $n(\omega)$ for the model ocean as 1.5 resonances per radian per day. Comparison with Figs 1 and 2 shows this result to be reasonably correct. A similar equation may also be obtained for the Rossby wave resonances.

The resonant frequencies of a more realistic ocean have been calculated by Platzmann (1975). His model, which was of the Atlantic and Indian Oceans without friction, gave a more uniform distribution of resonances than the present model. This was probably a consequence of the irregular boundaries used. Platzmann’s model also showed a similar density of resonances near 10 rad day$^{-1}$ to the present model, the smaller area of the earlier model apparently being compensated for by a smaller mean depth and a larger perimeter.

THE STRENGTHS OF THE RESONANCES

Another important property of the resonances is their apparent widths or strengths. This is most simply defined by reference to the expression for the total energy (equation 3.12) as being the frequency separation from the resonance frequency at which the contribution of the resonance to the total energy equals the energy of the system at zero frequency. Thus if $E_0$ is the total energy at zero frequency, the strength of the $r$th resonance $S_r$ is,

$$S_r = \left| \omega_r \gamma_r / E_0^{1/2} \right|. \quad (4.3)$$

Because of the complexity of the equation for the rms amplitude the above definition will not exactly match the apparent widths of Figs 1 and 2, but the discrepancy should not be large. The most important term in equation (4.3) is the overlap integral $\gamma_r$, which for a frictionless ocean is a measure of how well the tidal elevation associated with the resonance matches the equilibrium tide. Thus

$$S_r = \left( \omega_r / E_0^{1/2} \right) \rho g \int \xi_r^* \xi \, dA. \quad (4.4)$$

The resonance strength can also be written in terms of how well the current velocities associated with each resonance match the tidal forces, for,

$$\omega_r \gamma_r = \omega_r \int \phi_r^* \xi \, dA,$$

$$= -i \int \phi_r^* L \xi \, dA.$$

If,

$$\phi_r = \begin{pmatrix} u_r \\ \xi_r \end{pmatrix}$$
then,

\[ S_r = \left( \frac{h}{E_0^{1/2}} \right) \left| \int \mathbf{u}_r^* \cdot \mathbf{F} \, dA \right| . \]  

(4.5)

At low frequencies, Figs 1 and 2 show that the strengths of the Rossby wave resonances are small. In terms of equation (4.4) this arises because the angular velocity \( \omega_r \) is small and also the surface elevation of a Rossby wave is small, so that the quantity \( \gamma_r \) is small. Alternatively, using equation (4.5), as the velocity field of a Rossby wave is mainly rotational, the non-zero part of the integral, which comes from the divergent part of the velocity field, is small.

At high frequencies, the widths of the resonances are again small because the integrand in equation (4.5) oscillates rapidly with no region of stationary phase, so the oscillations cancel except in a region of widths \( \lambda/2 \) near each boundary. (This behaviour can be readily checked for a rectangular ocean, which also shows that the integral is largest when the rapidly oscillating function is zero at the boundaries.) Thus for the two-dimensional integral of equation (4.5), the strength of a high frequency resonance will be proportional on average to \( \lambda^2 \) or assuming equation (4.2) to \( \omega^{-2} \). A similar result is obtained from equation (4.4), but as \( \xi_r \) is not zero at the boundary, cancellation effects are larger and this compensates for the extra factor of \( \omega \). Again comparison with Figs 1 and 2 show that the \( \omega^{-2} \) behaviour is reasonably correct.

At intermediate frequencies, and this includes the tidal bands, the wavelengths of the resonances become comparable with the size of the ocean and with the wavelength of the tidal force. The spatial behaviour of the resonances over the whole ocean, determining how well they match the tidal forces, then becomes critical.

The fortuitous spatial behaviour of a few resonances can then have a dramatic effect on the response. Thus in the model ocean two strong resonances, which clearly match the driving force of the \( Y_2^{-2} \) equilibrium tide at a frequency near 2 cycle day\(^{-1}\), give a broad region in which the tidal amplitude is very large. If the tide generating body travels in the opposite direction then, except just at the resonant frequencies, much lower tides are produced. Similarly much smaller tides are found near 1 cycle day\(^{-1}\) for forcing corresponding to the \( Y_2^{-1} \) equilibrium tide. As will be discussed later, this fortuitous effect that can arise from the spatial behaviour of the resonances becomes even more important once friction is introduced.

THE BACKGROUND

It is very noticeable in Figs 1 and 2 (and also in plots of the total energy) that in the range \(-20\) to \(+20\) rad day\(^{-1}\), the rms amplitude of the ocean rarely falls below its value at zero frequency. This may be the result of cooperation between the resonances, similar to that which gives the equilibrium response at zero frequency (equation 3.12), but as yet a satisfactory explanation has not been found.

5 Tidal dissipation

The next step is to introduce tidal dissipation into the model. Tidal dissipation is believed to occur mainly on the resonant continental shelves surrounding the deep ocean (Webb 1976a). However, fitting such shelves into the model introduces an extra level of complexity. Instead the model has been kept as simple as possible, by using a linear bottom friction term \(-\kappa u\) and by choosing the friction coefficient \( \kappa \), so that the decay time for energy in the model has a realistic value.
Figure 3. The rms amplitude of the model ocean, with friction, when excited by the force corresponding to the $Y_{3}^{-1}$ equilibrium tide with unit rms amplitude. The friction coefficients used correspond to decay times of 60, 30, 20, 15 and 12 hr (1/r day = 0.4, 0.8, 1.2, 1.6 and 2.0). The latter gives the lowest amplitude resonance peaks.

Figure 4. The rms amplitude of the model ocean, with friction, when excited by the force corresponding to the $Y_{3}^{-1}$ equilibrium tide with unit rms amplitude. The friction coefficient used corresponds to decay times of 60, 30, 20, 15 and 12 hr, the latter giving the lowest amplitude resonance peaks.
If \( \langle u^2 \rangle \) is the mean square velocity in our model ocean then the average rate at which the ocean works against the bottom friction is \( \kappa \langle u^2 \rangle \). Then if it is assumed that the average potential energy of the ocean equals its average kinetic energy, the decay time for the total energy is,

\[
\tau = \rho h \langle u^2 \rangle / \kappa \langle u^2 \rangle, \\
= \rho h / \kappa,
\]

or,

\[
\kappa = \rho h / \tau. \tag{5.1}
\]

Studies of the real ocean (Garrett & Munk 1971; Webb 1973), indicate that \( \tau \) lies between 24 and 60 hr. In the model \( \tau \) was first chosen and the value of the friction coefficient \( \kappa \) then obtained from equation (5.1).

The introduction of friction should change both the frequency and the shape of the resonances. If the friction is small, then its effect can be calculated using perturbation theory. Let \( \kappa F \) be that part of \( L \) (equation 3.1), dependent on the friction coefficient and let \( \phi^0_r \) and \( \omega^0_r \) be the \( r \)th eigenfunction and eigenvalue of equation (3.3) without friction. Using first order perturbation theory (Morse & Feshbach 1953), the \( r \)th eigenfunction and eigenvalue with friction are,

\[
\phi_r = \phi_r^0 + \kappa \sum_{s \neq n} \phi_s^0 \frac{1}{\omega_r^0 - \omega_s^0} \int (\phi_s^0)^* F \phi_r^0 \, dA. \tag{5.2}
\]

\[
\omega_r = \omega_r^0 - i \kappa \int (\phi_r^0)^* F \phi_r^0 \, dA,
\]

\[
= \omega_r^0 - i \kappa \int |u_r^2| \, dA. \tag{5.3}
\]

Thus the initial effect of friction on the shape of the resonances is to mix resonances with similar spatial behaviour in the dissipation region. This mixing is enhanced for resonances of similar frequency.

The initial effect of friction on the frequency of each resonance is to give it a small imaginary component. For the gravity wave resonances, the potential energy and kinetic energy parts of equation (3.8) are approximately equal, so the imaginary component of frequency becomes approximately \(- \kappa / (2 \rho h)\) or \(- 1 / (2 \tau)\). For the Rossby wave resonances, which have little potential energy, it becomes \(- 1 / \tau\).

For a given value of the energy decay time \( \tau \), the imaginary component of frequency should be the same whether dissipation occurs mainly on resonant continental shelves, as in the real oceans, or over the whole of the ocean floor, as in the model. However, the effect on the shape of the resonances could be different in the two cases. In the real ocean the dissipation occurring on the continental shelves may result in the eigenfunctions becoming progressive waves transporting energy out of the deep ocean. Unfortunately in the model this effect will not be represented correctly.

Also in the real ocean there may be resonances which are not coupled or only weakly coupled to the regions of energy dissipation. Such resonances will affect the rms amplitude but not the power throughput of the ocean.
6 Solutions with friction

The rms amplitude response of the ocean, with friction, is shown in Figs 3 and 4. The power throughput of the ocean, that is the work done by the Moon on the ocean and the work done by the ocean against friction, is shown in Figs 5 and 6. Because of our ignorance about the decay of energy in the deep ocean, calculations were made for a number of values of the decay time \( \tau \) between 12 and 60 hr. A decay time of 60 hr, gives each resonance an imaginary coordinate of about \(-0.2\) rad day\(^{-1}\). As the decay time is reduced, that is the friction is increased, the resonances move further from the real axis.

When the decay time is 60 hr many of the narrow resonances, seen when no friction was present, have disappeared, indicating that the effective width of the missing resonances must have been less than 0.2. The remaining peaks are due either to individual resonances whose effective width is sufficiently large or to the coalescence of nearby narrow resonances.

As the effect of friction is increased further, the remaining peaks are reduced in amplitude and become broader, with nearby peaks sometimes coalescing. This effect may be accentuated in the model ocean because of a clustering of the resonances. Increases in the friction coefficient also make more apparent the underlying smooth background which is due to the combined effect of all the distant resonances, including those that are apparently lost when friction is introduced.

The main effect of friction is thus seen to be the obliteration of the narrower isolated resonances. It is not enough for the ocean to have a resonance at a particular frequency. To be important, the resonance must be sufficiently excited so that, either by itself, or with its close neighbours, it stands out high above the background. For realistic values of the decay time only a few of the resonances satisfy this criteria and to distinguish them they will be called the key resonances. They are the resonances that best match the forces driving the tide.

**Figure 5.** The power throughput of the model ocean when driven by the force corresponding to the \( Y_2^{-1} \) equilibrium tide with unit rms amplitude. The friction coefficients used correspond to decay times of 60, 30, 20, 15 and 12 hr, the latter giving the lowest amplitude peaks.
Figure 6. The power throughput of the model ocean, when driven by the force corresponding to the $Y_1^{-1}$ equilibrium tide with unit rms amplitude. The friction coefficients used correspond to decay times of 60, 30, 20, 15 and 12 hr, the latter giving the lowest amplitude peaks.

**POWER THROUGHPUT**

The key resonances also dominate the power throughput of the ocean (Figs 5 and 6). The power throughput is zero at zero frequency when there are no currents in the ocean and it also drops off at high frequencies, but in the region of the tidal bands it has a large range of values, the magnitude depending mainly on how near the excitation frequency is to the frequency of the nearest key resonance.

When at a fixed frequency, the friction coefficient is increased from zero, the power throughput is at first proportional to $\kappa$ because for a small friction coefficient the velocities are not very different from their values without friction. As the friction is increased further the power throughput near the resonances declines and the background between the peaks rises due to the broadening of the peaks. At a decay time of 60 hr all but the key resonances are lost in the background, but it requires a decay time of less than 12 hr before all the key resonances merge with the background. Calculations made with decay times of below 6 hr showed that in this range the background itself is being reduced, so that as the decay time goes to zero the power throughput also goes to zero.

For small values of the friction coefficient the overall behaviour can be explained in terms of equation (3.15). If the frequency of the $r$th resonance is $\omega_r - \ell \Gamma$, and if the friction is small enough so that $\gamma_r$ approximately equals $\gamma^*$, then the power throughput due to the $r$th resonance at frequency $\omega$, is,

$$W_r(\omega) = 2\omega^2 |\gamma_r|^2 \frac{\Gamma}{(\omega - \omega_r)^2 + \Gamma^2}. \quad (6.1)$$
When the friction coefficient is small, \( \Gamma \) for the gravity wave resonances will be approximately equal to \( 1/2\pi \). As the friction is increased from zero, the power throughput due to the resonance will initially increase linearly with \( \Gamma \). The power levels off when \( \Gamma \) equals \( (\omega - \omega_r) \) and after this decreases eventually going as \( 1/\Gamma \).

7 Comparison with the world oceans

There have been a number of estimates of tidal dissipation in the world's oceans. Lambeck (1977) has recently made independent estimates using data from astronomical measurements, satellite measurements and numerical tidal models. He obtains a total dissipation rate of about \( 4 \times 10^{12} \) W of which about \( 3 \times 10^{12} \) W is due to the \( M_2 \) tide.

Estimates based on actual tidal observations have been reviewed by Cartwright (1977). Of these, the best is probably Miller's (1966) estimate of \( 1.7 \times 10^{12} \) W for the \( M_2 \) tide.

Now the equilibrium \( M_2 \) tide has a rms amplitude of 12.05 cm (McMurtree & Webb 1975). The model ocean has an area of 80 per cent of the actual deep ocean and is driven by an equilibrium tide with a rms amplitude of 1 m. Thus Lambeck's value of \( 3 \times 10^{12} \) W for the \( M_2 \) tide corresponds to a value of \( 1.9 \times 10^{14} \) W in our model ocean.

In fact, in Fig. 6, the model ocean shows a similar power throughput in the tidal band, between 12 and 12.5 rad day\(^{-1} \), when the decay time is 30 hr. This measure of agreement is somewhat surprising but it is very dependent on the position of the two key resonances near 11 rad day\(^{-1} \) and any change in the position of these resonances would greatly affect the power throughput.

It is noticeable from the figures of the power throughput, that in order to reach a figure comparable with Lambeck's estimate the excitation frequency must be near to the frequency of one of the key resonances. This suggests that in the real ocean the tidal frequency may also be near to the frequency of a key resonance.

8 Spatial behaviour

In the absence of any friction, the tidal patterns produced by the model are very similar to those illustrated by Proudman & Doodson (1936) and Doodson (1938). Thus for the \( Y_2^2 \) driving function, the two key resonances at 10.5 and 11.4 rad day\(^{-1} \) correspond to the resonances found by Doodson at ocean depths of 4.33 and 3.52 miles (Doodson 1938, Figs 14 and 17) which are predominantly standing waves trapped at the equator.

Longuet-Higgins & Pond (1970, see Fig. 1), also show that when rotation is removed these two resonances become the two symmetric long gravity modes of order 3, \( P_3^3(\cos \theta) \cos \phi \) and \( P_3^3(\cos \theta) \cos 3\phi \). They are also two of the basis functions used in the expansion equation (2.4). As the frequency of the two modes is degenerate the effect of even a small coriolis force will be to mix them strongly. Solving equation (2.14) with just these two modes gives the eigenfunctions,

\[
P_3^3(\cos \theta) \cos \phi + iP_3^3(\cos \theta) \cos 3\phi,
\]

and,

\[
P_3^3(\cos \theta) \cos \phi - iP_3^3(\cos \theta) \cos 3\phi.
\]

These functions show the four regions of maximum amplitude along the equator and the three amphriodromes at 30° N seen in Doodson's results. They do not give the fourth amphriodrome near the poles, so this must result from the mixing in of other basis functions.
Figure 7. Cotidal and cophase lines for the model ocean when driven by the force corresponding to the $Y_{1}^{-1}$ equilibrium tide of unit rms amplitude, at a frequency $1 \text{ cycle day}^{-1}$. The frictional decay time is $30 \text{ hr}$.

Figure 8. Cotidal and cophase lines for the model ocean when driven by the force corresponding to the $Y_{1}^{-2}$ equilibrium tide of unit rms amplitude, at a frequency $2 \text{ cycle day}^{-1}$. The frictional decay time is $30 \text{ hr}$.

However, the fact that a linear combination of just the two modes gives such a good representation of the solution, gives support to the observation made earlier that the Coriolis force is doing little more than acting as a perturbation of the gravity wave modes.

**BEHAVIOUR WITH FRICTION**

Figs 7 and 8 show the cotidal and cophase lines for the diurnal and semi-diurnal tides in the model ocean, when the decay time is $30 \text{ hr}$. These should be compared with cotidal charts of the Pacific based on harmonic constants (i.e. Dietrich (1944), also published in Defant (1961)).

The main effect of introducing friction is that the tidal waves are no longer symmetric about the central longitude of the ocean. This is most noticeable for the semi-diurnal tide, which in the west remains a standing wave but in the east becomes a progressive wave.
Figure 9. Cotidal and cophase lines for the model ocean when driven by the force corresponding to the $Y_2$ equilibrium tide of unit rms amplitude, at a frequency of 1.7 cycle day$^{-1}$. This is near the frequency of the strongest resonance seen in Figs 2, 4 and 6. The frictional decay time is 30 hr.

The model response at 2 cycle day$^{-1}$ has strong similarities with the $M_2$ tide in the North Pacific. Both show standing waves in the western equatorial region, a more progressive wave in the east and a Kelvin wave along the northern boundary. However, the observed tide is less complex than the model although a similar complexity to the latter is found in Zahel’s (1977) numerical model.

There is less agreement between the model and the southern Pacific, but then the model does not include the connections with the Atlantic and Indian oceans. For the diurnal tide the agreement between the model and the real ocean is also poor but they both show a gradual increase in phase from east to west with the tides in the northern and southern Pacific out of phase.

Fig. 9 shows the response of the model ocean to a force corresponding to the $Y_2$ equilibrium tide, at a frequency of 10.5 rad day$^{-1}$. This is at the frequency of the most important of the key resonances affecting the tidal bands and most of the observed response in this figure will be due to the resonance.

The resonance is predominantly an equatorially trapped wave with similar features to Doodson’s resonances but as a result of friction the largest amplitudes are found in the west of the ocean. This was also true for the other key resonance at 11.4 rad day$^{-1}$ and must be responsible for the standing wave observed in the west of the ocean for the semi-diurnal tide.

The energy is trapped in the west and because of this the group velocity, which could transport energy along the equator, must be small. This was an unexpected property, but Rossby waves on a $\beta$-plane ocean with small group velocities can exist and have been discussed by Moore & Philander (1977).

**The Power Throughput**

The work done by the semi-diurnal tidal force on the ocean is shown in Fig. 10. Most of the work is done in the regions along the equator where one would expect to find the strongest currents. There are also regions where the ocean is feeding energy back to the tide producing body.

The work done can also be represented as an overlap integral between the tide in the ocean and the equilibrium tide (equation 2.23) as shown in Fig. 11.

Before these diagrams were plotted it was thought that the work done on the ocean might be concentrated near to the latitude where the tidal wave moves at the same speed as the
Figure 10. The rate of working, $F_{HH}$, of the tidal forces on the model ocean. The force has a frequency of 2 cycle day$^{-1}$ and corresponds to the $Y_2^0$ equilibrium tide of unit rms amplitude. The frictional decay time is 30 hr. The rate of working is in units W m$^{-2}$.

Figure 11. The overlap integral (equation 2.23) estimate of the rate of working of the tidal force. The forcing and friction used are the same as in Figs 8 and 10.

equilibrium tide. This is because the tidal forces can then be in the same phase with the tidal velocities over a large area of ocean. The critical latitude $\theta_c$ is given by,

$$\theta_c = \cos^{-1}\left[(gh)^{1/2}/R\Omega\right].$$

For an ocean of depth 4400 m this latitude is 63° north or south of the equator. (This argument has been used to support the hypothesis that the Antarctic ocean is an important region for the generation of the tides.) However, as is seen in Figs 10 and 11 at 2 cycle day$^{-1}$ the critical latitude is not important. It was also not important in calculations made at other frequencies in and around the diurnal and semi-diurnal tidal bands.

To sum up this section on spatial behaviour, the key resonances of the model, which helped to give such good agreement with the estimated dissipation rate of the real ocean, are found to be equatorially trapped resonances, similar in shape to the standing wave observed in the semi-diurnal tide of the North Pacific.

9 Concluding remarks

The model has shown that, when there is no energy dissipation present, all resonances are important in the sense that each gives a large amplitude response over a reasonably large frequency range. This feature is seen to be partly responsible for the high semi-diurnal tides.
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calculated by Doodson but in addition, for an ocean of realistic depth, there is also a clustering of particularly strong resonances near 2 cycle day\(^{-1}\).

When a realistic amount of dissipation is introduced many of the resonances merge into the background. The response is then dominated by a few key resonances plus the smooth background term. In the frequency range considered, the amplitude of both the key resonances and the background is of order one, so that the height of the tide in the model ocean is of the same order as the height of the equilibrium tide.

In the model, the work done by the Moon on the ocean is mainly due to the key resonances. It therefore appears that the boundary conditions for the shelf models, referred to in the introduction, will have to model these resonances. This is still a difficult task, but it is undoubtedly easier than modelling the full set of resonances.

The key resonances are found to be the long gravity wave resonances near 1 and 2 cycle day\(^{-1}\), whose velocity field matches the tidal forces. For the semi-diurnal tide the most important resonances are the pair of equatorially trapped modes. Comparison with the North Pacific indicates that the tides there may be dominated by similar resonances.

The model is also in close agreement with Lambeck's estimate of the dissipation rate due to the M\(_2\) tide and the model has shown that such a dissipation rate is only likely if there is a key resonance within 2 rad day\(^{-1}\) of the tidal frequency. Consequently, the power throughput of an ocean will be strongly affected by any factors which change the frequency or shape of the resonances. Changes in the resonant frequency may move the key resonances away from the tidal bands. Changes in the shape of the resonances may mean that the key resonances near 2 cycle day\(^{-1}\) would no longer match the driving forces. In the past such changes will have occurred in the real ocean due to continental drift modifying the shape and depth of the ocean and because of the movement of the Earth's pole.

With these points in mind it is plausible that on average in the past, the power throughput in the tidal bands has been lower than its present value. An average power throughput of one third of the present value is not impossible and this value would put the Gerstenkorn event near the time of formation of the solar system.

References


Heaps, N. S., 1972. Tidal effects due to water power generation in the Bristol Channel, Tidal Power,
Appendix 1: further details of the method used

THE BASIS FUNCTIONS

The eigenfunctions of Laplace's equation on a hemisphere (equations 2.5 and 2.6), satisfying the conditions \( \nabla \phi \cdot \hat{n} = 0 \) and \( \psi_r = 0 \) on the boundary are,

\[
\phi_r = \alpha_n^m p_n (\cos \theta) \cos m\phi \left\{ \begin{array}{l} n = 1, 2, \ldots \\ m = 1, 2, \ldots , n \end{array} \right.
\]

and,

\[
\psi_r = \alpha_n^m p_n (\cos \theta) \sin m\phi \left\{ \begin{array}{l} n = 1, 2, \ldots \\ m = 1, 2, \ldots , n \end{array} \right.
\]

(A1)
with eigenvalues,
\[ \mu = \nu = n(n + 1)/R^2. \quad (A2) \]

The terms with \( n = 0 \) are left out as they do not affect the velocities or the tidal height. Normalizing the functions so that,
\[ \int \phi \phi \, dA = \int \psi \psi \, dA = 1, \]
then,
\[ \alpha_n^m = \left( \frac{2n + 1(n - m)!}{\pi R^2 (n + m)! (n + m + 1)} \right)^{1/2}. \quad (A3) \]

As described by Longuet-Higgins & Pond (1970), both the functions \( \phi \) and \( \psi \), and the sets of equations represented by equation (2.14), can be split into two sets. One of these sets describes motions symmetric about the equator and the other motions antisymmetric about the equator. The equilibrium tide corresponding to the \( Y_2 \) spherical harmonic produces tidal elevations that are symmetric about the equator, so only the symmetric set of equations need be solved. Similarly for the \( Y_2^{-1} \) equilibrium tide only the anti-symmetric set of equations need be solved.

**THE COEFFICIENTS \( \beta \)**

These were calculated by Longuet-Higgins & Pond, but their paper contains some misprints. If in \( \beta_{r,s} \), \( n \) and \( m \) are the pair of indices from equation (A1) corresponding to \( r \), and \( n' \) and \( m' \) are the pair of indices corresponding to \( s \), then,
\[ \beta_{r,s} = 0 \quad \text{when } (m + m') \text{ is even}, \]
and, when \( (m + m') \) is odd,
\[ \beta_{r,s} = \frac{n'(n' + 1) + m'(m' + 1)}{m' + 1} - \frac{2m}{(m' + 1)} \frac{n(n + 1) - n'(n' + 1) + m'}{m^2 - m'^2} I(n', n') \]
\[ + \frac{1}{m' + 1} I(n, n') \]
\[ \beta_{r,s} = \frac{2m}{(m^2 - m'^2) (n' + 1)} \left[ (n' + 1)(n' + m') I(n', n' - 1) \right. \]
\[ + n'(n' + 2)(n' - m' + 1) I(n, n' + 1) \]
\[ - \beta_{s,r}, \]
\[ \beta_{r,s} = \frac{2mm'}{m'^2 - m^2} I(n, n'). \quad (A4) \]

Where,
\[ I(n, n') = \int_{-1}^{1} P_n^m(\mu) P_n^{m'}(\mu) \, d\mu. \quad (A5) \]

A method for calculating the integral (A5) has been given by Crease (1966).
THE DRIVING TERM

The driving term \( \xi_r \) in equation (2.14) is given by,

\[
\xi_r = \int \xi \phi_r \, dA. \tag{A6}
\]

If the equilibrium tide \( \xi \), is that associated with the spherical harmonic \( Y_{n}^{m'}(\theta, \phi) \), and it has a rms amplitude of one, then,

\[
\xi = (2\pi)^{1/2} Y_{n}^{m'}(\theta, \phi) \exp (-i\omega t) + c.c.
\]

Considering the term with angular velocity \( \omega \),

\[
\xi = (2\pi)^{1/2} Y_{n}^{m'}(\theta, \phi) \exp (-i\omega t)
\]

\[
\times \left( \frac{\pi}{2} \right)^{1/2} \left[ (1 + \delta_{m'0})^{1/2} \phi_{n}^{m'} + i(1 - \delta_{m'0}) \text{sgn}(m') \psi_{n}^{m'} \right].
\]

Thus,

\[
\xi_r = \left( \frac{\pi}{2} \right)^{1/2} \left[ (1 + \delta_{m'0})^{1/2} \delta_{mn} \delta_{mm'} + i(1 - \delta_{m'0}) \text{sgn}(m') \int \psi_{n}^{m'} \phi_{n}^{m} \, dA \right]. \tag{A7}
\]

Where,

\[
\int \psi_{n}^{m'} \phi_{n}^{m} \, dA = 0 \quad \text{if} \quad (m' + m) \text{ is even},
\]

\[
= \alpha_{n}^{m'} \alpha_{n}^{m} \frac{2m'}{m'^2 - m^2} I \left( \frac{m'}{n'} \frac{m}{n} \right) \quad \text{if} \quad (m' + m) \text{ is odd}.
\]

Appendix 2

Webb (1974) showed that if the functions are analytic, the Green’s function of an operator \( [\mathcal{L}(\omega) - \lambda] \) is,

\[
G(x, x', \omega) = \sum_{r} \frac{\phi_{r}(x) \psi_{r}^{*}(x')}{(\omega - \omega_{r})(\partial \lambda / \partial \omega)_{\omega = \omega_{r}}}, \tag{B1}
\]

where the resonant frequencies \( \omega_{r} \) are the values of \( \omega \) for which the eigenfunction equation,

\[
[\mathcal{L}(\omega) - \lambda_{r}] \phi_{r}(x) = 0, \tag{B2}
\]

has eigenvalue \( \lambda_{r} \) equal to \( \lambda \). \( \psi_{r} \) is the eigenfunction of the corresponding Hermitian adjoint equation.

VECTOR NOTATION

For Laplace’s tidal equations written in matrix form (equation 3.2),

\[
(L - i\omega) \xi = L \xi, \tag{B3}
\]

the Green’s function is (Webb 1974),

\[
G_{1}(x, x', \omega) = i \sum_{r} \frac{\phi_{r}(x) \psi_{r}^{*}(x')}{\omega - \omega_{r}}, \tag{B4}
\]
and the solution of equation (B3) is,

$$\zeta(x) = -\sum_r \phi_r(x) \frac{\omega_r}{\omega - \omega_r} \Psi^*_r(x') \zeta(x') \, dx'.$$

(B5)

The functions $\phi_r$ and $\psi_r$ are normalized so that,

$$\int \psi^*_r(x) \phi_r(x) \, dx = 1.$$

If

$$\phi_r = \left( \begin{array}{c} u_r \\ \xi_r \end{array} \right)$$

and

$$\psi_r = \left( \begin{array}{c} u'_r \\ \xi'_r \end{array} \right),$$

then the tide height $\xi$ is given by,

$$\xi(x) = -\sum_r \xi_r(x) \frac{\rho g \omega_r}{\omega - \omega_r} \int \psi^*_r(x') \xi(x') \, dx'.$$

(B6)

SCALAR NOTATION

It is possible to eliminate the velocity $u$ from equation (B3) and obtain an equation for $\xi$ only. This gives,

$$[L''(\omega) - \epsilon] \xi(x) = L''(\omega) \tilde{\xi}(x).$$

(B7)

Where,

$$L''(\omega) = -\frac{\rho}{\lambda \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \frac{h'^2 \sin \theta}{\cos^2 \theta - \lambda^2} \left( -\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \right) - \frac{\partial}{\partial \phi} \left( \frac{h'}{\cos^2 \theta - \lambda^2} \left( -i \cot \theta \frac{\partial}{\partial \theta} + \lambda \frac{\partial}{\partial \phi} \right) \right) \right],$$

$$\epsilon = 4\Omega^2R^2/gh,$$

$$h' = h/h_0,$$

and,

$$\lambda = \omega/2\Omega.$$

(B8)

The mean depth $h_0$ is introduced so that for a constant depth ocean, equation (B8) is equivalent to equation (2.10) of Longuet-Higgins (1968).

The Green’s function is now,

$$G_2(x, x', \omega) = \sum_r \frac{\xi_r(x) \xi^*_r(x')}{(\omega - \omega_r)(\partial \epsilon/\partial \omega)_{\omega = \omega_r}}.$$

(B9)
We can choose \( \zeta_r \) so that it is the same as \( \zeta_r \) of equation (B6). \( \chi_r \) will have the same spatial behaviour as \( \zeta_r \) used earlier, but it is normalized so that,

\[
\int \chi_r^*(x) \zeta_r(x) \, dx = 1. \tag{B10}
\]

The solution of equation (B7) is now,

\[
\xi(x) = - \sum_r \xi_r(x) \frac{\epsilon}{(\omega - \omega_r)(\partial \epsilon/\partial \omega)_{\omega=\omega_r}} \int \chi_r^*(x') \tilde{\xi}(x') \, dx'. \tag{B11}
\]

**DISCUSSION**

Comparing equations (B11) and (B6), one sees that the term \( (\partial \epsilon/\partial \omega) \) is there to correct for the different normalizations used for the functions \( \xi_r \) and \( \chi_r \). From this one concludes that the role of the term \( (\partial \lambda/\partial \omega) \) in equation (B1) is to act as a normalizing constant.

If the equilibrium tide has the form of the function \( \xi_s(x) \), then from equation (B6),

\[
\xi(x) = \xi_s(x) \frac{\rho g \omega_s}{\omega - \omega_s} \int \xi_s^*(x') \tilde{\xi}_s(x') \, dx'. \tag{B12}
\]

and from equation (B11),

\[
\xi(x) = - \xi_s(x) \epsilon/[(\omega - \omega_s)(\partial \epsilon/\partial \omega)_{\omega=\omega_s}]. \tag{B13}
\]

Thus,

\[
\frac{\epsilon}{(\partial \epsilon/\partial \omega)_{\omega=\omega_s}} = \omega_s \rho g \int \xi_s^*(x') \tilde{\xi}_s(x') \, dx'.
\]

If there is no friction then the integral is just the ratio of the potential energy of the resonance to its total energy. Thus,

\[
\frac{\text{P.E.}}{\text{E.}} = \frac{\frac{\epsilon}{\partial \omega}}{\omega_s \frac{\partial \epsilon}{\partial \omega}}.
\]

\( (\partial \omega/\partial \epsilon) \) is a measure of the change in the resonance frequency as the depth of the ocean or the rotation rate of the Earth is changed. Thus we have shown that there is a connection between this change and the energy partition within the resonance.

Physically this is a reasonable connection. At the Rossby wave end of the spectrum the motion of the water is mainly horizontal. Thus both the potential energy and the effect on the resonances of changes in the depth of the ocean are small. Conversely at the high frequency end of the spectrum, the resonances have almost the same amount of kinetic and potential energy, and the wavelength of the resonances, and thus their frequency, depend on the depth of the ocean.

The validity of the equation is confirmed by the calculations of Longuet-Higgins (1968). He plots both the energy partition and the frequency of the resonances as a function of \( e^{-1/2} \). In terms of the variables Longuet-Higgins uses,

\[
\frac{\text{P.E.}}{\text{E.}} = -\frac{\partial \log (\omega/2\Omega)}{\partial \log e^{-1/2}}. \tag{B15}
\]