Indefinite Finsler Type Metrics and Their Peculiar Properties

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Usually a Finsler metric tensor is a tensor derived from a given fundamental function \( L(x, y) \) by the equation

\[
\bar{g}_{ij} = g_{ij} - 2n_i n_j \quad (n_i \text{ is a unit vector with respect to } g)
\]

is also the Riemann metric. However this is not the case for Finsler geometry. To discuss this fact we first define the concept “generalized Finsler metric”.

We shall call a symmetric tensor \( g_{ij}(x, y) \) of positively homogeneous of degree zero, i.e., \( g_{ij}(x, ky) = g_{ij}(x, y) \) for any \( k > 0 \), a generalized Finsler metric if \( g_{ij} \) is non-degenerate. Perhaps it will be kind to give the condition for a given \( g_{ij} \) being a usual Finsler metric and the result is as follows:

Let us define a tensor \( C_{ijk} \) correspond-
ing to $g_{ij}$ by the equation

$$2C_{ijk} = g_{ij}b + g_{jk}k - g_{ik}j,$$

then we can prove the following Lemma.

**Lemma** $g_{ij}(x, y)$ is a usual Finsler metric if $C_{ijk}y^i = C_{ijk}y^k = 0$.

**Proof** Necessity of the condition follows from a usual equation for the metric given by $L(x, y)$. Sufficiency can be proved as follows. Let us define $F^2(x, y)$ by $g_{ij}(x, y)y^iy^j$, then if the condition of the lemma is satisfied, $g_{ij}$ is precisely the metric derived from $F(x, y)$. q.e.d.

As an application of the lemma we can prove the following theorem.

**Theorem** If we define an Finsler type metric $g_{ij}$ from a given Finsler metric $g_{ij}$ by the equation

$$g_{ij} = g_{ij} - 2n_in_j,$$

where $n_i(x, y)$ is a contravariant vector with unit length with respect to $g^{ij}$ and degree zero in $y$, then $g_{ij}$ is a usual Finsler metric if $n_i$ is $y$-independent.

**Proof** If we construct the tensor $C_{ijk}$ corresponding to $g_{ij}$, then it is a straightforward matter to show that if $g$ is a usual Finsler metric, $n_i$ is $y$-independent. If $n_i$ is $y$-independent, the given $g$ is a Finsler metric derived from $L - (n_iy^i)^2$ where $L$ is the fundamental function of $g$. q.e.d.

As an example of the generalized Finsler metric, we consider the following type metric:

$$g_{ij} = LL_i|_j - L_j|_i,$$

where $L(x, y)$ is a given function of degree one in $y$. This tensor is precisely $g_{ij}$ in (3) for $n_i$ corresponding to the case $n_i = L_i$, so it is not a usual Finsler metric.

The metric $g_{ij}$ given in (4) has the following interesting and important property. By the definition of $g_{ij}(x, y)$, $g_{ij}(x, y)y^iy^j < 0$ for any tangent vector $y$ in $M_x$ (tangent vector space at $x$), so any vector $y$ is timelike with respect to $g_{ij}$.

If we define a causal structure by using a curve $x^t(\xi)$ with definite sign tangent vector (for example, $g_{ij}(x, y)y^iy^j < 0$ for any timelike curve), above metric $g_{ij}$ does not have past or does not have future. This kind of property is not peculiar to $g_{ij}$ defined by (4), but also for any non-symmetric Finsler metric, i.e., for the metric such that $g_{ij}(x, -y) = g_{ij}(x, y)$. This kind of future-past non-symmetric causal structure will be appropriate in discussing the beginning of our universe (Big Bang). Details of the causal structure in Finsler space will be discussed elsewhere.

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