Indefinite-Metric Quantum Field Theory of General Relativity. VI

Commutation Relations in the Vierbein Formalism

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The canonical commutation relations are analyzed in detail in the indefinite-metric quantum field theory of gravity based on the vierbein formalism. It is explicitly verified that the BRS charge, the local-Lorentz-BRS charge and the Poincaré generators satisfy the expected commutation relations.

§ 1. Introduction

In a previous paper, we have extended our manifestly-covariant canonical formalism of quantum gravity to the coupled Einstein-Dirac system. Because the generally-covariant Dirac theory is most conveniently described in terms of vierbein, we should regard the vierbein itself, rather than the gravitational field $g_{\mu\nu}$, as the fundamental quantity. Since the Dirac Lagrangian is invariant under the local Lorentz (LL) transformation, it is necessary to introduce the LL-gauge-fixing term and the corresponding Faddeev-Popov (FP) ghost one. For this purpose, we have introduced three antisymmetric fields $s_{ab}$, $t_{ab}$ and $\tilde{t}_{ab}$. We have emphasized that, in contrast with the path-integral formalism, the choice of the gauge-fixing term is almost unique in the covariant canonical formalism. A new feature of the LL-gauge-fixing term is that the auxiliary field $s_{ab}$ is not a Lagrange multiplier field; its six components must be regarded as canonical variables. Owing to this novel situation, our theory provides a counterexample to the so-called "ghost-counting rule" stating that the number of the non-physical components of a gauge field should be equal to that of the associated FP ghosts. If one wishes to keep this rule, one must count the six components of $s_{ab}$ as the non-physical components.

Now, the purpose of the present paper is to analyze the canonical commutation relations in the vierbein formalism. In § 2, we present the expressions for the canonical conjugates and set up the canonical commutation relations. In § 3, we calculate the equal-time commutators involving a time derivative of a Heisenberg field. We show there that the commutators between the "old" fields $(g_{\mu\nu}, b_\mu, c^\alpha, \tilde{c}_\alpha)$ remain unchanged despite the fact that the ten components of $g_{\mu\nu}$ are no longer regarded as canonical variables. In § 4, it is explicitly shown that the conserved
charges and the Poincaré generators satisfy the expected commutation relations.
In the Appendix, a remark is made on the commutator between \( b_\mu \) and the energy-
momentum tensor density.

We refer to Ref. 3) as II, to Ref. 4) as III and to Ref. 1) as V. We employ
the notation employed in II, III and V. For example,

\[
\partial_\mu A \cdot B = (\partial_\mu A) B ,
\]

\[
[A, B'] = [A(x), B(x')] \quad \text{at } x^0 = x'^0 ,
\]

\[
\partial^k = \partial / \partial x^k , \quad (k = 1, 2, 3)
\]

\[
\dot{g}^{\mu \nu} = h g^{\mu \nu} , \quad h = - \det h_{\mu \nu} = (- \det g_{\mu \nu})^{1/2}
\]

\[
h_{\mu \nu} = g_{\mu \nu} h_{\nu} , \quad h_{\mu \nu} = \gamma_{\mu \nu} h_{\rho \sigma}
\]

\[
\gamma^\nu h^\rho_{\nu} . \quad \{ \gamma^\nu , \gamma^\rho \} = 2 \delta_{\nu \rho}
\]

\[
\bar{\psi} = \psi^* , \quad \bar{\delta}_{ab} = \frac{1}{2} [ \bar{\gamma}^0 , \bar{\gamma}^1 ] ,
\]

\[
2 \Gamma^\mu_{ab} = (h^b \partial_\mu h^a - h^a \partial_\mu h^b - h^{ab} h^\mu \partial_\mu h_{ab}) - (a \leftrightarrow b)
\]

\[
\Gamma^\mu_{ab} = \frac{1}{2} \delta_{ab} \Gamma^\mu_{ab}
\]

\[
\frac{3}{2} \to
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§ 2. Canonical commutation relations

Our Lagrangian density of the vierbein formalism is written as

\[
\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{LLGF}} + \mathcal{L}_{\text{LFP}} + \mathcal{L}_{\text{D}} ,
\tag{2.1}
\]

where \( \mathcal{L} = \mathcal{L}_E + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} \) with \( \mathcal{L}_E , \mathcal{L}_{\text{GF}} \) and \( \mathcal{L}_{\text{FP}} \) being given in \([\text{II}, (2.12)] , \)

\[ \quad \text{[II, (2.13)] and [II, (2.4)]} \]

respectively, and

\[
\mathcal{L}_{\text{LLGF}} = \bar{g}^{\mu \nu} \Gamma^\nu_{\rho \delta} \partial_\rho s_{ab} ,
\tag{2.2}
\]

\[
\mathcal{L}_{\text{LFP}} = - i \bar{g}^{\rho \nu} \partial_\rho \bar{t}_{ab} \cdot (\partial_\nu \bar{t}_{cd} + 2 \Gamma^0_{\nu \rho} \bar{t}_{cd} ) ,
\tag{2.3}
\]

\[
\mathcal{L}_D = i h \bar{\psi} (\gamma^\rho \partial_\rho - \gamma^\nu \Gamma^\nu_{\rho \delta} + im) \psi .
\tag{2.4}
\]

The canonical variables are \( h_{\mu \nu} , s_{\mu} , c_{\mu} , \bar{t}_{ab} , \bar{t}_{ab} \) and \( \psi \), while \( b_\mu \) and \( \bar{\psi} \) are
not regarded as canonical variables. The canonical conjugates are as follows:

\[
\pi_{\mu}^{\cdot \alpha} = \partial L_{\text{tot}} / \partial h_{\mu \alpha}
\]

\[
= 2 h_{\alpha} \pi^{\cdot \alpha} + 2 \bar{g}^{\rho \nu} \partial_\rho \bar{t}_{cd} \left( \frac{1}{2} \partial_\nu \bar{t}_{cd} - i \partial_\nu \bar{t}_{cd} \cdot t_{cd} \right) - i h \bar{\psi} \gamma^\rho \partial_\rho \psi ,
\tag{2.5}
\]

where

\[
\pi^{\cdot \alpha} = \frac{1}{2} \left( \partial L / \partial \bar{g}^{\cdot \mu} + \partial L / \partial \bar{g}_{\cdot \mu} \right) = \pi^{\cdot \alpha}
\tag{2.6}
\]
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is given in [II, (2·22)] and [II, (2·23)];

\[
\begin{align*}
\pi_{\mathcal{E},b} &= \partial L_{\text{tot}} / \partial \mathcal{E}^b - i \tilde{g}^\alpha_{bc} \partial \mathcal{X}^c, \\
\pi_{\mathcal{X},b} &= \partial L_{\text{tot}} / \partial \mathcal{X}^b = - i \tilde{g}^{\alpha c} \partial \mathcal{E}^c, \\
\pi_{ab} &= \left( \partial L_{\text{tot}} / \partial \mathcal{E}^{ab} - \partial L_{\text{tot}} / \partial \mathcal{X}^{ab} \right) = \tilde{g}^\alpha_{ba} \Gamma_{\mathcal{E}}^{ab}, \\
\pi_{iab} &= \left( \partial L_{\text{tot}} / \partial \mathcal{E}^{iab} - \partial L_{\text{tot}} / \partial \mathcal{X}^{iab} \right) = - i \tilde{g}^{\alpha c} \partial \mathcal{X}^{iab}, \\
\pi_{i} &= \partial L_{\text{tot}} / \partial \mathcal{X}^i - \partial L_{\text{tot}} / \partial \mathcal{X}^i = - i \tilde{g}^{\alpha c} \partial \mathcal{X}^i, \\
\pi_{i} &= \partial L_{\text{tot}} / \partial \mathcal{X}^i - \partial L_{\text{tot}} / \partial \mathcal{X}^i = - i \tilde{g}^{\alpha c} \partial \mathcal{X}^i.
\end{align*}
\]

The canonical (anti-)commutation relations are as follows:

\[
\begin{align*}
\{ h_{ab}, \pi_{\mathcal{E},c} \} &= i \tilde{g}^\alpha_{bc} \delta^a_a \delta^b_b, \\
\{ \mathcal{E}^a, \pi_{\mathcal{E},c} \} &= i \tilde{g}^\alpha_{bc} \delta^a_a \delta^b_b, \\
\{ \mathcal{X}^a, \pi_{\mathcal{X},c} \} &= i \tilde{g}^\alpha_{bc} \delta^a_a \delta^b_b, \\
\{ s_{ab}, \pi_{\mathcal{E},c} \} &= \frac{1}{2} \left( \delta^\alpha_{a} \delta^b_b - \delta^\alpha_{b} \delta^a_a \right) \delta^c_c, \\
\{ \mathcal{E}_a, \pi_{\mathcal{E},c} \} &= \frac{1}{2} \left( \delta^\alpha_{a} \delta^b_b - \delta^\alpha_{b} \delta^a_a \right) \delta^c_c, \\
\{ \mathcal{X}_a, \pi_{\mathcal{X},c} \} &= \frac{1}{2} \left( \delta^\alpha_{a} \delta^b_b - \delta^\alpha_{b} \delta^a_a \right) \delta^c_c, \\
\{ \psi_a, \pi_{\mathcal{X},c} \} &= i \delta^a_a \delta^b_b,
\end{align*}
\]

and the other (anti-)commutators vanish.

It is evident that the (anti-)commutation relations concerning the FP ghosts \( \mathcal{E}^a \) and \( \mathcal{X}_a \) remain unchanged, whence hereafter we do not discuss them.

Finally, we note that

\[
\begin{align*}
[h^{ab}, X'] &= - h^{ab} h^{cd} [h_{cd}, X'], \\
[h, X'] &= h h^{ab} [h_{ab}, X'],
\end{align*}
\]

for any operator \( X \).

§ 3. Analysis of the commutation relations

We first note that since

\[
2 \partial \Gamma_{\mathcal{E}}^{cd} / \partial h_{ab} = \left( \delta^\alpha_{a} \delta^\beta_{b} \right) h^{cd} - \delta^\alpha_{b} \delta^\beta_{a} h^{cd} - h^{ac} h^{bd} - \left( c \leftrightarrow d \right),
\]

we have

\[
\partial \Gamma_{\mathcal{X}}^{cd} / \partial h_{ab} = 0.
\]
Hence (2.5) implies that

\[ \pi_h^{a_0} = 2h_y^{a_0}a_0, \quad \text{i.e., } \pi_h^{a_0} = \frac{1}{2} h_y^{a_0}a_0. \]  

(3.3)

Since, as emphasized in II, \( \pi_h^{a_0} \) depends on \( b_\rho \) but not on \( g_{sr} \), we find from [II, (2.22)]* that

\[ [h_{sa}, b'_\rho] = -i\kappa (g^{a_0})^{-1}h_{sa}\delta^a_0\delta^s, \]  

(3.4)

from which [II, (3.3)] is reproduced. From (3.4), (2.20) and (2.21), we see that

\[ [h^{a_0}, b'_\rho] = i\kappa (g^{a_0})^{-1}h^{a_0}\delta^a_0\delta^s, \]  

(3.5)

\[ [h, b'_\rho] = -i\kappa (g^{a_0})^{-1}\delta^a_0\delta^s, \]  

(3.6)

whence \( hh^{a_0} \) commutes with \( b_\rho \).

The canonical commutation relation \( [\pi_h^{a_0}, \pi_h^{a'_0}] = 0 \) together with (3.3) and (2.13) implies that

\[ [\pi_h^{a_0}, \pi_h^{a'_0}] = 0, \]  

(3.7)

from which [II, (3.6)], i.e., \([b_\rho, b'_\rho] = 0\), follows.

The following formulae are the direct consequences of the canonical (anti-)commutation relations:

\[-[s_{ab}, \delta^{ed}_r] = [s_{ab}, \delta^{ed}_r] = 0, \]  

(3.8)

\[-[s_{ab}, \delta^{ed}_r] = [s_{ab}, \delta^{ed}_r] = -\frac{1}{2}i(g^{a_0})^{-1}\left( (\delta^a_{a'_0}c_0 - \delta^a_0c_{a'}) - (c\leftrightarrow d) \right)\delta^s, \]  

(3.9)

\[ \{\ell_{ab}, \delta^{ed}_r\} = 0, \]  

(3.10)

\[ \{\ell_{ab}, \delta^{ed}_r\} = -\{\ell_{ab}, \delta^{ed}_r\} = -\frac{1}{2}(g^{a_0})^{-1}(\delta^a_{a'_0}\delta^e_0 - \delta^a_0\delta^e_{a'})\delta^s, \]  

(3.11)

\[ [h_{sa}, \delta^{ed}_r] = 0, \]  

(3.12)

From (3.3) and [II, (2.22)], we also have

\[ [s_{ab}, b'_\rho] = 0, \]  

(3.13)

\[ [s_{ab}, b'_\rho] = 0, \]  

(3.14)

\[ [s_{ab}, b'_\rho] = [s_{ab}, b'_\rho] = -i\kappa (g^{a_0})^{-1}\delta^a_0\delta^s, \]  

(3.15)

To proceed further, it is convenient to decompose \( h^{a_0}_a\pi_h^{a_0} \) into the symmetric and antisymmetric parts, that is, we set

\[ \pi_{sa}^{a_0} = \frac{1}{2}(h^{a_0}_a\pi_h^{a_0} + h^{a_0}_a\pi_h^{a_0}). \]  

(3.16)

* \( a_0 = g - (2e)^{-1}g^{a_0}\delta^{a_0}b_\rho \), where \( G^* \) depends on \( g_{sr} \) only.
On substituting (2·5) with (3·1) into (3·16), we have

\[ \pi_+^{\mu
u} = 2\pi^{\mu
u} - \left[ hh^{\mu\nu} \left( g^{\alpha\beta} h^{\alpha\beta} + g^{\alpha\beta} h^{\alpha\beta} \right) - (c \leftrightarrow d) \right] \left( \frac{1}{2} \partial_{\rho} s_{\alpha\rho} - i \partial_{\rho} \tilde{\varepsilon}_{\alpha} \cdot e^\nu \right) + i hh^{\mu\nu} \left( h^{\alpha\beta} h^{\alpha\beta} + h^{\alpha\beta} h^{\alpha\beta} \right) \bar{\varphi} \gamma_\mu \delta_{\nu} \psi, \]

\[ \pi_-^{\mu
u} = \bar{\varphi} \left( h^{\mu\nu} - h^{\alpha\beta} h^{\alpha\beta} \right) \left( \frac{1}{2} \partial_{\rho} s_{\alpha\rho} - i \partial_{\rho} \tilde{\varepsilon}_{\alpha} \cdot e^\nu \right) - i hh^{\mu\alpha} h^{\alpha\nu} \bar{\varphi} \gamma_\mu \delta_{\nu} \psi. \]

Here \( \bar{\varphi} \) can be expressed in terms of \( \pi_\phi \):

\[ \bar{\varphi} = -i ( \bar{\varphi} )^{-1} \pi_\phi. \]

From \([s_{ab}, \pi_+^{\mu
u}] = 0\) and (3·8), it is easy to see

\[ [s_{ab}, s_{cd}] = 0. \]

From (3·18) and (2·13) together with (3·12), we find

\[ [h_{ab}, s_{cd}] = \frac{1}{2} i ( \bar{\varphi} )^{-1} ( \eta_{ab} h_{cd} - \eta_{cd} h_{ab} ) \delta^d = -[h_{ab}, s_{cd}]. \]

Hence

\[ [g_{ab}, s_{cd}] = 0, \]

\[ [s_{ab}, \Gamma_+^{\mu\nu}] = \frac{1}{2} i ( \delta^a_{\beta} \delta^b_{\gamma} - \delta^a_{\gamma} \delta^b_{\beta} ) \left( \bar{\varphi} \right)^{-1} \delta^\gamma_{\delta}. \]

We note that (3·23) is consistent with (2·16), as it should be.

Next, from (3·16) and (2·13) together with (3·17), (3·21) and (3·12), we have

\[ [h_{ab}, \pi^{\alpha\beta}] = \frac{1}{2} i \left\{ \left[ \delta^\alpha_{\mu} h^\mu_a + \left( \bar{\varphi} \right)^{-1} \left( \delta^\alpha_{\mu} h^\mu_a - \delta^\alpha_{\mu} h^\mu_a \right) \right] + (c \leftrightarrow d) \right\} \delta^\alpha. \]

Hence

\[ [g_{ab}, \pi^{\alpha\beta}] = \frac{1}{2} i \left( \delta^\alpha_{\mu} \delta^\mu_{\alpha} + \delta^\alpha_{\mu} \delta^\mu_{a} \right) \delta^\alpha. \]

from which the expression for \([g_{ab}, \pi^{\alpha\beta}]\) presented in [II, (3·15)] is reproduced.

The canonical commutation relation \([\pi_h^{\alpha\beta}, \pi_h^{\gamma\delta}] = 0\) together with (2·13) yields

\[ [h_{a\gamma}^{\alpha\beta}, \pi_h^{\gamma\delta}] = \frac{1}{2} i g^{\alpha\nu} (2\pi^{\gamma\nu} - h^{\gamma\nu} \bar{\alpha}_h^{\gamma\nu}) \delta^\nu. \]

On substituting (2·5) with (3·1) into the right-hand side of (3·26), it becomes

\[ ig^{\alpha\nu} (h^{\gamma\nu} h^{\gamma\nu} - h^{\gamma\nu} h^{\gamma\nu}) \left( \frac{1}{2} \partial_{\rho} s_{\alpha\rho} - i \partial_{\rho} \tilde{\varepsilon}_{\alpha} \cdot e^\nu \right) + \frac{1}{2} g^{\alpha\nu} h^{\alpha\nu} \bar{\varphi} \gamma_\mu \delta_{\nu} \psi. \]

On the other hand, for the \( \psi \) term of \( h_{a\gamma}^{\alpha\beta} \), we obtain

\[ \left[ -i h_{a\gamma}^{\alpha\beta} (\partial \Gamma_\gamma / \partial h_{ab}) \psi, \pi^{\gamma\delta} \right] = \frac{1}{2} g^{\nu\alpha} h^{\mu\nu} \bar{\varphi} \gamma_\mu \delta_{\nu} \psi \]

after some calculation. Thus the contribution from the \( \psi \) term cancels out in
By applying the above result to \([\pi_\mu, \pi^{\alpha\nu}]\) and using (2·13), we find
\[
[\frac{1}{2} s_{ab} - i \tilde{E}_{ab} t', \pi^{\alpha\nu}] = \frac{1}{2} i \kappa (\frac{1}{2} \partial_\rho s_{ab} - i \partial_\rho \tilde{E}_{ab} t') \delta^\alpha.
\]
(3·29)

Hence, from [II, (2·22)], (3·22), (3·12) and (3·15), we have
\[
- [s_{ab}, b'_\nu] = [s_{ab}, b'_\nu] = - i \kappa (\tilde{g}^{\alpha\beta})^{-1} \partial_\rho s_{ab} \cdot \delta^\alpha.
\]
(3·30)

In a similar way, but with the aid of (3·29), \([\pi_\mu, \pi^{\alpha\nu}]\) yields
\[
[\pi^{\alpha\nu}, \pi_{\alpha\nu}] = 0.
\]
(3·31)

Accordingly, the expressions for \([g_{ab}, b'_\nu]\) and \([g_{ab}, b'_\nu]\) presented in [II, (3·16) \(\sim (3·21)]\) are reproduced. For example,
\[
[\tilde{g}_{ij}, b'_\nu] = i \kappa (\tilde{g}^{\alpha\nu})^{-1} \delta^\alpha \tilde{g}_{ij} \Gamma_{\nu}^{\alpha\beta} \delta^\beta.
\]
(3·32)

The derivation of the expression for \([b_{\mu}, b'_\nu]\) is again indirect, but a remark is made on the direct method based on the Einstein equation in the Appendix.

Thus we have established that all commutation relations between the old fields \((g_{ab}, b_\mu, c^\nu, \bar{c}_\nu)\) remain unchanged.

Now our next task is to find the commutators involving \(h_{ab}\) explicitly. Since it is not easy to work it out in the deductive way, we first postulate the expressions for the commutators involving \(h_{ab}\) and then verify that they are consistent with the canonical commutation relations.

From \([\pi_a^{\mu}, \pi_b^{\nu}] = 0\), we have
\[
[g^{\alpha\beta} \Gamma_{\nu}^{\alpha\beta}, b'_\nu] = i \kappa (\tilde{g}^{\alpha\nu})^{-1} \delta^\alpha g^{\nu} \Gamma_{\nu}^{\alpha\beta} \delta^\beta.
\]
(3·33)

Since \(g^{\alpha\beta} \Gamma_{\nu}^{\alpha\beta}\) involves no \(h_{ab}\), (3·33) imposes six conditions on \([h_{ab}, b'_\nu]\). Furthermore, since
\[
\tilde{g}_{ij} = h_{ab} h_j^a + h_{ij} h_{ab},
\]
(3·34)

(3·32) provides six conditions on \([h_{ab}, b'_\nu]\) independent of (3·33). Since \(h_{ab}\) has twelve components, the above conditions are necessary and sufficient for determining \([h_{ab}, b'_\nu]\) uniquely.

Instead of solving them, we make an ansatz
\[
[h_{ab}, b'_\nu] = i \kappa (\tilde{g}^{\alpha\nu})^{-1} \delta^\alpha h_{ab} \Gamma_{\nu}^{\alpha\beta} \delta^\beta.
\]
(3·35)

It is evident that (3·35) reproduces (3·32). Furthermore, from (3·35) we obtain
\[
[\Gamma_{\nu}^{\alpha\beta}, b'_\nu] = - i \kappa (\tilde{g}^{\alpha\nu})^{-1} \delta^\alpha \Gamma_{\nu}^{\alpha\beta} \delta^\beta.
\]
(3·36)

after some calculation. Then with the aid of [II, (3·3)] or (3·5), (3·33) is seen

\(\dagger\) The independence of those simultaneous linear equations for \([\hat{h}_{ab}, b'_\nu]\) is confirmed easily for \(\kappa \to 0\), whence for \(\kappa\) infinitesimal, and then for general \(\kappa\) by analytic continuation.
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to be reproduced. Thus (3.35) is established. It is noteworthy that (3.36) yields

\[ [h^\nu^\alpha, b^\nu] = 0, \] (3.37)

that is, \( \tau^\nu \Gamma_v \) commutes with \( b^\nu \).

The De Donder condition [II, (2·19)] is rewritten as

\[ g^{\alpha \beta} \dot{h}_{\alpha \beta} = -g^{\alpha \beta} \dot{h}_{\alpha \beta} + (h^\alpha h^\beta - h^\beta h^\alpha) \dot{h}_{\alpha \beta} \]
\[ -g^{\alpha \beta} \partial_\alpha h_{\beta \gamma} + (h^\alpha h^\beta - h^\beta h^\alpha) \partial_\beta h_{\alpha \gamma}. \] (3.38)

From (3.38) and (3.35), we find

\[ [\dot{h}_{\alpha \beta}, b^\nu] = -i \kappa \langle g^{\alpha \nu} \rangle^{-1} (2 \partial_\nu [\langle g^{\alpha \nu} \rangle^{-1} g^{\alpha \beta} h_{\alpha \beta}] + (h_{\alpha \beta} + \partial_\nu h_{\alpha \beta}) \delta^\nu \rangle. \] (3.39)

Therefore

\[ [\dot{h}_{\alpha \beta}, b^\nu] = -i \kappa \langle g^{\alpha \nu} \rangle^{-1} [\partial_\nu (h_{\alpha \beta} \delta^\nu) - \partial_\beta h_{\alpha \beta} \delta^\nu], \] (3.40)
\[ [\dot{h}_{\alpha \beta}, b^\nu] = i \kappa \langle g^{\alpha \nu} \rangle^{-1} (2 \partial_\nu [\langle g^{\alpha \nu} \rangle^{-1} g^{\alpha \beta} h_{\alpha \beta}] 
+ [\partial_\nu h_{\alpha \beta} - (g^{\alpha \nu} \delta^\nu) \delta^\beta] \delta^\nu \rangle. \] (3.41)

Next, we wish to determine \( [h_{\alpha \beta}, \dot{h}_{\alpha \beta}] \). From the expression for \( [g_{\alpha \beta}, \dot{g}_{\alpha \beta}] \) presented in [II, (3·15)], it is natural to make an ansatz

\[ [h_{\alpha \beta}, \dot{h}_{\alpha \beta}] = -\frac{i \kappa}{\langle g^{\alpha \beta} \rangle} \left[ h_{\alpha \beta} \dot{h}_{\alpha \beta} - h_{\alpha \beta} \dot{h}_{\alpha \beta} - g_{\alpha \beta} \gamma_{\alpha \beta} 
+ (g^{\alpha \beta})^{-1} (\delta^{\alpha \beta} \delta_{\alpha \beta} + \delta^{\alpha \beta} h_{\alpha \beta} + \delta^{\alpha \beta} h_{\alpha \beta} + h^{\alpha \beta} g_{\alpha \beta}) \delta^\alpha \right]. \] (3.42)

In the following, we verify that (3.42) is indeed the correct formula.

Since \( \pi^\gamma_{\alpha \beta} \) involves \( \dot{h}_{\alpha \beta} \) only through \( \pi^\gamma_{\alpha \beta} \), for the present purpose the canonical commutation relation (2·13) reduces to (3·24) with \( (\sigma \tau) = (k l) \). On substituting the explicit expression for \( \pi^\gamma_{\alpha \beta} \) ([II, (2·23)] and [II, (2·22)]), we can calculate the left-hand side of (3·24) by using (3·24). Then we find that the right-hand side of (3·24) is reproduced. Thus (3·42) is consistent with (2·13).

The crucial check for the validity of (3·42) is provided by the canonical commutation relation

\[ [h_{\alpha \beta}, \pi^\gamma_{\alpha \beta}] = 0. \] (3.43)

It is straightforward (but somewhat tedious) to derive

\[ [h_{\alpha \beta}, \Gamma_v^\nu] = -\frac{1}{2} i \kappa \langle g^{\alpha \nu} \rangle^{-1} \left[ h_{\alpha \beta} \dot{g}_{\alpha \beta} - h_{\alpha \beta} \dot{g}_{\alpha \beta} 
+ h_{\alpha \beta} h_{\mu} - (g^{\alpha \nu} \delta^\nu \delta^\mu + h^{\alpha \nu} h_{\alpha \beta} + h^{\alpha \nu} h_{\alpha \beta}) \right] \delta^\alpha \] (3.44)

by using (3·42). From (3·44), (3·43) is reproduced immediately.

By the above two checks, the validity of the expression for \( [h_{\alpha \beta}, \dot{h}_{\alpha \beta}] \) is confirmed. The expression for \( [h_{\alpha \beta}, \dot{h}_{\alpha \beta}] \) is verified by means of (3·42) with \( \sigma = i \) and the De Donder condition (3·38). Thus (3·42) is established.
Finally, we investigate the (anti-)commutators involving the Dirac field. By using (3.19), it is straightforward to see
\[
\{\psi_\alpha, \bar{\psi}_\beta \} = (\gamma^\mu)^{-1} (\gamma^\rho)_{\alpha\beta} \theta^\rho. \tag{3.45}
\]
Furthermore, we easily have
\[
[\psi, \bar{b}_\rho] = 0, \tag{3.46}
\]
\[
\{\psi, \bar{t}_{a\beta} \} = 0, \{\psi, \bar{\tau}_{a\beta} \} = 0. \tag{3.47}
\]
From the Dirac equation [cf., (2.4)]
\[
\gamma^\mu \psi = (\gamma^0)^{-1} \partial^\mu - i m \psi, \tag{3.48}
\]
we obtain
\[
[\psi, X'] = (g^{0\rho})^{-1} \gamma^\rho (\gamma^{\alpha a}) \partial_{\alpha a} \psi + [\gamma^a X, X'] \psi \tag{3.49}
\]
for any \(X\) commuting with \(\psi\). Since \(\gamma^a X\) commutes with \(b_\rho\), we find
\[
[\psi, b'_\rho] = i \kappa (g^{0\rho})^{-1} \partial_{\alpha a} \psi \theta^a = -[\psi, b'_\rho]. \tag{3.50}
\]
Similarly, we obtain
\[
- [\psi, s_{a\beta}'] = [\psi, s_{a\beta}'] = -\frac{1}{2} i (g^{0\rho})^{-1} \partial_{\alpha a} \psi \theta^a \tag{3.51}
\]
with the aid of (3.23), and
\[
- [\psi, h_{a\beta}'] = [\psi, h_{a\beta}'] = -\frac{1}{2} i \kappa (g^{0\rho})^{-1} [h_{a\rho} + 2 (g^{0\rho})^{-1} \partial^a h_{a\rho}] \psi \theta^a \tag{3.52}
\]
with the aid of (3.44). From (3.52) we have
\[
[\psi, \Gamma^\rho_{a\beta}] = \frac{1}{2} i \kappa (g^{0\rho})^{-1} (h^{0a} h_b^\beta - h^{0a} h^b) \psi \theta^a. \tag{3.53}
\]
Of course, we can calculate the commutators involving \(\bar{\phi}\) in the same way.

**Table 1. Equation numbers of the (anti-)commutators.**

<table>
<thead>
<tr>
<th></th>
<th>(b_\sigma)</th>
<th>(\bar{b}_\sigma)</th>
<th>(\bar{b}_\sigma)</th>
<th>(s_{a\alpha})</th>
<th>(i_{a\alpha})</th>
<th>(\bar{i}_{a\alpha})</th>
<th>(\bar{\phi})</th>
<th>(\Gamma^\rho_{a\beta})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h_{a\rho})</td>
<td>(3.4)</td>
<td>(3.42)</td>
<td>(3.40)</td>
<td>(3.21)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(3.52)</td>
</tr>
<tr>
<td>(b_\sigma)</td>
<td>0</td>
<td>{3.35}</td>
<td>{3.39}</td>
<td>{3.30}</td>
<td>(3.14)</td>
<td>(3.15)</td>
<td>(3.50)</td>
<td>(3.36)</td>
</tr>
<tr>
<td>(s_{a\alpha})</td>
<td>0</td>
<td>(3.21)</td>
<td>(3.30)</td>
<td>0</td>
<td>(3.9)</td>
<td>0</td>
<td>(3.51)</td>
<td>(3.23)</td>
</tr>
<tr>
<td>(t_{a\beta})</td>
<td>0</td>
<td>0</td>
<td>(3.14)</td>
<td>(3.9)</td>
<td>0</td>
<td>(3.11)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\bar{t}_{a\beta})</td>
<td>0</td>
<td>0</td>
<td>(3.15)</td>
<td>0</td>
<td>(3.11)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0</td>
<td>(3.52)</td>
<td>(3.50)</td>
<td>(3.51)</td>
<td>0</td>
<td>0</td>
<td></td>
<td>(3.53)</td>
</tr>
</tbody>
</table>
They are equivalent to the hermitian conjugates of the above formulae, as it should be.

For future convenience, we summarize the results of this section in Table I.

\section*{§ 4. Conserved quantities}

Corresponding to the conserved currents presented in § 3, there are the following conserved charges:

\begin{align}
Q_0 &= \int d^3x \bar{\psi} \gamma^0 \gamma^\mu \psi, \\
Q_c &= i \int d^3x \bar{\psi} \gamma^0 (\vec{\partial}_\mu c^\mu - \partial_\mu \vec{c} \cdot \vec{c}), \\
Q_t &= i \int d^3x \left[ \bar{\psi} (\partial_t \gamma^0 - 2\partial_0 \gamma^0 t^a) \right], \\
Q_b &= \int d^3x \bar{\psi} \gamma^0 (\vec{\partial}_a c^a - \partial_a \vec{c} \cdot \vec{c}), \\
Q_s &= \int d^3x \left[ \bar{\psi} (s_{ab} \partial_a t^b - \partial_a s_{ab} t^b) \right].
\end{align}

The first three charges correspond to the conservation of the three types of fermions. With the aid of the results obtained in § 3 (and in II), it is straightforward to show that

\begin{align}
[\psi, Q_0] &= \psi, & [\bar{\psi}, Q_0] &= -\bar{\psi}, \\
[c^\mu, Q_c] &= -ic^\mu, & [\bar{c}_\mu, Q_c] &= i\bar{c}_\mu, \\
[t_{ab}, Q_t] &= i t_{ab}, & [\bar{t}_{ab}, Q_t] &= -i \bar{t}_{ab},
\end{align}

and that all other commutators of the same kind vanish. Hence the constant operator

\begin{equation}
\lambda \equiv i (-1)^{q_0 + i q_c - i q_t}
\end{equation}

anticommutes with all fermion fields.

The BRS charge \(Q_b\) and the LL-BRS charge \(Q_s\) are the generators of the BRS transformation \(\delta^*\) and the LL-BRS transformation \(\delta_{LL}\), respectively. Indeed, by using the commutation relations presented in § 3, we can show that

\begin{align}
[\Phi, \lambda Q_0] &= -i i \delta^* (\Phi), \\
[\Phi, \lambda Q_c] &= i \delta_{LL} (\Phi)
\end{align}

for any field \(\Phi\); for instance,

\begin{equation}
[h_{\mu\nu}, Q_0] = -i \kappa (h_{\mu\nu} \partial_\mu c^\mu + c^\mu \partial_\mu h_{\mu\nu})
\end{equation}
from (3.4), (3.40) and (3.41), and
\[
[h_{ab}, Q_s] = -it^b_{ab},
\]
\[
l_{ab}, Q_s \} = +it^b_{ab}.
\]
from (3.21) and from (3.9) and (3.11), respectively. We also have
\[
Q_b^2 = 0, \quad [Q_b, Q_s] = -iQ_b,
\]
\[
Q_s^2 = 0, \quad [Q_b, Q_s] = iQ_s.
\]
As shown in III and in V, the Poincaré generators are as follows:
\[
P_\mu = \kappa^{-1} \int d^4x \bar{\phi}^\mu \partial_\mu \phi,
\]
\[
M_{ab} = \eta_{ab} M_\mu - \eta_{ab} \hat{M}_\mu + M_{LL,\mu},
\]
where
\[
\hat{M}_\mu = \kappa^{-1} \int d^4x \bar{\phi}^\mu [x^v \partial_\mu b_v - \delta^\mu_v b_v + i\kappa (\bar{c} \partial_\mu c^a - \partial_\mu \bar{c} c^a)],
\]
\[
M_{LL, ab} = 2 \int d^4x \{ \bar{\phi}^\mu \partial_\mu s^a - \pi^a_{\mu a} s^\mu s^a + \pi^a_{\mu b} s^\mu s^b - i [\hat{M}_b (\hat{M}_a)^\mu s^a - \hat{M}_a (\hat{M}_b)^\mu s^b - (a \leftrightarrow b)] \}.
\]
We note that $\hat{M}_\mu$'s are still the generators of the general linear group $GL(4)$, which play an important role in proving that gravitons are exactly massless.

It is easy to check that
\[
[\phi, P_\mu] = i\partial_\mu \phi
\]
for any field $\phi$. Furthermore, it is not difficult to show that the old fields $g_{\mu \nu}, b_\mu, c^a$ and $\bar{c}_a$ commute with $M_{LL, \mu}$ and that
\[
[h_{ab}, M_{LL, \mu}] = i (\eta_{ab} h_{\mu} - \eta_{ab} h_{\mu}),
\]
\[
[s_{ab}, M_{LL, \mu}] = i (\eta_{ab} s_{\mu} + \eta_{ab} s_{\mu}) - (\mu \leftrightarrow \nu),
\]
\[
[l_{ab}, M_{LL, \mu}] = i (\eta_{ab} l_{\mu} + \eta_{ab} l_{\mu}) - (\mu \leftrightarrow \nu),
\]
\[
[\bar{l}_{ab}, M_{LL, \mu}] = i (\eta_{ab} \bar{l}_{\mu} + \eta_{ab} \bar{l}_{\mu}) - (\mu \leftrightarrow \nu),
\]
\[
[\gamma_{\mu}, M_{LL, \mu}] = i \bar{\gamma}_{\mu}\eta_{ab}.
\]
Then we see that every field has the right commutator with $M_{ab}$.

As shown in III, $P_\mu$ and $\gamma_{ab} M_{ab} - \gamma_{ab} \hat{M}_{ab}$ satisfy the commutation relations of the Poincaré algebra. Hence $P_\mu$ and $M_{ab}$ form the Poincaré algebra if
\[
[P_\mu, M_{LL, \mu}] = 0,
\]
\[
[\hat{M}_\mu, M_{LL, \mu}] = 0,
\]
\[
[M_{LL, \mu}, M_{LL, \nu}] = -i\eta_{\mu \nu} M_{LL, \lambda} + i\eta_{\mu \lambda} M_{LL, \nu} - i\eta_{\nu \lambda} M_{LL, \mu} + i\eta_{\lambda \mu} M_{LL, \nu}.
\]
We can verify \((4\cdot 27) \sim (4\cdot 29)\) by making use of the fact that \(\hat{\phi} M_{\mu\nu} = 0\) and by using the expression for the commutators \([\hat{\phi}, M_{\mu\nu}]\). It is trivial to confirm \((4\cdot 27)\) and \((4\cdot 28)\), but to confirm \((4\cdot 29)\) needs somewhat lengthy calculation.

Finally, of course, the Poincaré generators commute with the conserved charges.

**Appendix**

--- A Remark on the Energy-Momentum Tensor Density ---

Since no canonical conjugate involves \(\hat{b}_\mu\), one must make use of the Einstein equation explicitly if one wishes to calculate \([b_\mu, \hat{b}_\nu']\) directly. Such analysis was made in \(\Pi\), though it was not yet completely worked out. It was found there that in order to reproduce the known expression for \([b_\mu, \hat{b}_\nu']\), it is necessary and sufficient that the commutation relations \([\Pi, (5\cdot 17) \sim (5\cdot 20)\] hold. Because of the dependence on the independent fields, they must hold separately for the contribution from \(R_{\mu\nu}\) and for that from \(T_{\mu\nu}\). If one employs the energy-momentum tensor density rather than the tensor, a little manipulation shows that the \(T_{\mu\nu}\) part of \([\Pi, (5\cdot 17) \sim (5\cdot 20)\] can be rewritten into a very simple form:

\[
[b_\mu, T^\nu_{\cdot \nu'}] = i\kappa (\bar{g}^{00})^{-1} T^0_\rho \delta^\rho_{\nu'} \delta^\nu_{\rho',}
\]

(A-1)

Accordingly, in the present theory, we must have

\[
[b_\mu, T^0_{\cdot 0'}] = i\kappa (\bar{g}^{00})^{-1} T^0_\rho \delta^\rho_{0'} \delta^0_{\rho',}
\]

(A-2)

\[
[b_\mu, T^{1}_{\cdot 0'}] = i\kappa (\bar{g}^{00})^{-1} T^{1}_{\cdot 0'} \delta^0_{\rho} \delta^0_{\rho'}
\]

(A-3)

where \(T^0_{\cdot 0'}\) and \(T^{1}_{\cdot 0'}\) are given in \([V, (2\cdot 15)]\) and in \([V, (3\cdot 24) \sim (3\cdot 28)\], respectively. It is not difficult to confirm that \((A-2)\) is indeed true. Unfortunately, however, it is too cumbersome to check \((A-3)\) explicitly.

It is noteworthy that the commutation relations of the type

\[
[b_\mu, X_\nu'] = i\kappa (\bar{g}^{00})^{-1} X_\nu' \delta^\rho_{\nu} \delta^\rho_{\nu'}
\]

(A-4)

hold for many quantities \(X_\nu\). For instance, \((A-4)\) holds for \(h_{\mu\nu}, \Gamma_{\cdot \mu}^{\cdot \nu}, \partial_{\cdot \mu} e^\nu, \partial_{\cdot \mu} \bar{e}^\nu, \partial_{\cdot \nu} s_{\cdot \nu}, \partial_{\cdot \nu} t_{\cdot \nu}, \partial_{\cdot \mu} e_{\cdot \nu}, \text{ and } \partial_{\cdot \nu} \bar{e}_{\cdot \nu}\). The verification of \((A-2)\) is simplified considerably owing to this fact.

**References**


**Note added in proof:** The validity of \((A-3)\) has been explicitly confirmed. Furthermore, the expression for \([b_\mu, \hat{b}_\nu']\) has been proved by the direct method based on the Einstein equation. Detailed accounts will be presented in the eighth paper.