We investigate a classical description of spinning particles as a dynamical system consisting of one displacement variable (the center-of-mass variable) and associated internal string variables which produce Fermi operators after the canonical quantization.

The investigation is also made into the path integral expression to the propagator of such a spinning particle.

We further discuss briefly the extension of the above dynamical system to the case of a bi-local spinning particle.

§ 1. Introduction

In a previous paper,\textsuperscript{9} we discussed a method for constructing Fermi operators explicitly out of Bose operators from a general point of view and the result was applied to reformulate the pseudoclassical theory of spinning string\textsuperscript{2} described in terms of the Grassmann variables as a non-local and non-linear theory of the usual canonical variables. In such a construction of Fermi operators out of Bose operators, it played an important role that the Fermi operators are field variables in a one-dimensional parameter space and this is the reason why we had applied our result to the theory of spinning string.

However, it is still interesting to investigate a model of spinning particle of multi local type in order to understand the Dirac particles and their composite system in terms of the usual canonical variables. In the present paper, we investigate a model of spinning particle consisting of one displacement variable (the center-of-mass variable) and associated internal string variables which produce Fermi operators after the canonical quantization.

Now, as is well known, we can express the propagator of the particle in two different forms, that is, the form of path integral and the form of matrix element of the operator of time translation between appropriate initial and final states. The former is related explicitly with the classical action of the particle and can be obtained from the latter by the repeated use of the composition law (the equation of unitarity) satisfied by the propagator.

Using this flexibility in expressing the propagator, we try to derive the

\textsuperscript{9} This work was completed while he stayed at the Institute for Theoretical Physics, University of Karlsruhe as a Research Fellow of the Humboldt Foundation.
classical action of the spinning particle described in terms of the usual canonical variables according to the following line:

In another previous work,\textsuperscript{3} we investigated an operator formalism to the path integral in the case of spinless particles. First, in the next section, we review shortly this formalism and give further discussion on the treatment of the particle’s proper time in this formalism.

Secondly, in § 3, we extend this formalism to the case of pseudoclassical spinning particle described in terms of the Grassmann variables\textsuperscript{4} in order to obtain an operator expression to the propagator of the Dirac particle. In this expression, Fermi operators come out as the $q$-number counterpart of the Grassmann variables.

In § 4, thirdly, we try to rewrite this propagator in terms of Bose quantities applying the result in Ref. 1) and derive the classical action of this dynamical system by reexpressing this propagator in the form of path integral.

Section 5 is devoted to brief discussion on the extension of the above dynamical system to the case of a bi-local spinning particle.

\section{2. An operator approach to the path integral method}

As is well known, we can express the propagator of the Klein-Gordon particle under a potential field $V(x)$ in the following form of path integral:

$$K^V(2, 1; \tau_{2,1}) = \int [d^4x] \exp \left[ -i \int_{\tau_1}^{\tau_2} d\tau \left( \frac{1}{2} \dot{x}^2 - V(x) \right) \right], \quad (\equiv 0 \text{ for } \tau_{2,1}<0)$$

where $\tau$ denotes particle’s proper time and we have abbreviated $\tau_i$ and $x^\mu(\tau_i)$, $(i=1, 2)$ simply as $i$, $(i=1, 2)$.

The right-hand side of Eq. (2·1) is usually defined as a limit of all integrals by the coordinates of the particle at a large number of specified (proper) times separated by very small interval. On the other hand, in Ref. 3), it was shown that the right-hand side of Eq. (2·1) can also be defined as follows:

$$K^V(2, 1; \tau_{2,1}) = \langle 0 | \hbar^{(0)} (x_2 - \bar{x}_{2,1}) \exp \left[ i \int_{\tau_1}^{\tau_2} d\tau \ V(\bar{x}_{\tau,1}) \right] | 0 \rangle,$$

$(\equiv 0$, for $\tau_{2,1}<0)$ \hspace{1cm} (2·2)

where

$$\bar{x}_{\tau,1} = x_1 + \int_{\tau_1}^{\tau} d\tau' \sqrt{i} \left( a^\ast(\tau') + a(\tau') \right),$$

and

$$[a^\ast(\tau), a(\tau')] = -g^{\mu\nu} \delta(\tau-\tau'), \ (\text{other commutators are zero})$$

$^\text{a)}$ $\mu, \nu = 0, 1, 2, 3; \ \text{diag}(g^{\mu\nu}) = (1, -1, -1, -1)$. We also use the units; $\hbar=c=1$.\hspace{1cm}
and $|0\rangle$ is the ground state for the operators $a^\dagger(\tau)$, $(\tau_1 \leq \tau \leq \tau_2)$.

It should be noticed that the right-hand side of Eq. (2.2) has its meaning without depending on a limiting procedure and on the order of the operators $\hat{x}_{\tau_1\tau}$, $(\tau_1 \leq \tau \leq \tau_2)$. From Eq. (2.2) directly, we can further verify that $K^V(2,1;\tau_{s,1})$, $(\tau_{s,1} > 0)$ satisfies a Schrödinger-like equation regarding $\tau$ as the time and the following composition law:

$$K^V(3,1;\tau_{s,1}) = \int d^3x_2K^V(3,2;\tau_{s,2})K^V(2,1;\tau_{s,1}) \quad (\tau_1 < \tau_2 < \tau_3) \quad (2.3)$$

The path integral form of the propagator in Eq. (2.1), then, can be understood as the result of repeated use of Eq. (2.3).

The operator $\hat{x}_{\tau_1}$ used in Eq. (2.2) has an intuitive physical meaning such that each spectrum of $x_{\tau_1}$ represents possible trajectory of the particle in the time interval $[\tau_1,\tau]$ starting $x_{\tau_1}^a$ at time $\tau_1$. This interpretation may be rather strange because of the non-hermiticity of the operator $\hat{x}_{\tau_1}$. However, it can be shown, in general, that the operator obtained by replacing the factor $\sqrt{\mathcal{I}}$ in $\hat{x}_{\tau_1}$ with $e^{\delta t}$, $(|\delta| \leq \pi/4)$ can be always expanded by a set of complete basis so that $\hat{x}_{\tau_1}$ has real spectra. The cases of $\delta = 0$ and $\delta = \pi/4$ correspond respectively to the Wiener process and the present quantum process, and both processes are related with each other by an analytic continuation maintaining the meaning of the operator $\hat{x}_{\tau_1}$. Thus, the $\delta$-function in Eq. (2.2) plays the role restricting the trajectories of the particle so as to end $x_{\tau_1}^a$ at time $\tau_2$. This completes the main results in Ref. 3).

Now, the usual Feynman propagator $D_F(x_2-x_1; m^2)$ is related with $K^0(2,1;\tau_{s,1})$ as follows:

$$iD_F(x_2-x_1; m^2) = \int_0^\infty d\tau d\left(\frac{\tau_{s,1}}{2}\right)K^0(2,1;\tau_{s,1})e^{-i(m^2-\delta E)\tau_{s,1}/\mathcal{I}} \quad (\mathcal{I} = +0) \quad (2.4)$$

The formal aspect of this equation is simply that, under the above integration, the Schrödinger-like equation satisfied by $K^0(2,1;\tau_{s,1})$ is transformed into the equation of the stationary state of "energy $m^2/2"$, that is, the definition of $D_F(x_2-x_1; m^2)$.

Another interesting way of looking at Eq. (2.4) is to regard the integration with respect to $\tau$ as a part of path integral of the classical action, in which $\tau$ plays a dynamical variable ordered by a true time parameter $s$. The propagator corresponding to such a classical action will satisfy a composition law with respect to the parameter $s$.

A simple example of such a propagator is the following:

$$K^s(2,1;\tau_{s,1}) = \int_0^{2\pi} d\theta e^{-i\theta(\check{x}_{\tau_1} - \check{x}_{\tau_2})} \delta(\tau_1 - \tau_{s,1})|0\rangle e^{-i m^2\tau_{s,1}/\mathcal{I}}$$

(\(=0\) for $\tau_{s,1} < 0$) \quad (2.5)
Canonical Approach to Spinning Particles

\[ \tau_{\tau_1} = \tau_1 + \int_{\tau_1}^t ds' \sqrt{1/i\kappa} (\bar{a}^4(s') + \bar{a}^4(s')) \]
\[ x_{\tau_1}^a = x_1^a + \int_{\tau_1}^t ds' \sqrt{i \frac{d^2}{ds'^2} (a^{4\tau}(s') + \bar{a}^{4\tau}(s'))} \]
\[ \left( \frac{d^2}{ds} = \frac{1}{\sqrt{i\kappa}} (\bar{a}^4(s) + \bar{a}^4(s)) \right) \]

(2·6)

and

\[ [\bar{a}^4(s), \bar{a}^{4\tau}(s')] = (-1)^{\delta_{4\tau, 4}\delta_{\tau, \tau'}} (s - s') \]
(A, B = 0, 1, ..., 4; other commutators are zero.)

The state \(|0\rangle\), in this case, is of course the ground state for the operators \(\bar{a}^4(s), (A = 0, 1, ..., 4)\). We have also introduced the real parameter \(\kappa\) for convenience. It should be noticed that (2·5) is a kind of five-dimensional extension of the propagator \(K^a(2, 1; \tau_2, 1)\).

Then, it is not difficult to verify the following composition law:

\[ k^a(2, 1; s_{\tau_1}) = \int d^4x \int d(\epsilon \delta_{s_{\tau_1} \sqrt{\frac{i\kappa}{2\pi}}}) k^a(3, 2; s_{\tau_2}) k^b(2, 1; s_{\tau_1}) \]

\[ (e_2 = \frac{\tau_{\tau_2}}{s_{\tau_1}}) \]

(2·7)

and

\[ k^a(2, 1; s_{\tau_1}) = i(2\pi i \epsilon \delta_{s_{\tau_1}})^{-1/2} \exp \left[ -\frac{i}{2s_{\tau_1}} \left( \frac{x'^2}{e} + m^2e + \kappa e^2 \right) \right] \]

\[ (x'^a = \frac{dx'^a}{ds}) \]

(2·8)

for small \((x'^a - x^a) \sim \tau_{\tau_1} \sim s_{\tau_1}\). Combining Eqs. (2·7) and (2·8), we thus obtain the expression:

\[ k^a(2, 1; s_{\tau_1}) = \int [d^4x][de] \delta(\tau_{s_{\tau_1} - \int_1^2 ds e}) \exp \left[ -\frac{i}{2} \int_1^2 ds \left( \frac{x'^2}{e} + m^2e + \kappa e^2 \right) \right] \]

(2·9)

where the factor \(e^{-\frac{s_{\tau_1}}{2}}\) appearing in the normalization coefficient in Eq. (2·8) has been included in the definition of the measure \([de]\).

Therefore, noticing the relation:

\[ k^a(2, 1; s_{\tau_1}) \rightarrow s^{-1/2}_{s_{\tau_1}} K^a(2, 1; \tau_{s_{\tau_1}}) e^{-i\tau_{s_{\tau_1}}/s^2}, \quad (\kappa \rightarrow 0) \]

we can finally arrive at the following expression:
In the above equation, the limit $\xi \to 0$ must be taken after all integrations. It is, however, interesting to observe that, in this limit, the classical action appearing in Eq. (2·10) tends to the (gauge) invariant action, in which $e$ plays the role of the einbein variable under the reparametrization of $s$. In other words, $\kappa e^2/2$ acts as the gauge fixing term, which has been introduced in order to obtain a propagator satisfying a composition law with respect to the parameter $s$.

§ 3. Extension to the case of pseudoclassical spinning particle

In this section, we investigate a counterpart of Eq. (2·10) in the case of Dirac particle as the pseudoclassical dynamical system described in terms of the Grassmann variables.

We start with the following generalized definition for the Feynman propagator of Dirac particle:

$$S_F^s(2, 1) = \frac{1}{2} \int_0^\infty d\tau_{\tau_1} \langle \langle 2| \exp\left[\frac{i}{\tau_1} \left( m\xi^s + i\xi^s \cdot \partial_2 \right) \right] 1 \rangle \rangle K^0(2, 1; \tau_{\tau_1}) e^{-i(m^2 - i\phi)\tau_1/\xi}, \quad (3·1)$$

where $\xi^s$ and $\xi^s$ are the operators satisfying the same anti-commutation rules as $r^a$ and $\gamma^b r^a$ respectively, and $|i\rangle$, $(i=1, 2)$ denote the elements of a set of complete basis for these operators. If we take $r^a$ and $\gamma^b r^a$ themselves as $\xi^s$ and $\xi^s$, then $S_F^s(2, 1)$ will be reduced obviously to $i\tau^a S_F(x_2 - x_1; m^2)$.

Now, the function under the integration in Eq. (3·1) does not satisfy a composition law such as Eq. (2·3), which is important to express the propagator in the form of path integral. In order to relate $S_F^s(2, 1)$ with a propagator satisfying a composition law, let us rewrite, first, the right-hand side of Eq. (3·1) in the following form:

$$S_F^s(2, 1) = \int_0^\infty d\tau_{\tau_1} d(\tau_{\tau_1} \xi^b) K_F^s(2, 1),$$

where

$$K_F^s(2, 1) = \langle \langle 2| \exp\left[\frac{1}{\tau_1} \left( m\xi^s + i\xi^s \cdot \partial_2 \right) \right] 1 \rangle \rangle K^0(2, 1; \tau_{\tau_1}) e^{-i(m^2 - i\phi)\tau_1/\xi}, \quad (3·2)^*)$$

* The integration of the Grassmann variables is defined, according to Berezin, as a linear functional satisfying $\int d\theta = 1$ and $\int d\theta_1 = 0$. 
and $\xi^i$ is the Grassmann variable anticommuting with both of $\hat{\xi}^s$ and $\hat{\xi}^s$. For a fixed $\xi^i$, $K_F^s(2,1)$ satisfies a composition law which reproduces the form of propagator by summing the intermediate states $\{|2\rangle\}$ in addition to the integration with respect to $x_2^s$.

In $K_F^s(2,1)$, $\xi^i$ still remains as a parameter and, next, let us relate $K_F^s(2,1)$ with a propagator such that $\xi^i$ plays a dynamical variable ordered by the time $\tau$. This step is the same as the fact that we have regarded $\tau$ as a dynamical variable in the previous section. As a simple example of such a propagator, we here investigate the following:

$$k_F^s(2,1) = \langle 2, 0 | \hat{\beta}^{(0)} (x_3 - \hat{x}_1^s) \exp \left\{ - \frac{i}{2} (m^2 + im \hat{\xi}^s \hat{\xi}^s) \tau_2 \right\} |0, 1\rangle,$$

(3.3)*)

where

$$\hat{x}_1^s = x_1^s + \int d\tau' \{ \sqrt{i} (a^s(\tau') + a^s(\tau)) - \frac{i}{2} \hat{\xi}^s \xi^s \},$$

(3.4)

and we also assume that $\hat{\xi}^i$ is the operator anti-commuting with both of $\hat{\xi}^s$ and $\hat{\xi}^s$ and satisfying $(\hat{\xi}^i)^2 = 0$. In this case, of course, $|i\rangle, (i = 1, 2)$ denote the elements of complete basis for the operators $\hat{\xi}^A, (A = 0, 1, \cdots, 5)$.

Now, let us represent these operators by using anti-commuting oscillator variables as follows:

$$\hat{\xi}^s = (b^s + b^s), \quad \hat{\xi}^s = \frac{1}{\sqrt{2}} (b^s + b^s), \quad \hat{\xi}^s = \frac{1}{\sqrt{2} \kappa'} b^s,$$

(3.5)

where $\kappa' (>0)$ is a parameter and

$$\{b^A, b^B\} = (-1)^{A+B} \delta_{A,B}, \quad \{b^A, b^B\} = \{b^B, b^B\} = 0.$$ (A, B = 0, \cdots, 5)

One possible choice of the complete basis for these operators is that

$$|i\rangle = \exp (\hat{\xi}^i b^i - \sqrt{\kappa'} \hat{\xi}^s \hat{\xi}^s b^s - \hat{\xi}^i b^s) |O_b\rangle,$$

$$\langle i| = \langle O_b | \prod_A (\sqrt{\kappa'} \hat{\xi}^A b^A - b^A), \quad (\prod_A f^A = f^s f^s \cdots f^s)$$

(3.6)

where $|O_b\rangle$ is the ground state for the operators $b^A, (A = 0, \cdots, 5)$. Another convenient choice of the complete basis is the following:

$$|i'\rangle = \exp (-\sqrt{\kappa'} \hat{\xi}^i b^s) U(\zeta, \zeta_i) |O_b\rangle,$$

$$\langle i'| = \langle O_b | U(-\hat{\xi}^i, -\zeta_i) (\sqrt{\kappa'} \hat{\xi}^i - b^i),$$

(3.7)

where

$$U(\zeta, \zeta_i) = \exp \left[ (\zeta^s b^s + \zeta^s b^s) - (\zeta^s b^s + \zeta^s b^s) \right].$$

*) For the sake of simplicity, we hereafter write $m^s - i\xi$ as $m^s$ simply.
As can be verified easily, the states defined in (3.6) and (3.7) form a complete basis in the following sense:

\[
\frac{1}{\sqrt{\kappa}} \int (\prod_{A}^{b} d\xi^{A}) |i\rangle \langle i| = \frac{1}{\sqrt{\kappa'}} \int d\xi' \prod_{A=0}^{b} d\xi_{A} d\zeta_{A} |i'\rangle \langle i'| = -I. \quad (3.8)^{9}
\]

We can also verify that

\[
k_{\eta}^{b}(2, 1) = -i \int (\prod_{A}^{b} d\eta_{A} d\eta'_{A}) k_{\eta}^{b}(2', 1'), \quad (3.9)
\]

where

\[
\xi^{A} = \frac{1}{2} (\xi^{A} + \xi'), \quad \eta^{A} = \frac{i}{2} (\xi^{A} - \xi'), \quad (A = 0, 1, 2, 3, 5)
\]

and \(k_{\eta}^{b}(2, 1)\) and \(k_{\eta}^{b}(2', 1')\) are the propagators defined in Eq. (3.3) by using the basis given respectively in (3.6) and (3.7).

Equation (3.8) says that the composition law for the propagator \(k_{\eta}^{b}(2', 1')\) can be written as follows:

\[
k_{\eta}^{b}(3', 1') = \int \frac{d\xi^{A}}{\sqrt{\kappa}} (\prod_{A=0}^{b} d\xi_{A} d\zeta_{A}) \int d^{4}x_{z} k_{\eta}^{b}(3', 2') k_{\eta}^{b}(2', 1'). \quad (3.10)
\]

For small \(\tau_{z,1}\), we can further verify that

\[
k_{\eta}^{b}(2', 1') = i (2\pi i \tau_{z,1})^{-2} \int \frac{d\eta^{A}}{\sqrt{\kappa'}} \exp [i \tau_{z,1} L], \quad (3.11)
\]

where

\[
L = -\frac{1}{2} \left\{ \hat{x}^{2} + i \hat{\xi}^{2}(\sqrt{2} \hat{\xi} \cdot \hat{x} + m \hat{\xi}) - i (\hat{\xi} \cdot \hat{\xi} + \hat{\eta} \cdot \hat{\eta}) + 2i \kappa' \eta^{A} \xi^{A} \right. \\
\left. + i (\hat{\xi}^{2} \hat{\xi}^{2} + \hat{\eta}^{2} \hat{\eta}^{2}) + m^{2} \right\}
\]

by regarding \(\xi_{2}^{A} - \xi_{1}^{A}\) and \(\eta_{2}^{A} - \eta_{1}^{A}\) as the same order amounts as \(\tau_{z,1}\). From Eqs. (3.10) and (3.11), we thus arrive at the expression:

\[
k_{\eta}^{b}(2', 1') = \int \prod_{A}^{b} d\xi^{A} d\eta^{A} \int \frac{d\eta' d\xi'}{\kappa'} \int d^{4}x \left[ (\sqrt{\kappa'} \hat{\xi}^{2}) \exp \left[ i \int_{1}^{2} d\tau L \right] \right]. \quad (3.12)
\]

The \(k_{\eta}^{b}(2', 1')\) is the propagator in phase space with respect to the Grassmann variables and the path integral expression to \(k_{\eta}^{b}(2, 1)\) (the propagator in \(\xi\)-space) can be obtained by substituting the above expression of \(k_{\eta}^{b}(2', 1')\) for the right-hand

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side of Eq. (3.9). It is, of course, possible to obtain a path integral expression of \( k_{p}(2, 1) \) by using the first completeness in Eq. (3.8) directly, but the use of Eq. (3.12) is more convenient because, in this expression, the integral measures of the Grassmann variables always appear in the forms of even amount.

In both the cases of \( k_{p}(2, 1) \) and \( k_{p}(2', 1') \), the relation between the propagators \( K_{p} \) and \( K_{p}' \) can be given by

\[
\int d\xi^{2} K_{p}^{0}(2, 1) = -\left( \frac{2}{k'} \right)^{1/2} \int d\xi^{2} d\xi^{1} k_{p}^{0}(2, 1). \tag{3.13}
\]

Thus, using Eqs. (3.2), (3.12) and (3.13), we can obtain, finally the path integral expression to \( S_{p}(2, 1) \).

We have so far regarded \( \tau \) as a parameter but if we follow the prescription used in the previous section, we can also obtain the formulation regarding \( \tau \) as a dynamical variable, that is, a function of a true time ordering parameter \( s \). We do not repeat, here, this procedure and only refer to the following result:

The propagator defined by

\[
\bar{K}_{p}^{0}(2, 1; s_{1, 1}) = \sqrt{\frac{2\pi}{i\kappa}} \langle 0, 2 | \bar{\phi}(x_{2} - x_{1}) \bar{\phi}(\tau_{2} - \tau_{1}) \exp \left\{ -\frac{i}{2} (m^{2} s_{1, 1} + im \xi s_{1, 1}) \right\} | 0, 1 \rangle ,
\]

where

\[
\tilde{\xi}^{2}_{1, 1} = x_{1}^{2} + \int_{1}^{2} ds \left\{ \sqrt{2} \left[ (d\xi^{2}) (\bar{a}^{n}(s) + a^{n}(x)) - \frac{i}{2} \xi^{r} \xi^{s} \right] \right\} , \tag{3.14}
\]

satisfy a composition law with respect to the parameter \( s \). Then, we can derive the following action:

\[
S = -\frac{1}{2} \int d\xi^{2} \left( e^{-\xi^{2}} + \sqrt{2} i e^{-\xi^{2}} \cdot \xi^{2} \cdot \xi^{r} - i \xi^{2} \cdot \xi^{r} + i \xi^{2} \xi^{s} + im \xi^{2} \xi^{s} \right)
\]

form the above propagator in \( \xi \)-space.

In the limit \( \kappa, \kappa' \rightarrow 0 \), the above action tends to the one of the spinning particle discussed by Brink et al., which is invariant under a supersymmetric transformation mixing the usual canonical variables and the Grassmann variables in addition to the reparametrization of \( s \).

\[\text{§ 4. Bose description of a spinning particle}\]

In the previous section, we have seen that \( k_{p}^{0}(2, 1) \) defined in Eq. (3.3) is the

\[\text{*) Since } \xi^{A} \text{ and } \gamma^{A}, (A=0, 1, 2, 3, 5) \text{ appear in a decoupled way in } L, \text{ the effective action, in this case, becomes a function of } \xi^{A} \text{ and } \gamma^{A}.\]
operator expression to the propagator of a spinning particle and can be related with the path integral of the classical action introduced by Brink et al. by choosing the states defined in Eq. (3·6) as the complete basis for the operators \((b^A, b^A\dagger)\), \((A=0, 1, \ldots, 5)\).

The form of the classical action, however, depends on the choice of the complete basis for the above operators explicitly and we here investigate the possibility of deriving a classical action described in terms of Bose variables only by an appropriate choice of the complete basis.

Now, as was shown in the previous paper, the field operators, in the one-dimensional parameter space \(\sigma_2 \leq \sigma \leq \sigma_1\), defined as

\[
\beta^A(\sigma) = e^{i4\pi A(\sigma)}\alpha^A(\sigma + \varepsilon), \quad (A=0, 1, \ldots, 5; \varepsilon = +0)
\]

where

\[
N(\sigma) = -\sum_{A=0}^{5} (-1)^{2A+1} \int_{\sigma} d\sigma' \alpha^A(\sigma') \alpha^A(\sigma'),
\]

satisfy the following anti-commutation rules:**

\[
\{(\beta^A, f), (\beta^B\dagger, g)\} = (-1)^{2A+B} \delta_{A,B}(f, g),
\]

\[
\{(\beta^A, f), (\beta^B, g)\} = \{(\beta^A\dagger, f), (\beta^B\dagger, g)\} = 0
\]

providing that

\[
[\alpha^A(\sigma), \alpha^B\dagger(\sigma')] = (-1)^{2A+B} \delta_{A,B}(\sigma - \sigma'),
\]

\[
[\alpha^A(\sigma), \alpha^B(\sigma')] = [\alpha^A\dagger(\sigma), \alpha^B\dagger(\sigma')] = 0.
\]

The above transformation from \(\{\alpha^A(\sigma)\}\) to \(\{\beta^A(\sigma)\}\) is just a representation of Fermi operators in terms of Bose operators.

We hereafter put the range of \(\sigma\) as \(0 \leq \sigma \leq 1\) without loss of generality and, then, the operators

\[
\beta^A = (\beta^A, 1), \quad \beta^A\dagger = (\beta^A\dagger, 1), \quad (A=0, 1, \ldots, 5)
\]

satisfy the same anti-commutation rules as the operators defined in Eq. (3·5). Substituting these operators into the right-hand side of Eq. (3·3) and regarding \(\{|\nu\rangle\}\) as a set of complete basis for these operators, we can again obtain a propagator satisfying a composition law.

Let us try to derive the classical action associated with such a propagator by taking the following states as the above complete basis:

\[
|\nu\rangle = \exp \left[ \sum_{A=0}^{5} \int_{\sigma} d\sigma \{ (\alpha^A, \bar{\xi}_f^A) - (\alpha^A, \bar{\xi}_f\dagger^A) \} + \sqrt{m} (\alpha^A, \bar{\xi}_f^A) \right] |0\rangle,
\]

** After the publication of Ref. 1, the author found that the representation of Fermi operators out of Bose operators in this form had been already discussed by Schroer in Ref. 7.

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\(\beta^A(\sigma) = e^{i4\pi A(\sigma)}\alpha^A(\sigma + \varepsilon), \quad (A=0, 1, \ldots, 5; \varepsilon = +0)\)

\(\{(\beta^A, f), (\beta^B\dagger, g)\} = (-1)^{2A+B} \delta_{A,B}(f, g),\)

\(\{(\beta^A, f), (\beta^B, g)\} = \{(\beta^A\dagger, f), (\beta^B\dagger, g)\} = 0\)

\([\alpha^A(\sigma), \alpha^B\dagger(\sigma')] = (-1)^{2A+B} \delta_{A,B}(\sigma - \sigma'),\)

\([\alpha^A(\sigma), \alpha^B(\sigma')] = [\alpha^A\dagger(\sigma), \alpha^B\dagger(\sigma')] = 0.\)

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\(*\) After the publication of Ref. 1, the author found that the representation of Fermi operators out of Bose operators in this form had been already discussed by Schroer in Ref. 7.

\(**\) \((f, g) \equiv \int_{\sigma_2} d\sigma f(\sigma)g(\sigma), \) etc.
Canonical Approach to Spinning Particles

\[ \langle j | = \langle 0 | \exp \left\{ \sum_{A=0}^{5} \frac{i}{2} \lambda_r A^A \right\} \left( \alpha^A, \bar{\beta}^A, \alpha^A, \bar{\beta}^A \right) \]

\[ - \frac{1}{2} \left( \sqrt{\kappa} \bar{\xi}^A_j \left( \alpha^A - \alpha^A \right), \sqrt{\kappa} \bar{\xi}^A_j \left( \alpha^A - \alpha^A \right) \right) \tag{4.5} \]

where \( |0\rangle \) is the ground state for the operators \( \alpha^A (\sigma), (A=0,1,\ldots,5) \). These states are complete in the following sense:

\[ \int \left\{ \prod_{A=0}^{5} \frac{d\xi^A_j}{\sqrt{2\pi}} \right\} \left\{ \sqrt{\kappa} \frac{d\bar{\xi}^A_j}{\sqrt{2\pi}} \right\} |j\rangle \langle j| = (\text{unit operator}), \tag{4.6} \]

where \( \xi^A (\sigma) = \xi^A (\sigma) + i\bar{\eta}^A (\sigma), (A=0,1,2,3,5) \) and the functional measure should be understood, by using a set of complete orthonormal functions \( \{ \phi_n (\sigma) \} \) in \( 0 \leq \sigma \leq 1 \), in such a sense as \( \langle ad\bar{\xi}^A \rangle = \prod_n a (d\xi^A, \phi_n) \).

Now, we can verify that

\[ k_F^g (2,1) = i \left( 2\pi i \tau_{2,1} \right)^{-1} \int \left\{ \sqrt{\kappa} \frac{d\bar{\xi}^A_j}{\sqrt{2\pi}} \right\} \exp \left\{ i \tau_{2,1} \left( \sum_{A=0}^{5} \frac{1}{2} \left( -1 \right)^{\lambda_r A^A} \lambda_r A^A \right) \right\} \]

\[ \tag{4.7} \]

for small \( \tau_{2,1} \sim \bar{\xi}^A_j - \xi^A_j \sim |\bar{\xi}^A_j - \xi^A_j|, (A=0,1,2,3,5) \), where

\[ A = -2 (\bar{\eta}^A_0, \bar{\xi}_0) + 2 \kappa' (\eta^A, \bar{\xi}^A) + 2 (\bar{\eta}^A_0, \bar{\xi}^A) \]

\[ - \frac{1}{2} \left( \bar{x} - \frac{i}{\sqrt{2}} (\bar{\xi}^A_0, \theta_\kappa \bar{\xi}^A) \right)^2 - \frac{1}{2} m (\bar{\xi}^A, \theta_\kappa \bar{\xi}^A) - \frac{1}{2} m^2, \tag{4.8} \]

\[ (\xi^A (\sigma) = \xi^A (\sigma), \text{etc.,}) \]

and \( \theta_\kappa \) is the matrix in \( \sigma \)-space defined as,

\[ (\varepsilon (\sigma) = |\sigma| \sigma^{-1}) \]

\[ \theta_\kappa (\sigma'', \sigma') = \exp \left\{ -2 \varepsilon (\sigma'' - \sigma') \int_{\sigma''}^{\sigma'} d\sigma \sum_{A=0}^{5} \left( -1 \right)^{\lambda_r A^A} \lambda_r A^A (\eta^A (\sigma)^2 + \bar{\xi}^A (\sigma)^2) \right\} 

\[ + 4i \kappa' \varepsilon (\sigma'' - \sigma') \int_{\sigma''}^{\sigma'} d\sigma \eta^A (\sigma) \bar{\xi}^A (\sigma) + i\pi \theta (\sigma'' - \sigma'). \tag{4.9} \]

Using Eqs. (4.6) and (4.7), we can express \( k_F^g (2,1) \) as follows:

\[ k_F^g (2,1) = \int \left[ d^6x \right] \int \left[ \prod_{A=0}^{5} \frac{d\xi^A_j}{\sqrt{2\pi}} \right] \left[ \sqrt{\kappa} \frac{d\bar{\xi}^A_j}{\sqrt{2\pi}} \right] \]

\[ \times \exp \left\{ -i \left( \sum_{A=0}^{5} \left( -1 \right)^{\lambda_r A^A} \lambda_r A^A \right) A \right\} + i \int_{\tau} A d\tau, \tag{4.10} \]

where we have used the bold-faced bracket in order to express the measure of

\[ k_F^g (2,1) \]

\[ \text{if we use the states which diagonalize the operators } \xi^A, (A=0,1,2,3,5) \text{ as the states } |f_i \rangle, (i=1,2), \text{ then the factor } [\cdots],^2 \text{ in Eq. (4.10) will disappear.} \]
path-integral by the field variables. Noticing, further, that
\[
\langle 0 | 2^{4-4} | 0 \rangle = 1, \quad \langle 1^{4-4} | \sqrt{2} \kappa b^4 | 0 \rangle = -2 \sqrt{2} \pi \partial \{ \partial \left( \sqrt{2} \xi^4 \right) \}/\partial \langle \xi^4 | 1 \rangle,
\]
where \(| j^{4-4} \rangle = | j \rangle |_{x^4=0}, (j=1, 2)\), we can also obtain the following:
\[
S_{+}(2, 1) = -2 \sqrt{2} \pi \int_{0}^{\infty} \frac{d \tau_{2,1}}{\tau_{2,1}} \int \left( \frac{\kappa'}{2\pi} \right) d\xi_1^0 d\xi_2^0 k_{F}^0(2, 1) \frac{\partial}{\partial \langle \xi^4 | 1 \rangle} \left( \sqrt{2} \xi^4 \right).
\]
Equations (4·10) and (4·11) say that \(f_{i}^{A} d\tau A\) plays the part of classical action of this dynamical system.

We have regarded \(\tau\) as a parameter so far. However, if we start from the propagator \(k_{F}^0(2, 1; z_{1,1})\) defined in Eq. (3·14), for example, instead of \(k_{F}^0(2, 1)\), we can always arrive at the formulation regarding \(\tau\) as a dynamical variable. Then, it is not difficult to obtain the following as the classical action associated with such a propagator:
\[
S = -\int_{0}^{\infty} ds \left( 2(\eta^4, \xi^4') - 2\kappa' (\eta^4, \xi^4') - 2(\eta^4, \xi^4') + \frac{\kappa}{2} e^2 \right) + \frac{1}{2} e \left( x^2 - \frac{i}{\sqrt{2}} (\xi^4, \theta \xi^2) \right) + \frac{i}{2} m (\xi^4, \theta \xi^0 \xi^6) + \frac{1}{2} m^2 e \right].
\]
The action \(S_0\) which is (gauge) invariant under the reparametrization of \(s\) is obtained from \(S\) by taking the limit \(\kappa, \kappa' \rightarrow 0\). We finally comment on the problem appearing in the quantization of the dynamical system described by \(S_0\).

From the definition of \(S_0\), we can see that
\[
\Pi^A(\sigma) = (-1)^{s_{A}s_{A}} 2\eta^A(\sigma), \quad (A=0, 1, 2, 3, 5)
\]
\[
P_{s} = -e^{-1} \left\{ x_{s}^2 - \frac{i}{\sqrt{2}} (\xi^4, \theta \xi^2) \right\},
\]
are the momenta conjugate to \(\xi^A(\sigma), (A=0, 1, 2, 3, 5)\) and \(x^s\) respectively. Then, after the canonical quantization,
\[
\alpha^A(\sigma) = \xi^A(\sigma) + \frac{i}{2} \Pi^A(\sigma), \quad (A=0, 1, 2, 3, 5)
\]
can be regarded as the operators defined in Eq. (4·3).

Now, the variation of \(e\) and \(\xi^A(\sigma)\) in \(S_0\) leads to the constraints:
\[
(P^2 - m^2) T = 0,
\]
\[
(\sqrt{2} P \cdot (\partial_{\phi} \xi) (\sigma) - m (\partial_{\phi} \xi^6) (\sigma)) T = 0,
\]
\[
\delta (\xi) \equiv \Pi_{a} \delta_{a} (\xi, \phi_{a}).
\]
where $\hat{\theta}_0$ is the $q$-number counterpart of $\theta_0$, in which the order of the operators $(\xi^A(\sigma), \Pi^A(\sigma))$, ($A=0,1,2,3,5$) is appropriately defined. However, how should we define the ordering of the operators? The answer is taking the normal ordering with respect to $(\alpha^A(\sigma), \alpha^{A\dagger}(\sigma))$, that is to say,

$$\hat{\theta}_0(\sigma'', \sigma') \equiv: \exp \left[ -2\varepsilon (\sigma'' - \sigma') \int d\sigma \sum_{A=0}^{1} \frac{1}{2} (-1)^{A+\alpha} \alpha^A(\sigma) \alpha^A(\sigma) + i\pi(\sigma'' - \sigma') \right] ; \tag{4.14}$$

and, then, it can be verified that (see (A·4))

$$(\hat{\theta}_0 \xi^A)(\sigma) = \frac{1}{2} [ (\beta^A, 1) + (\beta^A, 1) ] e^{i\pi N(\sigma)}. \quad (A=0,1,2,3,5)$$

Thus, the second of Eq. (4·13) can be transformed into the following Dirac-like equation:

$$(P \cdot \Gamma - m \Gamma^\theta) \Psi = 0, \quad (\Psi^\theta = e^{i\pi N(\sigma)} \Psi) \tag{4·15}$$

where

$$\Gamma^\theta = [(\beta^0, 1) + (\beta^0, 1)], \quad \Gamma^\theta = \frac{1}{\sqrt{2}} [ (\beta^0, 1) + (\beta^0, 1) ].$$

Since $N(\sigma)$ is commutable with $P^0$, Eq. (4·15) is compatible with the first of Eq. (4·13). Equation (4·15) says that the spectra of the operator $\xi^\sigma(\sigma)$ are infinitely degenerate and so, in other words, greater part of the internal string variables remains as hidden variables in this model.

§ 5. Discussion

We have been able to obtain a classical model of spinning particle described in terms of one displacement variable $x^a$, two kinds of internal dynamical variables $(\xi^a(\sigma), \xi^b(\sigma))$, and two kinds of auxiliary variables $(e, \xi^a(\sigma))$ which become gauge variables in the limit $\epsilon, \epsilon' \to 0$.

Though the internal dynamical variables are the usual canonical variables, after the canonical quantization, they have been able to produce a set of Fermi operators as the result of their non-local and non-linear character. In this sense, we could derive a Dirac-like equation from the classical action in which the gauge invariant limit $\epsilon, \epsilon' \to 0$ had been taken.

We can extend this model easily so that mass spectra may appear by introducing more displacement variables $x_i^a$, ($i=1, \cdots, N$) and potentials among them. We here show only an example in case $N=2$ such that the classical action is given as follows: ($\lambda =$const)

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*We can always eliminate $e$ from the classical action using the constraint obtained by varying the action with respect to $e$. 
\[ S_0 = - \int_0^2 ds \left[ 2(\xi', \xi'') - 2(\eta', \xi'') + \sum_{i=1}^2 \frac{1}{2\epsilon} \left( x_i' - \frac{i}{\sqrt{2}} \xi_i'' \right)^2 \right. \\
\left. + \lambda i e^{2} \xi_i' x_i' + \lambda (i \sqrt{2} (\xi_1' - \xi_2') \cdot \xi_3' - \lambda \xi_3' e) \right], \]  
(5.1)

where we have put \( \xi^i(\sigma) = \xi^i(\sigma) \) and

\[ x^s = (x_1^s - x_2^s), \]
\[ \xi_i^s = \int_0^1 d\sigma \theta_0(0, \sigma) f_i(\sigma) \xi^i(\sigma), \quad (i = 1, 2, 3) \]

and \( f_i(\sigma), \ (i = 1, 2, 3) \) are appropriate orthonormal functions in \( 0 \leq \sigma \leq 1 \).

Varying the above action with respect to \( e \) and \( \xi^i \), we can obtain the following constraints among the canonical variables:

\[ P_1^s + P_2^s - 2\lambda (i \sqrt{2} (\xi_1' - \xi_2') \cdot \xi_3' - \lambda \xi_3') = 0, \]
\[ P_1 \xi_1^s + P_2 \xi_2^s + \sqrt{2} \lambda \xi_3^s = 0, \]  
(5.2)

where \( P_i^s, \ (i = 1, 2) \) are the momenta conjugate to \( x_i^s, \ (i = 1, 2) \). Then by defining

\[ P^s = P_1^s + P_2^s, \quad \bar{P}^s = \frac{1}{2} (P_1^s - P_2^s), \]
\[ \bar{a}^s = \frac{1}{\sqrt{2} \lambda} (\lambda \bar{x}^s + i \bar{P}^s), \quad a^s = \frac{1}{\sqrt{2} \lambda} (\lambda x^s - i P^s), \]
\[ \bar{b}^s = \frac{i}{2} (\xi_1'^s - \xi_2'^s) + \frac{1}{\sqrt{2}} \xi_3'^s, \quad \bar{b}^s = - \frac{i}{2} (\xi_1'^s - \xi_2'^s) + \frac{1}{\sqrt{2}} \xi_3'^s, \]  
(5.3)

the canonical quantization leads to the following (anti-)commutation rules:

\[ [\bar{a}_s, \bar{a}_s] = -g_{ss}, \quad [\bar{a}_s, a_s] = [\bar{a}_s, a_s] = 0, \]
\[ \{\bar{b}_s, \bar{b}_s\} = -g_{ss}, \quad \{\bar{b}_s, b_s\} = \{\bar{b}_s, \bar{b}_s\} = 0. \]  
(5.4)

In the quantized system, we can also express Eq. (5.2) by the use of the variables defined in Eq. (5.3) in the following form:

\[ (\frac{1}{2} P^s + 2\lambda [\bar{a}_s, a^s] + 2\lambda [\bar{b}_s, \bar{b}^s]) \Psi = 0, \]
\[ (\frac{1}{2} P \cdot \Gamma + \sqrt{2} \lambda (\bar{a} \cdot \bar{b} + a \cdot b)) \Psi = 0. \quad (\Gamma^s = \xi_1^s + \xi_2^s) \]  
(5.5)

Since \( \Gamma^s \) satisfies the same anti-commutation rules as \( \bar{r}_s r^s \) and anti-commutes with \( (\bar{b}_s, \bar{b}_s) \), the consistency of Eq. (5.5) can be verified easily by using Eq. (5.4).

Equations (5.5) are the wave equations for the bi-local spinning particle consisting of two displacement variables \( x_i^s, \ (i = 1, 2) \) coupled with the same internal string-like variables \( \xi^s(\sigma) \) by respective weights \( f_i(\sigma), \ (i = 1, 2) \). Similarly in the above, we can extend this model further to the case \( N > 2 \) without changing the
freedom of internal string variable. Thus, in the limit \( N \to \infty \), this dynamical system tends to the one of spinning string and the degeneracy of the spectra for the internal string variable is removed completely.

We finally notice that, in the above treatment of the bi-local spinning particle, we have disregarded the constraint playing an important role in the elimination of the negative norm states. In order to derive such a constraint from the classical action, it is necessary to modify the definition of the classical action slightly from the one in (5·1).

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Appendix

It is always necessary to represent each operator defined in Eq. (4·1) in the form of well-defined matrix element in order to examine their anti-commutation rules. For the sake of simplicity, let us investigate the case of one oscillator field variable such as

\[
[\alpha(\sigma), \alpha'(\sigma')] = \delta(\sigma - \sigma'), \quad (\sigma, \sigma' \in [0, 1])
\]

\[
\beta(\sigma) = \exp \left[ i\pi \int_{\sigma}^{\sigma'} d\sigma' \alpha(\sigma') \alpha'(\sigma') \right] \alpha'(\sigma + 0). \tag{A·1}
\]

As the convenient complete basis, we can take the following coherent states:

\[
|z\rangle = e^{i\pi a^{+}}|0\rangle. \quad (=\langle z^{+}\rangle)
\]

(A·2)

Then, we can verify easily that

\[
\langle z_{1}|\beta'(\sigma)\beta'(\sigma')|z_{2}\rangle = \langle z_{1}'|\alpha'(\sigma') \alpha(\sigma) |z_{2}\rangle^{*} = z_{1}(\sigma)^{*}z_{2}(\sigma') \exp(z_{1}{^{\ast}a^{+}}, z_{2}{^{\ast}a}),
\]

\[
\langle z_{1}|\beta'(\sigma')\beta'(\sigma)|z_{2}\rangle = \langle z_{1}'|\beta'(\sigma - \sigma') + \alpha'(\sigma') \alpha(\sigma) |z_{2}\rangle^{*} = z_{1}(\sigma)^{*}z_{2}(\sigma') \exp(z_{1}{^{\ast}a^{+}}, z_{2}{^{\ast}a}),
\]

(A·3)

\[
(z^{\ast}(\sigma') = (-1)^{\delta(\sigma - \sigma')}z(\sigma'),
\]

provided that both the sides of Eq. (A·3) are smeared by the well-behaved functions \( f(\sigma), g(\sigma') \) as in Eq. (4·2). Thus, noticing \( (z_{1}{^{\ast}a^{+}}, z_{2}{^{\ast}a}) = (z_{1}{^{\ast}a^{+}}, z_{2}{^{\ast}a}) \), we can arrive at the anti-commutation rules of Eq. (4·2).

It should be noticed that the rightest side of each equation in (A·3) is rather the definition of the operators \( \beta'(\sigma)\beta'(\sigma') \) and \( \beta(\sigma')\beta'(\sigma) \). We finally notice that the formal expression of the operator having matrix element \( e^{i\pi a^{+}a^{\ast}} \) between the
basis defined in (A·2) is not unique as can be seen in the following:

\[
\langle z_1 \exp \left[ \pm i \pi \int_0^\sigma d\sigma' \alpha'(\sigma') \alpha(\sigma') \right] | z \rangle
\]

\[= \langle z_1 \rangle \exp \left[ -2 \int_0^\sigma d\sigma' \alpha'(\sigma') \alpha(\sigma') \right] | z \rangle = \exp \left( z_2^* \cdot z_1 \right). \tag{A·4}
\]

References

   See also the following: T. Goto, Prog. Theor. Phys. 60 (1978), 1298.
   See also the following:
7) B. Schroer, Lectures presented at Cargese summer school, 1976.
   See also the following (in §§ 3 and 4) and papers cited therein: