The Lorentz invariant theory of gravitation is reexamined when the source of gravity is spin-1/2 particles and gauge fields such as the electromagnetic field. Starting with conventional field theory of a symmetric, massless tensor field coupled to the symmetrized energy-momentum tensor of the source, we notice that in addition to the symmetrized tensor field, a new antisymmetric tensor field is required to ensure consistency of the field equation. It is then shown that there are two satisfactory nonlinear theories of gravitation, which are degenerate in the classical limit. One of them is General Relativity based on the Riemann space-time. The other is a gravitational theory based on the Weitzenböck space-time with absolute parallelism. In the latter theory, which is called New General Relativity, an antisymmetric field produce physically observable effects.

§ 1. Introduction

The Lorentz invariant theory of gravitation, which describes gravitation by conventional field theory of a symmetric tensor field \( \{ h^{\mu \nu} \} \) coupled to the symmetrized energy-momentum tensor of matter, has a long history.\(^{1a,1b,2a,2b} \) It is known by now that such a theory necessarily becomes nonlinear one in order to ensure gauge invariance to all orders in \( h^{\mu \nu} \), and that the Minkowski metric initially assumed is unobservable in principle. In fact it was pointed out,\(^3 \) by taking as probes light rays and classical point particles, that real clocks and measuring rods are effected by gravity in such a manner that the metric measured by them corresponds to a Riemann space-time. Thus the Lorentz invariant theory of gravitation gives the same concepts and consequences as Einstein's General Relativity.

It seems, therefore, that there is no room for any new theory of gravitation for classical phenomena. In the microscopic world, however, the situation is different and a new possibility arises, because fundamental constituents of matter such as leptons and quarks are all spin-1/2 particles described by spinor wave functions obeying the Dirac equation. In the Lorentz invariant theory thus far considered, due attention has not been paid to the gravitational interaction of spin-1/2 particles.

\(^{2a} \) The Lorentz invariant theory of gravitation can be divided into two classes: (a) classical field theory of a symmetric tensor field,\(^1a \) and (b) quantum theory of massless particles with spin 2.\(^{1b} \) The present paper is an extension of the first approach to the gravitational interaction of spinor fields.
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which may be considerably different from that of classical point particles.

It is the purpose of this paper to study Lorentz invariant theory of gravitation for spin-1/2 particles interacting with gauge fields such as the electromagnetic field. These particles and fields are very likely to be fundamental constituents of matter according to contemporary high energy physics. In this paper the gravitational and gauge fields are treated as if they were classical, while spin-1/2 particles are described by spinor wave functions obeying the Dirac equation.

Let us briefly outline the main points of this paper. We start with the assumption that the gravitational potential is a symmetric tensor field \{h^{\mu\nu}\} in Minkowski space-time coupled to the symmetrized energy-momentum tensor \{T^{\mu\nu}\} of matter. The total action \[I^{(0)}\] is then given by

\[I^{(0)} = \int d^4x L^{(0)} = \int d^4x \left\{ L_G^{(0)} + L_V^{(0)} + L_D^{(0)} - \frac{1}{2} h^{\mu\nu} T_{\mu\nu}\right\},\]  

where \[L_V^{(0)}\] and \[L_D^{(0)}\] are the Lagrangian densities of gauge fields and spin-1/2 particles, respectively, in the absence of the gravitational interaction. The Lagrangian density of the symmetric field, \[L_G^{(0)}\], is unambiguously determined by the requirement that energy be positive definite. \[L_G^{(0)}\] is invariant up to a divergence under gauge transformation,

\[h'_{\mu\nu} = h_{\mu\nu} + \alpha_{\mu\nu} + \alpha_{\nu\mu},\]  

where \[\alpha^a\] are four small but otherwise arbitrary functions. Here a comma denotes differentiation with respect to \[x'.\]

As is well known, \[\partial_{\mu} T^{\mu\nu} \neq 0\] when the gravitational interaction is taken into account. This is due to the fact that the last term of (1.1), which describes the gravitational interaction of matter, is not invariant under gauge transformation (1.2) when \[\partial_{\mu} T^{\mu\nu} \neq 0.\] Thus we must revise the action (1.1). It turns out impossible, however, to construct such an action that is invariant under (1.2) to all orders in \[h^{\mu\nu}\], so we must also revise gauge transformation (1.2) at the same time: The revised transformation law, under which the revised action is invariant, will still involve four small but otherwise arbitrary functions.

In order to make the action gauge invariant to lowest order* in \[h^{\mu\nu}\], we must introduce a new antisymmetric field \[\{A^{\alpha\beta}\}\] coupled to the antisymmetric part of the non-symmetric energy-momentum tensor of spin-1/2 particles. The total action \[I^{(0)}\] thus becomes

\[I^{(1)} = \int d^4x L^{(1)} = \int d^4x \left\{ L_G^{(0)} + L_V^{(0)} + L_D^{(0)} - \frac{1}{2} h^{\mu\nu} T_{\mu\nu} + A^{\alpha\beta} T^{(0)}_{\alpha\beta}\right\}.\]  

* Here “lowest order” means to ignore terms proportional to \[h \cdot A.\]
This action is invariant to first order in \( h^{\alpha\nu} \) and \( A^{\alpha\nu} \) under gauge transformation

\[
h'_{\mu\nu} = h_{\mu\nu} + A_{\mu\nu} + A_{\nu\mu}, \quad A'_{\mu\nu} = A_{\mu\nu} - \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu}).
\]  
(1.3)

combined with coordinate transformation

\[
x'^{\alpha} = x^{\alpha} + A^\alpha (x).
\]  
(1.4)

An antisymmetric field \( \{ A^{\alpha\nu} \} \) is not required so long as we are concerned with classical particles, the electromagnetic field and other tensor fields: Its necessity is a new feature peculiar to spin-1/2 particles, or more generally speaking, to particles described by spinor wave functions. It is this new freedom of an antisymmetric field that opens a possibility for a gravitational theory other than General Relativity.

Since the total action (1.1') is not yet fully invariant under (1.3) and (1.4) to higher orders in \( h^{\alpha\nu} \) and \( A^{\alpha\nu} \), the field equation is still contradictory, and consequently we must still revise the action (1.1'). Before going further, however, we must notice that there are two possible ways to treat an antisymmetric field.

The first way is to regard \( \{ A^{\alpha\nu} \} \) as a purely mathematical entity without any physical content. For this purpose we add a new coupling term \( (1/2) A_{\mu\nu} S^{\alpha\nu} \) to the action, where \( \{ S^{\alpha\nu} \} \) is the spin tensor of spin-1/2 particles: The total action is now

\[
I^{(1)} = \int d^4 x L^{(1)}_A = \int d^4 x \left( L_G^{(0)} + L_V^{(0)} + L_D^{(0)} - \frac{1}{2} h^{\alpha\nu} T_{\mu\nu} + A^{\alpha\nu} T_{(x\alpha\nu)}^{(0)} + \frac{1}{2} A_{\mu\nu} S^{\alpha\nu} \right),
\]  
(1.5)

and is invariant under another gauge transformation*

\[
A'_{\mu\nu} = A_{\mu\nu} + \omega_{\mu\nu}(x), \quad \phi' = \phi + \frac{i}{2} \omega_{\mu\nu}(x) S^{\mu\nu}\phi,
\]  
(1.6)

to lowest order** in \( h^{\alpha\nu} \) and \( A^{\alpha\nu} \), where \( \omega_{\mu\nu} = -\omega_{\nu\mu} \) are six small but otherwise arbitrary functions and \( \phi \) is Dirac spinor wave function of spin-1/2 particles. An antisymmetric field \( \{ A^{\alpha\nu} \} \) can be made identically vanishing by choosing \( \omega_{\mu\nu} = -A_{\mu\nu} \).

The second way is to regard \( \{ A^{\alpha\nu} \} \) as a physically meaningful dynamical

* Our convention of the gamma matrices in special relativity is as follows:

\[
[r^\alpha, r^\beta] = -2\eta^{\alpha\beta}, \quad (\tau^\alpha) = \text{diag} (-1, +1, +1, +1),
\]

\[
S^\alpha = \frac{i}{4} [r^\alpha, r^\beta], \quad \tau_3 = i\tau_1^1 \tau_1^{\text{I}} \tau_1^{\text{I}}.
\]

** "Lowest order" means to ignore terms proportional to \( h \cdot \omega \) or \( A \cdot \omega \).
variable with its own kinetic action. The total action \((1 \cdot 1')\) then becomes

\[
I^{(0)}_M = \int d^4x L^{(0)}_M
= \int d^4x \left\{ L_G^{(0)} + L_V^{(0)} + L_D^{(0)} + L_A^{(0)} - \frac{1}{2} h^{\mu\nu} T_{\mu\nu} + A^{\mu\nu} T^{(D)}_{\mu\nu} \right\}, \tag{1.7}
\]

where the Lagrangian density \(L_A^{(0)}\) of an antisymmetric field is uniquely fixed by reasonable assumptions (see the postulates of \([1] \sim [4]\) of \(\S\ 3\)): An antisymmetric field describes a massless meson with spin-parity \(0^+\).

By making the total action completely gauge invariant to all orders in \(h^{\mu\nu}\) and \(A^{\mu\nu}\), we are led to two nonlinear theories of gravitation: One is obtained by the first way, and the other by the second way. The nonlinear theory obtained by the first way is just the usual General Relativity.

On the other hand, the second way leads to another theory of gravitation which contains a new parameter \(\lambda\) besides Einstein's gravitational constant \(\kappa\). The underlying geometry of space-time in this nonlinear theory of gravitation is the Weitzenböck space-time with absolute parallelism.\(^5\) This theory is investigated in detail in Ref. 6), and is given the name, New General Relativity, since Einstein in 1928 introduced absolute parallelism into physics for the first time.\(^7\)

In \(\S\ 2\) we briefly review the gravitational interaction of classical point particles. We then discuss in \(\S\ 3\) the gravitational interaction of spin-1/2 particles and gauge fields in lowest order of the gravitational coupling constant. In \(\S\ 4\) we revise the action following the first way, and derive General Relativity. The second way of revising the action is then analyzed in \(\S\ 5\), and New General Relativity is derived. We close the paper in \(\S\ 6\).

\section*{\(\S\ 2\). Gravitational interaction of classical point particles}

In this section we briefly review how the Lorentz invariant theory of a symmetric tensor field \(\{h_{\mu\nu}\}\), coupled to the energy-momentum tensor \(\{T_{\mu\nu}\}\) of classical point particles, leads to Einstein's General Relativity. We take as the Lagrangian density of the symmetric tensor field, \(L_G^{(0)}\), the following expression:\(^8\)

\[
L_G^{(0)} = \frac{1}{2\kappa} \left[ (\delta_{\mu}^{\alpha} - \frac{h}{2} \eta_{\mu}^{\alpha}) (\partial_{\nu} \Gamma_{\mu}^{\alpha} - \partial_{\mu} \Gamma_{\nu}^{\alpha}) + \eta_{\mu}^{\alpha} (\Gamma_{\nu}^{\mu} \Gamma_{\sigma}^{\nu} - \Gamma_{\nu}^{\mu} \Gamma_{\sigma}^{\nu}) \right], \tag{2.1}
\]

with \(h\) defined by \(h = h_{\mu}^{\mu}\). Here \(\{\Gamma_{\mu}^{\nu}\}\) is symmetric with respect to \(\mu\) and \(\nu\), and is varied independently of \(\{h_{\mu\nu}\}\): It is to be noticed that \(\{\Gamma_{\mu}^{\nu}\}\) appears only in \(L_G^{(0)}\). Our convention of the Minkowski metric is \((\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \text{diag}(-1, +1, +1, +1)\). The Euler derivative of \(L_G^{(0)}\) with respect to \(h^{\mu\nu}\) is given by

\[
\frac{\partial L_G^{(0)}}{\partial h^{\mu\nu}} = \frac{1}{2\kappa} G^{(0)}_{\mu\nu} = \frac{1}{2\kappa} \left[ (\partial_{\nu} \Gamma_{\mu}^{\alpha} - \partial_{\mu} \Gamma_{\nu}^{\alpha}) - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} (\partial_{\alpha} \Gamma_{\beta}^{\nu} - \partial_{\beta} \Gamma_{\alpha}^{\nu}) \right], \tag{2.2}
\]
while the Euler-Lagrange equation of $\Gamma^i_{\mu\nu}$ is obtained by taking variation of $L_{\varepsilon}^{(0)}$ with respect to $\Gamma^i_{\mu\nu}$,

$$
\Gamma^i_{\mu\nu} = -\frac{1}{2} \eta^{i\rho} (h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\rho\mu,\nu}).
$$

(2.3)

Here we denote symmetrization of tensor indices by a round bracket ( ), while antisymmetrization by a square bracket [ ]. Using (2.3) in (2.2), we find that $G_{\mu\nu}^{(1)}$ is just the first order Einstein tensor,

$$
G_{\mu\nu}^{(1)} = \frac{1}{2} (\Box h_{\mu\nu} + \eta_{\mu\nu} \bar{h}_{\rho\sigma} - 2 \bar{h}_{(\rho,\sigma)\nu}),
$$

(2.4)

formed of the metric tensor

$$
g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu},
$$

(2.5)

where $\Box \equiv \partial^\rho \partial_\rho$, and we define as usual

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h,
$$

(2.6)

Notice the minus sign in front of $h_{\mu\nu}$ in (2.5); $h_{\mu\nu}$ of this paper differs from the usual definition\(^\text{*1}\) of $h_{\mu\nu}$ by a sign. The Lagrangian density $L_{\varepsilon}^{(0)}$ of (2.1) is invariant under gauge transformation

$$
\delta h_{\mu\nu} = A_{\mu,\nu} + A_{\nu,\mu},
$$

(2.7)

$$
\delta \Gamma^i_{\mu\nu} = - A^i_{\rho,\mu\nu},
$$

(2.8)

up to a divergence. This invariance property of $L_{\varepsilon}^{(0)}$ is reflected in the fact that the divergence $\partial^\rho G_{\mu\nu}^{(1)}$ vanishes identically.

A classical particle moving along a classical trajectory $z^\alpha(\tau)$ under the influence of the symmetric tensor field is described by the action

$$
I_M = \frac{m}{2} \int \eta_{\mu\nu} \dot{z}^\mu \dot{z}^\nu d\tau - \frac{1}{2} \int h_{\mu\nu} T^{\mu\nu} d^4x
$$

$$
= \frac{m}{2} \int g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu d\tau,
$$

(2.9)

since $\{T^{\mu\nu}\}$ is given by

$$
T^{\mu\nu} = m \int \delta (z(\tau) - x) \dot{z}^\mu \dot{z}^\nu d\tau.
$$

(2.10)

Here a dott represents differentiation with respect to the proper time $\tau$, and our units are $\hbar = c = 1$. The field equation of $h_{\mu\nu}$ reads

$$
G_{\mu\nu}^{(1)} = \kappa T_{\mu\nu}.
$$

(2.11)

\(^\text{*1}\) Usually $h_{\mu\nu}$ is defined by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. 

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This equation is inconsistent, because $\partial_\nu T^{\mu \nu} \neq 0$ due to the gravitational interaction. This inconsistency arises from the fact that the matter action $I_M$ of (2.9) is not invariant under gauge transformation (2.7). Therefore, we must revise the transformation law (2.7), and also the gravitational Lagrangian density (2.1) in order to maintain its invariance under the revised transformation.

The action $I_M$ of (2.9) can be made invariant under (2.7) to lowest order in $h_{\mu \nu}$, by making coordinate transformation

$$\delta x^\nu = \Lambda^\nu, \quad \delta x^\nu = \Lambda^\nu$$

(2.12)

at the same time, so we make coordinate transformation (2.12). The action $I_M$ of (2.9) is invariant under (2.12) to all orders in $h_{\mu \nu}$, if $h_{\mu \nu}$ transforms like

$$\delta h_{\mu \nu} = A_{\mu, \nu} + A_{\alpha, \mu \nu} - A_{\nu, \alpha \mu} + h_{\mu \nu} A_{\alpha, \delta \nu} + h_{\nu, \alpha \mu},$$

(2.13)

or equivalently

$$\delta g_{\mu \nu} = -A_{\nu, \mu} g_{\mu \nu} - A_{\alpha, \mu \nu} g_{\mu \nu},$$

(2.13')

where $\{g_{\mu \nu}\}$ is defined by (2.5). We thus take (2.13) as the revised, correct transformation law of $h_{\mu \nu}$: (2.13') is just the transformation law of a covariant second-rank tensor under small coordinate change (2.12). The inverse of $\{g_{\mu \nu}\}$, $\{g^{\mu \nu}\}$, which is given by $g^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu}$ to first order in $h_{\mu \nu}$, then transforms like a contravariant second-rank tensor.

The gravitational Lagrangian density $L_G^{(0)}$ is not invariant under the revised transformation (2.12) and (2.13), so it must be revised by rewriting in terms of those quantities that obey the transformation law of tensors under (2.12). Accordingly, $(\eta^{\mu \nu} + h^{\mu \nu})$ and $\eta^{\mu \nu}$ are to be replaced by $g^{\mu \nu}$. The trace $h = h_{\mu}^\mu$ cannot be replaced by a scalar because $g_{\mu}^\mu = 4$; its role becomes clear, if one notices the relation

$$\sqrt{-g} = 1 - \frac{1}{2} h + O(h^2)$$

(2.14)

with

$$g = \det (g_{\mu \nu}),$$

(2.15)

and the well-known fact that $\sqrt{-g} d^4 x$ is an invariant volume element. Since $L_G^{(0)}$ can be rewritten within accuracy up to second order of $h^{\mu \nu}$ as

$$L_G^{(0)} = \frac{1}{2\kappa} \left(1 - \frac{h}{2}\right) \left[ (\eta^{\mu \nu} + h^{\mu \nu}) (\partial_\rho \Gamma^\rho_{\mu \nu} - \partial_\mu \Gamma^\rho_{\nu \rho} + \eta^{\rho \nu} (\Gamma^\rho_{\mu \rho} \Gamma^\rho_{\nu \rho} - \Gamma^\rho_{\mu \rho} \Gamma^\rho_{\nu \rho})) \right],$$

(2.1')

the revised, correct Lagrangian density $L_G$ must be

---

*Throughout this paper, $A^\nu$ are four small but otherwise arbitrary functions, so second and higher order terms of $A^\nu$ can be ignored.*
This is just the Hilbert action in General Relativity.

§ 3. Gravitational interaction of fundamental particles and fields
——Lowest order——

We now turn to the gravitational interaction of spin-1/2 fundamental particles and gauge fields, assuming as in the previous section that the gravitational potential \( \{h^{\mu \nu}\} \) is coupled to the symmetrized energy-momentum tensor \( \{T^{\mu \nu}\} \). As the simplest model we take an interacting system of a charged Dirac field \( \psi \) and the electromagnetic field \( \{A_\mu\} \); generalization to non-abelian gauge fields is straightforward.

The gravitational interaction Lagrangian density \( L_{\text{int}}^{(1)} \) is

\[
L_{\text{int}}^{(1)} = -\frac{1}{2} h^{\mu \nu} T_{\mu \nu}^{(D)} - \frac{1}{2} h^{\mu \nu} \{T^{(D)} + T^{(EM)}\}
\]

with \( \{T^{(D)}_{\mu \nu}\} \) and \( \{T^{(EM)}_{\mu \nu}\} \), the symmetrized energy-momentum tensor of a Dirac field and the electromagnetic field, respectively (see the Appendix for the explicit expressions for them). The matter Lagrangian densities including the gravitational interaction, \( L_D^{(1)} \) and \( L_{\text{EM}}^{(1)} \), are then given by

\[
L_D^{(1)} = L_D^{(0)} - \frac{1}{2} h^{\mu \nu} T_{\mu \nu}^{(D)} = L_D^{(1)} - \frac{h}{2} L_D^{(0)},
\]

\[
L_{\text{EM}}^{(1)} = L_{\text{EM}}^{(0)} - \frac{1}{2} h^{\mu \nu} T_{\mu \nu}^{(EM)} = L_{\text{EM}}^{(1)} - \frac{h}{2} L_{\text{EM}}^{(0)},
\]

where \( L_D^{(1)} \), \( L_{\text{EM}}^{(1)} \) and \( \{F_{\mu \nu}\} \) are defined by

\[
L_D^{(1)} = \frac{i}{2} \left( \partial^\mu + \frac{1}{2} h^\mu \right) \{\overline{\psi} \gamma^\mu (\partial_\mu - ie A_\mu) \psi - (\partial_\mu + ie A_\mu) \overline{\psi} \gamma^\mu \psi\} - m \overline{\psi} \psi,
\]

\[
L_{\text{EM}}^{(1)} = -\frac{1}{4} (\eta^{\mu \nu} \eta_{\rho \sigma} + \eta^{\mu \sigma} \eta_{\rho \nu} + \eta^{\nu \rho} \eta_{\mu \sigma}) F_{\mu \rho} F_{\nu \sigma},
\]

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

As in the previous section, we assume that the gravitational Lagrangian density \( L_G^{(0)} \) is given by (2.1). The field equation of \( h^{\mu \nu} \) derived from the action

\[
I^{(1)} = \int d^4 x L^{(1)} = \int d^4 x \{L_G^{(0)} + L_D^{(1)} + L_{\text{EM}}^{(1)}\}
\]

reads

\[
G^{(1)}_{\mu \nu} = \kappa T_{\mu \nu},
\]
which is contradictory, however, since the divergence of $T^{\mu \nu}$ does not vanish when the gravitational interaction is taken into account. This inconsistency occurs because $L_D^{(0)}$ and $L_{BM}^{(0)}$ are not invariant under gauge transformation (2.7), which leaves the gravitational Lagrangian density $L_G^{(0)}$ invariant up to a divergence. Therefore, we must revise $L_D^{(0)}$ and $L_{BM}^{(0)}$ so that they become invariant under (2.7) or properly revised version of it.

Let us begin with $L_D^{(0)}$. By transformation (2.7) with $\psi$ and $\partial \psi$ kept unchanged, a term, $h_{\mu}^{\mu} \bar{\psi} \gamma^\mu \partial_{\mu} \psi$, in $L_D^{(0)}$ of (3.2b) changes like

$$\delta (h_{\mu}^{\mu} \bar{\psi} \gamma^\mu \partial_{\mu} \psi) = (A_{\mu,v} + A_{v,\mu}) \bar{\psi} \gamma^\mu \partial^\nu \psi,$$

which must be canceled in some way by a variation of $\bar{\psi} \gamma^\mu \partial_{\mu} \psi$ due to change of $\psi$ and $\partial \psi$. In order for such cancellation to occur, $\psi$ and $\partial \psi$ have to transform as

$$\partial \psi = 0; \quad \delta (\partial \psi) = \xi \gamma^\mu A_{\mu} \partial_{\mu} \psi$$

with $\xi$ appropriate numerical constants. The transformation (3.7a) and (3.8) of $\psi$ and $\partial \psi$ is realized, if and only if we make coordinate transformation

$$x^\nu = x^\nu + A^\nu.$$

Thus gauge transformation (2.7) should be combined with coordinate transformation: Just the same situation occurred in the gravitational interaction of classical point particles. The equation (3.7a) means that $\psi$ must transform like a scalar under coordinate transformation (3.9); therefore, $\partial \psi$ changes like

$$\delta (\partial \psi) = - A^\nu \partial_{\mu} \psi.$$

We assume that gamma matrices are invariant under gauge transformation, because their elements are merely numerical constants which have nothing to do with the gravitational interaction. A term, $(\partial^\nu + \frac{1}{2} h_{\nu}^{\nu} \bar{\psi} \gamma^\nu \partial_{\mu} \psi$, of $L_D^{(0)}$ then varies under (2.7) and (3.7a, b) as

$$\delta \left[ (\partial_{\mu} + \frac{1}{2} h_{\mu}^{\nu} \bar{\psi} \gamma^\nu \partial_{\mu} \psi) \right] = \frac{1}{2} (A_{\mu,v} - A_{v,\mu}) \bar{\psi} \gamma^\mu \partial^\nu \psi.$$

Consequently $L_D^{(0)}$ of (3.2b) is not gauge invariant as it stands. This difficulty is overcome by introducing an antisymmetric field $\{A^{\mu \nu}\}$, which transforms as

$$\delta A_{\mu \nu} = - \frac{1}{2} (A_{\mu,v} - A_{v,\mu}),$$

in addition to the symmetric field $\{h^{\mu \nu}\}$, and then making the following replacement in $L_D^{(0)}$:
Hereafter we denote by $L_D$ the newly obtained Lagrangian density,

$$L_D = L_D^{(1)} - \frac{h}{2} L_D^{(0)},$$

(3.13a)

$$L_D^{(1)} = \frac{i}{2} \left( \partial^\mu \frac{1}{2} h^\mu - A^\mu \right) \left( \bar{\psi} (\partial_\tau - ie A_\tau) \psi - (\partial_\tau + ie A_\tau) \bar{\psi} \psi \right) - m \bar{\psi} \psi,$$

(3.13b)

The electromagnetic potential $\{A_\mu\}$ has to transform in the same way as $\partial_\tau \psi$, so we define

$$\delta A_\mu = - A_\mu \cdot A_\nu.$$

(3.14)

The $L_D$ of (3.13b), which we call the Dirac Lagrangian, is then invariant to lowest order in $h^{\mu \nu}$ and $A^{\mu \nu}$ under the combined transformation, (2·7), (3·7a, b), (3·9), (3·11) and (3·14). The action of a Dirac field,

$$I_D^{(1)} = \int d^4 x L_D^{(1)},$$

(3.15)

is also invariant to lowest order in $h^{\mu \nu}$ and $A^{\mu \nu}$ under the same transformation, since $(1 - (h/2)) d^4 x$ is an invariant volume element to lowest order in $h^{\mu \nu}$.

From (3·3) and (3·14) it follows that the electromagnetic action

$$I_{EM}^{(1)} = \int d^4 x L_{EM}^{(1)},$$

(3.16)

is invariant to lowest order in $h^{\mu \nu}$ under transformation (2·7), (3·9) and (3·14). It is to be noted that an antisymmetric field $\{A^{\mu \nu}\}$ need not be used in $I_{EM}^{(1)}$.

Next let us turn to another symmetry property of the Dirac Lagrangian density $L_D$ of (3·13a). Consider the following transformation:

$$\delta \psi = \frac{i}{2} \omega_\mu S^{\mu \nu} \psi,$$

(3·17a)

$$\delta x^\tau = 0, \quad \delta A_\tau = 0,$$

(3·17b)

$$\delta h^{\mu \nu} = 0, \quad \delta A^{\mu \nu} = \omega^{\mu \nu}.$$

(3·17c)

with $\omega_\mu + \omega_\nu = 0$ and $|\omega_\mu| \ll 1$. Equation (3·17a) represents infinitesimal Lorentz transformation of $\psi$. Equation (3·17b) is assumed here because the transformation law under coordinate transformation has already been treated above, while (3·17c) is postulated in order to ensure that the Dirac Lagrangian density $L_D$ be invariant under global transformation for which $\omega_\mu$ do not depend on $x$. For local transformation with $\omega_\mu$ being small functions of $x$, $L_D^{(1)}$ changes to lowest order in
with the spin tensor of a Dirac field given by
\[ S^{\mu \nu} = -i \left( \frac{\partial L_D^{(8)}}{\partial \dot{\psi}^\rho} - \overline{\psi} S^{\rho \sigma} \frac{\partial L_D^{(8)}}{\partial \dot{\psi}^\rho} \right) = -\frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \dot{\psi} \gamma_\rho \gamma_\sigma \psi. \] (3.19)

Here the totally antisymmetric Levi-Civita symbol in special relativity is defined by \( \varepsilon^{0123} = 1 \). If \( \omega_\mu \) are four-dimensional rotation,
\[ \omega_\mu = H_\mu - H_{\nu \rho} \] (3.20)
with \( H_\mu \) four small but otherwise arbitrary functions, \( L_D^{(1)} \) is invariant under (3.17a−c). An antisymmetric field \( \{A^\mu\} \) has six degrees of freedom, but three of them can be eliminated by this transformation: Thus the Lagrangian density \( L_D^{(1)} \) involves three extra degrees of freedom associated with \( \{A^\mu\} \).

Now arises the question: How should we treat these three extra degrees of freedom associated with an antisymmetric field \( \{A^\mu\} \)? There are two answers to this problem.

(i) The first is to regard \( \{A^\mu\} \) as purely mathematical, unphysical entity. In order to realize this possibility, we shall introduce a new, additional interaction
\[ L_{\text{int}}^{(1)} = \frac{1}{2} A_{\mu \nu \rho} S^{\mu \nu}, \] (3.21)
which we call the nonminimal interaction. The total Lagrangian density of a Dirac field, \( L_D^{(1)} + L_{\text{int}}^{(1)} \), is then invariant to lowest order in \( h^{\mu \nu} \) and \( A^{\mu \nu} \) under transformation (3.17a−c), with \( \omega_\mu \) six small but otherwise arbitrary functions. Therefore, an antisymmetric field \( \{A^\mu\} \) can be made identically vanishing by a transformation (3.17a−c) with \( \omega_\mu = -A_\mu \). The nonminimal interaction of (3.21) is invariant under transformation (3.7a) and (3.11), since the spin tensor \( \{S^{\mu \nu}\} \) of (3.19) is totally antisymmetric. The total action, which we denote by \( I_1^{(1)} \) in this case,
\[ I_1^{(1)} = \int d^4x L_1^{(1)} = \int d^4x \{ L_D^{(8)} + L_D^{(1)} + L_{\text{EM}}^{(1)} + L_{\text{int}}^{(1)} \}, \] (3.22)
is thus invariant to lowest order in \( h^{\mu \nu} \) and \( A^{\mu \nu} \) under two kinds of transformations:
\[ \delta h_{\mu \nu} = A_{\mu \nu} + A_{\nu \mu}, \quad \delta A_{\mu \nu} = -\frac{1}{2} (A_{\mu \nu} - A_{\nu \mu}), \] (3.23a)
\[ \delta x^\nu = A^\nu, \] (3.23b)
\[ \delta \phi = 0, \quad \delta (\partial_\mu \phi) = -A_\mu \partial_\mu \phi, \quad (3.23c) \]
\[ \delta A_\mu = -A_\nu \partial_\mu A_\nu, \quad (3.23d) \]

with \( A_\mu \) four small but otherwise arbitrary functions, and the transformation \((3.17a-c)\) with \( \omega_\mu = -\omega_\nu \) six small but otherwise arbitrary functions. It should be stressed here that the nonminimal interaction of \((3.21)\) is needed only to make the invariance group of the action larger.

(ii) The second is to regard an antisymmetric field \( \{ A^\mu \} \) as dynamical variable with its own kinetic Lagrangian density \( L_A^{(0)} \). The total action, which we denote by \( I_1^{(0)} \) in this case, is then given by
\[ I_1^{(0)} = \int d^4x L_1^{(0)} = \int d^4x \{ L_G^{(0)} + L_0^{(1)} + L_{EM} + L_A^{(0)} \}. \quad (3.24) \]

The Lagrangian density \( L_A^{(0)} \) can be constructed by requiring the following postulates:

[1] Invariance under the Lorentz group.
[2] Invariance under gauge transformation \((2.7)\) and \((3.11)\).
[4] \( L_A^{(0)} \) be a quadratic expression of first-order derivative of an antisymmetric field, \( \{ A^{\mu \nu} \} \), but at most linear with respect to first-order derivative of the symmetric field, \( \{ h^{\mu \nu} \} \).

The postulate of [4] is required, because we assume in this paper that the symmetric field is described by the Lagrangian density \( L_G^{(0)} \) of \((2.1)\) to lowest order in \( h^{\mu \nu} \). If \( L_A^{(0)} \) contained quadratic terms of \( \{ h^{\mu \nu} \} \), the field equation of \( h^{\mu \nu} \) would not be the linearized Einstein field equation \((3.5)\). From the postulate of [2] it follows that \( L_A^{(0)} \) should contain \( h^{\mu \nu} \) and \( A^{\mu \nu} \) only through the combination,
\[ T_{\mu \nu} = -\frac{1}{2} h_{\mu \nu} + A_{\mu \nu} - (\mu \phi^\nu - \nu \phi^\mu), \quad (3.25) \]

which is decomposed into three irreducible parts under the Lorentz group:\(^{10}\)
\[ t_{\mu \nu} = T_{(\mu \nu)} + \frac{1}{3} \eta_{(\mu} \nu_{\nu)} = \frac{1}{3} \eta_{(\mu} \nu_{\nu)}, \quad (3.26) \]
\[ \eta_\mu = T_{(\mu 0)}, \quad (3.27) \]
\[ a^\mu = \frac{1}{6} \epsilon^{\mu \rho \sigma \tau} T_{\rho \sigma \tau}, \quad (3.28) \]

The most general expression for \( L_A^{(0)} \) is then given by
\[ L_A^{(0)} = c_1 (t_{\mu \nu} t_{\mu \nu}) + c_2 (\eta_\mu \eta_\mu) + c_3 (a^\mu a_\mu), \quad (3.29) \]

with three unknown parameters, \( c_1, c_2 \) and \( c_3 \). The first two terms in \((3.29)\),
however, involve quadratic terms of \( \{h^\nu_\rho,\} \), so we put

\[
c_1 = 0 = c_2 \tag{3.30}
\]

because of the postulate of [4]. Thus the kinetic Lagrangian density of an antisymmetric field becomes

\[
L_A^{(6)} = \frac{9}{4\lambda} (a^\pi a_\pi) - \frac{1}{2\lambda} A^{\mu\nu}(A_{\mu\nu}^\tau + A^\tau A^\nu_{\pi\nu} - A^\mu_{\nu\pi\tau}) \tag{3.31}
\]

where we introduce a new parameter \( \lambda \) instead of \( c_3 \); \( \lambda = 9/4c_3 \). This Lagrangian density is invariant under gauge transformation (3.11), so we can put the gauge condition,

\[
A^{\mu\nu} = 0 \tag{3.32}
\]

The antisymmetric field described by (3.31) represents a massless meson with spin-parity \( 0^+ \). If we lift up the postulate of [3], we can add to (3.31) a parity violating term \( (\tau a_{\pi \rho}) \); This possibility is discussed in Ref. 6).

The total action \( I_1^{(6)} \) of (3.24) is invariant to lowest order in \( h^\nu_\pi \) and \( A^\nu_\pi \) under two kinds of transformations: One is (3.23a \(-d) \), and the other is (3.17a \(-c) \) with \( \omega_{\mu\nu} \) given by (3.20).

The invariant property of the total actions, \( I_1^{(6)} \) and \( I_6^{(6)} \), mentioned above holds true only to lowest order in \( h^\nu_\pi \) and \( A^\nu_\pi \), so the field equations derived from them are not yet consistent. Therefore, we must still revise \( I_1^{(6)} \) and \( I_6^{(6)} \) so that they become invariant to all orders in \( h^\nu_\pi \) and \( A^\nu_\pi \).

§ 4. Nonlinear theory. I

--- Einstein's general relativity ---

Let us revise the action \( I_1^{(6)} \) of (3.22) and the transformation of (3.23a \(-d) \) so that the revised action be invariant to all orders in \( h^\nu_\pi \) and \( A^\nu_\pi \) under revised transformation. Equation (3.23b) means a general coordinate transformation, \( x^\nu \rightarrow x'^\nu = x^\nu + A^\nu \), so it does not need any modification. Equations (3.23c \(-d) \), which define the transformation law of \( \phi \) and \( \{A^\pi\} \) under coordinate transformation, should not be changed either, because they indicate that \( \phi \) and \( \{A^\pi\} \) transform like a scalar and a covariant vector, respectively, under general coordinate transformation. Therefore, only Eq. (3.23a), i.e., the transformation law of \( \{h^\nu_\pi\} \) and \( \{A^\nu_\pi\} \), need be revised.

The problem is thus the following: (1) Define the correct transformation law of \( \{h^\nu_\pi\} \) and \( \{A^\nu_\pi\} \) under general coordinate transformation, and then (2) revise the action \( I_1^{(6)} \) by adding to it second and higher order terms of \( h^\nu_\pi \) and \( A^\nu_\pi \) so that the revised action be invariant under general coordinate transformation.
T. Shirafuji

Since spin-1/2 particles are fundamental constituents of matter, we define the transformation law of \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \) by requiring that the Dirac Lagrangian \( L_0 \) of (3·13b), without any further corrections, is invariant to all orders in \( h^{\mu}_r \) and \( A^{\mu}_r \) under general coordinate transformation:

\[
L_0^{(4)} (\phi', A^r, h^{\mu}_r, A^{\mu}_r) = L_0^{(4)} (\phi, A^r, h^{\mu}_r, A^{\mu}_r),
\]

where \( \phi' = \phi' (x') \) and so on. We assume that the gamma matrices are not subject to coordinate transformation, since their components are merely numerical constants. Then inspection of (3·13b) shows that \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \) should transform like

\[
\delta_\mu^* + \frac{1}{2} h^{\mu}_r + A^{\mu}_r = (\delta_\mu^* + A^{\mu}_r) \left( \delta_\mu^* + \frac{1}{2} h^{\mu}_r + A^{\mu}_r \right).
\]

Namely, \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \) do not transform independently, but in a pair: This transformation law reproduces (3·23a) in lowest order of \( h^{\mu}_r \) and \( A^{\mu}_r \). We adopt (4·2) as the correct transformation law of \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \). An important property of (4·2) is that one of the indices, \( \mu \), of \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \) does not take part in coordinate transformation. We use Latin letters for those indices that are not subject to coordinate transformation; thus, replacing \( \mu \) by \( k \), let us define sixteen quantities \( \{e^k_r\} \) by

\[
e^k_r = \delta^r_k + \frac{1}{2} h^{k}_r + A^{k}_r.
\]

which can be taken as the gravitational field variable, instead of \( \{h^{\mu}_r\} \) and \( \{A^{\mu}_r\} \). Then (4·2) reads

\[
e^{k'}_{r'} = (\delta^{k'}_{r'} + A^{r'}_{k'}) e^k_r = (\partial / \partial x^{k'}) e^k_r,
\]

showing that \( \{e^k_r\} \) for each \( k \) transforms like a contravariant vector under coordinate transformation. The gamma matrices have Latin indices, so the Dirac Lagrangian \( L_0^{(4)} \) of (3·13b) reads

\[
L_0^{(4)} = \frac{i}{2} e^* k \bar{\psi} \gamma^k (\partial_r - ie A_r) \psi - (\partial_r + ie A_r) \bar{\psi} \gamma^k \psi - m \bar{\psi} \psi.
\]

It will be found convenient to introduce the inverse of \( \{e^k_r\} \), \( \{e^*_k\} \), and the metric tensor, \( \{g^{\mu}_r\} \) and \( \{g^{\mu}_r\} \), by

\[
\phi^*_\mu e^k_r = \delta^*_\mu^r, \quad e^k_r e^*_k = \delta^* r l,
\]

\[
g^{\mu}_r = \eta^{\mu}_l e^*_k e^l, \quad g^{\mu}_r = \eta^{\mu}_l e^*_k e^l.
\]

The \( \{e^k_r\} \) for each \( k \) transforms like a covariant vector,

\[
e^{k'}_{r'} = (\delta^{k'}_{r'} - A^{r'}_{k'}) e^k_r = (\partial_r / \partial x^{k'}) e^k_r.
\]
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and \( \{g_{\mu\nu}\} \) and \( \{g^{\mu\nu}\} \) change like covariant and contravariant tensors of second rank, respectively. From (4·3), (4·5) and (4·6) we see that

\[
e^k_\nu = \delta^k_\nu - \frac{1}{2} h^k_\nu + A^k_\nu + O(h^2, A^2, hA),
\]
\[
g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + O(h^2, A^2, hA),
\]
\[
g^{\mu\nu} = \gamma^{\mu\nu} + h^{\mu\nu} + O(h^2, A^2, hA),
\]

where \( O(h^2, A^2, hA) \) represents second and higher order terms of \( h^{\mu\nu} \) and \( A^{\mu\nu} \). Here both Latin and Greek indices of \( \{h^{\mu\nu}\} = \{h^{\nu\mu}\} \) and \( \{A^{\mu\nu}\} = \{A^{\nu\mu}\} \) are raised or lowered by the Minkowski metric. We define as usual

\[ g = \det (g_{\mu\nu}) < 0, \]

then (4·9) gives

\[ \sqrt{-g} = 1 - \frac{h}{2} + O(h^2, A^2, hA). \]

We are now ready to derive the revised, correct action \( I_i \). Since \( \sqrt{-g} d^4x \) is an invariant volume element, \( I_i \) can be expressed as

\[ I_i = \int d^4x \sqrt{-g} L_i, \]

where \( L_i \) is the total Lagrangian and consists of four terms; the gravitational Lagrangian \( L_G \), the electromagnetic Lagrangian \( L_{EM} \), the Dirac Lagrangian \( L_D \) and the nonminimal interaction term \( L'_{int} \), which are all form invariant under coordinate transformation:

\[ L_i = L_G + L_{EM} + L_D + L'_{int}. \]

In lowest order of \( h^{\mu\nu} \) and \( A^{\mu\nu} \), the Lagrangian densities, \( L_G = \sqrt{-g} L_{EM} = \sqrt{-g} L_D = \sqrt{-g} L'_{int} \) should coincide with \( L_G^{(0)}, L_{EM}^{(0)}, L_D^{(0)} \) and \( L'_{int}^{(0)} \) of (3·22), respectively. It is seen from (2·1'), (3·3a, b) and (3·13a, b) with the help of (4·3) and (4·8)~(4·10) that \( L_G, L_{EM} \) and \( L_D \) are given by

\[ L_G = \frac{1}{2\kappa} g^{\mu\nu}(\partial_\mu \Gamma^i_\nu - \partial_\nu \Gamma^i_\mu + \Gamma^i_\rho \Gamma^\rho_\mu - \Gamma^i_\nu \Gamma^\rho_\mu), \]
\[ L_{EM} = -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}, \]
\[ L_D = L_D^{(0)} = \frac{i}{2} \epsilon^i_\kappa \bar{\psi} \gamma^i (\partial_\kappa - i e A_\kappa) \psi - (\partial_\kappa + i e A_\kappa) \bar{\psi} \gamma^i \psi - m \bar{\psi} \psi. \]

We assume that the totally antisymmetric Levi-Civita symbol \( \epsilon \), which takes the
values 0 and \( \pm 1 \), is not subject to coordinate transformation. Thus it has Latin indices,

\[
\varepsilon = \{ \delta^{ijmn} \}, \quad \varepsilon^{(0)(1)(2)(3)} = +1
\]

(4.18)

with Latin indices enclosed in parentheses. Consequently the spin tensor of (3.19) should also have Latin indices,

\[
S^{ijm} = -\frac{1}{2} \delta^{ijmn} \bar{\psi} \gamma_i \gamma_j \gamma_k \psi.
\]

(4.19)

In view of this property of the spin tensor, inspection of (3.21) shows that \( L_{\text{int}} \) is given by

\[
L_{\text{int}}' = \frac{1}{2} e_{j}^{*} e_{m}^{*} e_{\nu, \nu} S^{ijm}.
\]

(4.20)

The total Dirac Lagrangian \( L_D' \) is the sum of two terms; (4.17) and the nonminimal interaction term (4.20),

\[
L_D' = L_D + L_{\text{int}}.
\]

(4.21)

It is convenient to rewrite \( L_D' \) by using the Ricci rotation coefficients formed of \( \{ e_{k}^{*} \} \),

\[
A_{ijm} = e_{m}^{*} A_{ij} = \frac{1}{2} (C_{ijm} - C_{jim} - C_{mij})
\]

(4.22)

with \( C_{ijm} \) defined by

\[
C_{ijm} = -e_{k}^{*} e_{m}^{*} (e_{\nu, \nu} - e_{\nu, \nu}).
\]

(4.23)

Use of (4.19), (4.22) and (4.23) in (4.20) gives

\[
L_{\text{int}}' = \frac{1}{8} \delta^{ijmn} C_{ijm} \overline{\psi} \gamma_i \gamma_j \gamma_k \psi = \frac{1}{4} e_{j}^{*} e_{m}^{*} A_{ijm} \overline{\psi} \gamma_i \gamma_j \gamma_k \psi
\]

\[
= -\frac{1}{4} A_{ijm} \overline{\psi} \{ S^{ij}, \gamma^m \} \psi.
\]

(4.24)

Using (4.17) and (4.24) in (4.21), we get

\[
L_D' = \frac{i}{2} e_{k}^{*} \{ \overline{\psi} \gamma^{-k} (F_{\nu} - i e A_{\nu}) \psi - (F_{\nu} + i e A_{\nu}) \overline{\psi} \gamma^{k} \psi \} - m \overline{\psi} \psi,
\]

(4.25)

where \( F_{\nu} \) denotes the covariant derivative defined by the Ricci rotation coefficients of (4.22),

\[
F_{\nu} \psi = (\partial_{\nu} + \frac{i}{2} A_{ij \nu} S^{ij}) \psi, \quad F_{\nu} \overline{\psi} = \overline{\psi} \left( \partial_{\nu} - \frac{i}{2} A_{ij \nu} S^{ij} \right).
\]

(4.26)
The action (4·13) with (4·14) ~ (4·17) and (4·20) is just the action used in General Relativity. The symmetric tensor \( \{g_{\mu}\} \) defines the metric of the Riemann space-time, and the \( \{e_k^i\} \) is the tetrad fields, to which spinor fields are referred.

Now let us turn to another transformation (3·17a~c) which keeps the action \( I_i^{(0)} \) of (3·22) unchanged to lowest order in \( h^{\mu\nu} \) and \( A^\mu \). The problem is to find a new transformation which leaves the action \( I_i \) of (4·13) invariant and which reduces to (3·17a~c) in lowest order of \( h^{\mu\nu} \) and \( A^\mu \). First of all, we assume that (3·17b) is retained, since we are now familiar with the transformation law of all the dynamical variables, \( \{e_k^i\} \), \( \psi \) and \( \{A_i\} \), under coordinate transformation. We also retain (3·17a), which shows that the spinor field \( \psi \) is subject to an infinitesimal local Lorentz transformation. Since the index \( k \) of \( \{e_k^i\} \) contracts with the index \( k \) of the gamma matrices in the total Dirac Lagrangian \( L_D' \) of (4·25), the quantity \( \{e_k^i\} \) for each \( \nu \) should transform like a Lorentz vector. Thus we define as follows:

\[
\begin{align*}
\delta e_k^i &= \omega_k^m e_m^i, & \delta e_k^i &= \omega_k^m e_m^i, & (4·27a) \\
\delta \psi &= \frac{i}{2} \omega_{ij} S^{ij} \psi, & (4·27b) \\
\delta x^\nu &= 0, & \delta A_i &= 0, & (4·27c)
\end{align*}
\]

where \( \omega_{ij} = \eta_{ik} \omega^k_j = \eta_{jk} \omega^k_i \). From (4·6) it follows that the metric tensor, \( \{g_{\mu\nu}\} \) and \( \{g^{\mu\nu}\} \), is not changed,

\[
\delta g_{\mu\nu} = 0, \quad \delta g^{\mu\nu} = 0. & \tag{4·27d}
\]

This transformation reduces to (3·17a~c) in lowest order of \( h^{\mu\nu} \) and \( A^\mu \). As is well known, the covariant derivative \( F_{\nu\sigma} \) of (4·26) defined by the Ricci rotation coefficients changes like

\[
\delta (F_{\nu\sigma}) = \frac{i}{2} \omega_{ij} S^{ij} (F_{\nu\sigma}) & \tag{4·28}
\]

under (4·27a, b): Thus the total Dirac Lagrangian \( L_D' \) of (4·25) is invariant under (4·27a~d). The gravitational and electromagnetic Lagrangians, \( L_G \) and \( L_{EM} \), are also invariant by virtue of (4·27c, d). The total action \( I_i \) of (4·13) is therefore invariant. This is just the local Lorentz invariance of General Relativity.

Einstein's General relativity thus follows, if one makes an assumption that an antisymmetric field \( \{A^\nu\} \) required by consistency of the field equation should be regarded as a purely mathematical entity. This assumption, however, is brought in from outside: Therefore if one does not assume it, one is led to another theory of gravitation. This we do in the next section.
§ 5. Nonlinear theory. II

—New general relativity—

We shall now revise the action $I_\Pi^{(1)}$ of (3.24), following the second possibility stated in § 3. The Lagrangian densities, $L_G^{(0)}$, $L_{EM}$ and $L_D^{(1)}$ can be revised in the same manner as in the previous section. Since an antisymmetric field $\{A^{\mu\nu}\}$ describes a massless field with spin-parity $0^+$ in this case, we denote $\left(\delta^{\nu\nu}_\mu + \frac{1}{2} h^{\nu\nu}_\mu + A^\nu_\mu\right)$ as $b^{\nu\nu}_\mu$, instead of using $\{e^{\nu\nu}_\mu\}$ of (4.3): $\{b^{\nu\nu}_\mu\}$ and its inverse $\{b^{\nu\nu}_\mu\}$ should transform in the same way as $\{e^{\nu\nu}_\mu\}$ and $\{e^{\nu\nu}_\mu\}$, respectively, under coordinate transformation. The revised, correct, gravitational and electromagnetic Lagrangians, $L_G$ and $L_{EM}$, are given by (4.15) and (4.16), respectively, while the Dirac Lagrangian $L_D$ is given by (4.17) with $\{e^{\nu\nu}_\mu\}$ replaced by $\{b^{\nu\nu}_\mu\}$. The revised, correct action $I_\Pi$ can thus be represented as

$$I_\Pi = \int d^4x \sqrt{-g} L_\Pi,$$  \hspace{1cm} (5.2)

where

$$L_\Pi = L_G + L_{EM} + L_D + L_A,$$  \hspace{1cm} (5.3)

$$L_G = \frac{1}{2\kappa} g^{\mu\nu}(\partial_{\nu}F_{\mu\alpha} - \partial_{\mu}F_{\nu\alpha} - \Gamma^\beta_{\mu\nu}F_{\nu\alpha} - \Gamma^\beta_{\nu\mu}F_{\mu\alpha}),$$  \hspace{1cm} (5.4)

$$L_{EM} = -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F^{\mu\nu} F_{\rho\sigma},$$  \hspace{1cm} (5.5)

$$L_D = \frac{i}{2} b^{\nu\nu}_\mu \{\bar{\psi} \gamma^k (\partial_{\nu} - i e A_{\nu}) \psi - (\partial_{\nu} + i e A_\nu) \bar{\psi} \gamma^k \psi\} - m \bar{\psi} \psi,$$  \hspace{1cm} (5.6)

and $L_A$ is the revised Lagrangian of an antisymmetric field.

The form of $L_A$ can be fixed by the following requirements: (1) $L_A$ is form invariant under coordinate transformation, and (2) the Lagrangian density $L_A = \sqrt{-g} L_A$ coincides with $L_A^{(0)}$ of (3.31) in lowest order of $h^{\mu\nu}$ and $A^{\mu\nu}$. First of all we must redefine $\{a^{\mu\nu}_\alpha\}$ of (3.28) so that it become a scalar or a vector under coordinate transformation. For this purpose, we assume as in the previous section that the totally antisymmetric Levi-Civita symbol $\varepsilon$ has Latin indices (see (4.18)). Then Eq. (3.28) should be replaced by

$$a^{\mu\nu}_\alpha = \frac{1}{3} \varepsilon^{\mu\nu\rho\sigma\kappa} b^{\rho\sigma\kappa}_m b^{\mu\nu}_n b^{\rho\sigma\kappa}_{\mu\nu},$$  \hspace{1cm} (5.7)

which is a scalar, so the Lagrangian (3.31) becomes
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\[ L_A = \frac{9}{4 \lambda} (a^i a_i). \]  

This is the correct Lagrangian of an antisymmetric field.

Since \( \{b_k^i\} \) is the gravitational field variable of this case, the sum \( L_{G'} = L_G + L_A \) should be regarded as the gravitational Lagrangian. The action \( I_{\|} \) of (5.2) is exactly of the same form as that in New General Relativity based on the Weitzenböck space-time with \( \{b_k^i\} \) taken as the parallel vector fields. See Ref. 6) for details of New General Relativity; the action (5.2) is the action of the case \( c_1 = 0 = c_2 \) of this reference.

It is shown in § 3 that the action \( I_{\|}^{(3)} \) of (3.24) is invariant to lowest order in \( h_{\mu\nu} \) and \( A_{\mu\nu} \) under two kinds of transformations; one is (3.23a~d) and the other is (3.17a~c) with \( \omega_{\mu\nu} \) given by (3.20). The action \( I_{\|} \) of (5.2) is invariant under general coordinate transformation which is the revised form of (3.23a~d). Let us now try to revise another transformation (3.17a~c) with \( \omega_{\mu\nu} \) given by (3.20). As has been shown in the previous section, the revised form of (3.17a~c) is an infinitesimal local Lorentz transformation,

\[ \delta b_k^i = \omega_{k}^i (x) b_m^i, \quad \delta b_k^0 = \omega_{k}^0 (x) b_m^0, \]

\[ \delta \psi = \frac{i}{2} \omega_{\mu\nu} (x) S_{\mu\nu} \psi, \]

\[ \delta x' = 0, \quad \delta A_{\nu} = 0, \quad \delta g_{\mu\nu} = 0, \quad \delta g^{\mu\nu} = 0. \]

The gravitational and electromagnetic Lagrangians, \( L_G \) and \( L_{EM} \), are both invariant under (5.9a~c), while \( L_A \) and \( L_D \) are varied as

\[ \delta L_A = - \frac{3}{2 \lambda} \varepsilon^{i j m n} \omega_{j i} b_{m}^a a_{n}, \]

\[ \delta L_D = - \frac{i}{2} \varepsilon^{i j m n} \omega_{j i} b_{m}^a \tilde{\psi} \gamma_{\nu} \gamma_{\mu} \psi, \]

because \( a^i \) and \( \nu \psi \) change like

\[ \delta a^i = \omega^i_j a^j + \frac{1}{3} \varepsilon^{i j m n} \omega_{j i} b_{m}^a a_{n}, \]

\[ \delta (\nu \psi) = \frac{i}{2} \omega_{j i} S_{\mu\nu} \nu \psi + \frac{i}{2} \omega_{j i} S_{\mu\nu} \psi. \]

Thus the action \( I_{\|} \) of (5.2) is invariant under a local Lorentz transformation (5.9a~c) if and only if \( \{\omega_{ij} (x)\} \) satisfies the condition:

\[ \varepsilon^{i j m n} \omega_{j i} b_{m}^a = 0. \]

For global Lorentz transformation with \( \omega_{ij} = 0 \), this condition is satisfied independ-
ently of \( \{ b_k^r \} \), and therefore the field equation derived from the action is invariant under global Lorentz transformation.

Now consider a local Lorentz transformation \((5\cdot 9a\sim c)\) with \( \omega_{ij}(x) \) satisfying \((5\cdot 14)\). The Maxwell and Dirac equations derived from the action \( I_a \) is invariant under this transformation. However, the situation is different for the gravitational field equation derived by taking variation of the action with respect to \( b_k^r \). In fact, for an arbitrary, small variation of \( b_k^r \),

\[
\delta b_k^r \rightarrow b_k'^r = b_k^r + \delta b_k^r ,
\]

the condition \((5\cdot 14)\) is in general violated for \( b_k'^r \),

\[
\varepsilon^{ijmn} \omega_{ij}, b_k'^r \neq 0 .
\]

Therefore, the variation of the action \( \delta I_a \) due to \((5\cdot 15)\) is not necessarily invariant under \((5\cdot 9a\sim c)\), even if the condition \((5\cdot 14)\) is satisfied for \( b_k^r \). Thus the invariance of the action \( I_a \) under the local Lorentz transformations constrained by \((5\cdot 14)\) does not necessarily ensure that the gravitational field equation is also invariant under the same transformation.

§ 6. Conclusion

When the source of gravity is spin-1/2 particles, there are two different ways to formulate the Lorentz invariant theory of gravitation according as an antisymmetric field, which ensures consistency of the gravitational field equation, is treated as purely mathematical, unphysical entity or not. If we choose the former possibility, we are led to the usual General Relativity based on the Riemann space-time. If an antisymmetric field is physically significant, on the other hand, it describes a massless meson with spin-parity \( 0^+ \): In this case we are led to New General Relativity based on the Weitzenböck space-time. Whether an antisymmetric field is physically significant or not should be decided by experiment: At present there is no experimental evidence for an antisymmetric, massless meson with spin-parity \( 0^+ \).

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Appendix

--- Symmetrized Energy-Momentum Tensor in Special Relativity ---

For a dynamical system \( q = \{ q_A; A = 1, 2, \cdots \} \) described by a Lagrangian \( L = L(q, \partial_t q) \), the symmetrized energy-momentum tensor \( \langle T^{\mu\nu} \rangle \) is given by

\[
T^{\mu\nu} = \mathcal{T}^{\mu\nu} - \frac{1}{2} \partial_\mu (S^{\mu\nu} + S^{\nu\mu} + S^{\rho\sigma}) ,
\]

(A.1)
Lorentz Invariant Theory of Gravitation

where

\[ \mathbf{T}^a_{\alpha\beta} = - (\partial L/\partial q_{a,\alpha}) q_{a,\beta} + \delta^a_{\alpha} L, \]  
\[ \mathbf{S}^{\mu\nu} = - i (\partial L/\partial q_{a,\mu}) \mathbf{S}^{a,\nu} q. \]  

Here \( \{S^a\} \) is the Lorentz generator for \( q \). The symmetry property of \( \{T^a\} \) can be derived by means of the identity

\[ (\partial L/\partial q) \frac{i}{2} S^{a,\mu} q + \mathbf{T}^{(1a)} - \frac{1}{2} \partial_{\mu} S^{a,\nu} = 0, \]  

which follows from the Lorentz invariance of \( L \).

Now suppose that \( q \) consists of matter fields \( \phi = \{\phi_A; A = 1, 2, \ldots\} \) and the electromagnetic field \( \{A_{\mu}\} \), interacting by the minimal coupling,

\[ L = L_M(\phi, (\partial_{\mu} - ieA_{\mu}) \phi) + L_{EM}, \]  
\[ L_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \]

Let us define the gauge invariant non-symmetric energy-momentum tensor \( \{T_{iM}^{(M)}\} \) of matter by

\[ T_{iM}^{(M)} = - (\partial L_M/\partial q_{i,\alpha}) (\partial_{i} - ieA_{i}) \phi + \delta^a_{i} L_M. \]

Then we find after a little algebra that the symmetrized energy-momentum tensor \( \{T^a\} \) of (A·1) is given by

\[ T^a_{\mu\nu} = T^{(M)}_{\mu\nu} - \frac{1}{2} \partial_{\nu} (S^M_{\mu\rho} + S^M_{\rho\mu} + S^M_{\rho\nu}) + T^{(EM)}_{\mu\nu}, \]  

where \( \{S^M_{\mu\nu}\} \) is the spin tensor of matter fields,

\[ S^M_{\mu\nu} = - i (\partial L_M/\partial q_{i,\mu}) S^{a,\mu} \phi, \]

and \( \{T^{(EM)}_{\mu\nu}\} \) is the symmetrized energy-momentum tensor of the electromagnetic field,

\[ T^{(EM)}_{\mu\nu} = F^{\mu\rho} F_{\nu\rho} + \gamma^{\mu\nu} L_{EM}. \]

The sum of the first two terms in (A·8) is the symmetrized energy-momentum tensor of matter fields. The identity (A·4) applied to \( L_M \) gives

\[ (\partial L_M/\partial \phi) \frac{i}{2} S^{a,\mu} \phi + T^{(M1)}_{\mu\nu} - \frac{1}{2} \partial_{\mu} S^M_{\rho\nu} = 0. \]

When matter is a Dirac field \( \psi \), \( L_M \) is

\[ L_D = \frac{i}{2} \{\bar{\psi} \gamma^\mu (\partial_{\mu} - ieA_{\mu}) \psi - (\partial_{\mu} + ieA_{\mu}) \bar{\psi} \gamma^\mu \psi\} - m \bar{\psi} \psi, \]
so \( \{T^{(D)*}_i\} \) and \( \{S^{(EM)}_D\} \) are given by

\[
T^{(D)*}_i = -\frac{i}{2} \left( \overline{\psi} \gamma^a (\partial_i + ie A_i) \psi - (\partial_i + ie A_i) \overline{\psi} \gamma^a \psi \right) + \partial_i L_D, \quad (A\cdot13)
\]

\[
S^{(EM)}_D = \frac{1}{2} \varepsilon^{ij\mu\nu} \overline{\psi} \gamma^i \gamma^j \psi. \quad (A\cdot14)
\]

Using (A·13) and (A·14) in (A·8), we find that the symmetrized energy-momentum tensor of a Dirac field is just the symmetric part of \( \{T^{(D)*}\} \): Therefore we get

\[
T^{\text{sym}} = T^{(D)*ij} + T^{\text{EM}ij}. \quad (A\cdot15)
\]

Although we considered only the electromagnetic field, all the expressions obtained above are valid also for nonabelian gauge fields.

References

1. a) See for example,
   See also,
4) See, for example,
   Earlier references can be found therein.
8) See, for example,
   S. Deser, cited in Ref. 1).
9) See, for example, Ref. 4), chap. 18.
11) K. Hayashi, Phys. Letters 44B (1973), 497. See also Ref. 6).
12) See, for example,