Note on the Ising Model with Second-Neighbour and Four-Body Interactions Pertaining to Universality

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(Received July 2, 1979)

By making use of exact results on the Ising systems investigated by Vaks et al. and by Fisher, thermostatistical properties of the Ising lattice with second-neighbour and four-body interactions are studied. Transition temperatures of such systems can be obtained exactly and the following facts are proved. In the first place, the critical singularities of the systems with these interactions are the same as those of the ordinary Ising model. Secondly, the critical temperature of the ordinary antiferromagnet on the square lattice is a decreasing function of the magnitude of applied magnetic field. By making use of a dual transformation, it is also shown that second-neighbour and four-body interactions are irrelevant to the critical indices for the system with non-vanishing nearest-neighbour interaction.

§ 1. Introduction

In this paper we study the Ising system with second-neighbour and four-body interactions as well as usual nearest-neighbour interaction. In this respect we are interested in the works by Vaks et al. and by Fisher, and make use of their results to investigate the system mentioned above. Vaks et al. investigated a certain Ising lattice with some second-neighbour as well as nearest-neighbour interactions (referred to as VLO model, hereafter) on the basis of exact calculation. By applying a partial summation with some Ising variables or the cross-square transformation to the partition function of VLO model, we can get the partition function for the system with some second-neighbour and four-body interactions as well as nearest-neighbour interaction. The same transformation applied to Fisher’s antiferromagnetic model, which was proposed to study antiferromagnet exactly, is also to the purpose. Thus, we can get the exact transition temperature of our system in the case of certain special sets of interaction parameters, in which critical indices are found to be exactly the same as the usual Ising cases.

Another type of system in our interest is Baxter’s eight-vertex model, which is equivalent to double Ising layers coupling with each other by four-body interaction. This Baxter’s system is quite remarkable in that the critical indices change continuously with the strength of interactions and therefore the universality hypothesis breaks down.

On the basis of the results of cross-square transformation mentioned above and also of a dual relation in the blocked Ising systems investigated by Kasai, we can
show that the universality hypothesis holds even in the presence of four-body interaction so far as the usual nearest-neighbour interaction is present, where the former is irrelevant to the critical indices.

By applying the cross-square transformation in § 2, it is shown that VLO model is reduced to an Ising system with second-neighbour and four-body interactions, and the critical temperature and the critical indices of this system are obtained. In § 3, Fisher's model is studied in a similar way, where the phase diagram of antiferromagnet on magnetic field versus temperature plane is also discussed. Finally in § 4, discussion and concluding remarks are given, in which the universality hypothesis is verified by investigating the specific heat.

§ 2. Cross-square transformation applied to VLO model

We consider one spin $\sigma_0$ interacting pairwise with each of the surrounding four spins $\sigma_1, \sigma_2, \sigma_3$, $\sigma_4$ as shown in Fig. 1(a). In the partition function $Z(K)$ for this system, we first sum up with $\sigma_0$ to obtain

$$Z(K) = \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_4=\pm 1} \exp \{K\sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)\}$$

$$= C \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_4=\pm 1} \exp (X \sum_{i<j} \sigma_i \sigma_j + Y \sigma_i \sigma_j \sigma_0 \sigma_4), \quad (1)$$

where $K=\beta J$, with the exchange energy $J$, $\beta$ is the inverse temperature $(kT)^{-1}$ and $X$, $Y$ and $C$ are given by

$$X = \ln \cosh 4K/8,$$

$$Y = (\ln \cosh 4K - 4 \ln \cosh 2K)/8,$$

$$C = 2 (\cosh 4K)^{1/8} (\cosh 2K)^{1/2}. \quad (2)$$

![Fig. 1. By the cross-square transformation a five spin system with the nearest-neighbour interaction $J$ is reduced to a four spin system with the nearest-neighbour, second-neighbour and four-body interactions $J_1=kJX$, $J_2=kJX$ and $J_3=kJY$, respectively.](https://academic.oup.com/ptp/article-abstract/62/5/1225/1857635)
Equation (1) shows the thermodynamical equivalence between the system of five spins only with pair interaction and that of four spins with four- as well as two-body interactions such as shown in Figs. 1(a) and (b), respectively. The interaction parameters in Fig. 1(b) are $\beta^{-1}X$ and $\beta^{-1}Y$ respectively, corresponding to the interaction parameter $J$ in Fig. 1(a). It is noted that

$$N \geq 0, \ Y \leq 0,$$

where the equalities hold only at infinitely high temperature $\beta = 0$. In the following, by making use of this transformation, we obtain exact transition temperatures and critical indices of the system with Hamiltonian of the form

$$\mathcal{H} = -J_1 \sum \sigma_i \sigma_j - J_2 \sum \sigma_i \sigma_k - J_4 \sum \sigma_i \sigma_j \sigma_k \sigma_l,$$  \hspace{1cm} (4)

which consists of pair interactions $J_1, J_2$ between the nearest-neighbour and between some second-neighbours, respectively, and of four-body one $J_4$.

Vaks, Larkin and Ovchinnikov\(^0\) investigated such an Ising lattice (Union-Jack lattice) as shown in Fig. 2(a), where in addition to the nearest-neighbour interaction $J$, a second-neighbour one $J'$ exists only between half of such pairs. The transition temperature of this system is determined from equation

$$(y + 1)^2 (x^2 + 1)^2 - 2 (x^2 - 1)^2 = 0$$  \hspace{1cm} (5)

or

$$(y + 1)^2 (x^2 + 1)^2 - 2y^2 (x^2 - 1)^2 = 0,$$  \hspace{1cm} (6)

where $x$ and $y$ are the $\beta J$ and the $\beta J'$, respectively.

By making use of the cross-square transformation, VLO model is reduced to a lattice system with the Hamiltonian (4) such as shown in Fig. 2(b) (referred to as Model A, hereafter), in which $J_1, J_2$ and $J_4$ are determined by

$$K_1 = \tilde{\beta} J_1 = 2X + K',$$ $$K_2 = \tilde{\beta} J_2 = X, \ K_4 = \tilde{\beta} J_4 = Y.$$  \hspace{1cm} (7)

Here $\tilde{\beta}$ denotes the inverse temperature $(kT)^{-1}$ of this model and $K'$ is equal to $\beta J'$. The partition function $Z_A(K_1, K_2, K_4)$ of Model A is related to the one $Z_{VLO}(K, K')$ of VLO model with the lattice points of total number $N$ by

$$C^{N/2} Z_A(K_1, K_2, K_4) = Z_{VLO}(K, K').$$  \hspace{1cm} (8)
The critical temperature $kT_c/J_1$ and values of $J_2/J_1$ and $J_4/J_1$ are equal to $(2X + K')^{-1}$, $X(2X + K')^{-1} \text{ and } Y(2X + K')^{-1}$, respectively, where $X$, $Y$ and $K'$ are determined from Eqs. (2), (5) and (6). For example, we obtain $kT_c/J_1 = 3.6410$ with $J_2/J_1 = 1/2$ and $J_4/J_1 = -0.1309$ from the values $x = \sqrt{2} - 1$, $y = 0$. The transition temperature $T_c$ is shown by omitting the tilde for some cases of $J_2/J_1$ and $J_4/J_1$ values in Fig. 3. In case $J_2/J_1 = 0$ and $J_4/J_1 > 0$ which corresponds to the case $J = -J' > 0$ for VLO model, we obtain $T_c = 0$, where Model A is reduced to a special case of the eight-vertex model.\(^3\),\(^4\) The condition that Model A is reduced to the eight-vertex model,

\[
\tilde{\beta}J_1 = 2X + K' = 0,
\]

is neither consistent with Eq. (5) nor with Eq. (6) except for the case $T = 0$. Moreover, for every available value of $|J_4/J_1| \leq 1$, $T_c = J_2/kX$ is higher than the transition temperature $T^*_{\text{VLO}}$ of the eight-vertex model determined by

\[
\exp(2J_4/kT_{\theta}) \sinh 2J_2/kT_{\theta} = 1,
\]

at which higher order phase transition occurs. In Fig. 3, the transition temperatures for $J_4 = 0$ obtained by various approximate methods\(^8\),\(^9\),\(^10\) are also shown.

The critical indices of Model A are proved to be identical with those of VLO model in the following. From Eqs. (8) and (2), we obtain the relation

\[
\frac{1}{N} \frac{\partial^2 \ln Z_{\text{VLO}}}{\partial \beta^2} \left\{ (J' + J \text{ th } 4\beta J) \frac{\partial}{\partial \beta} \langle \sigma_i \sigma_j \rangle_{\beta} + \frac{J}{2} \text{ th } 4\beta J \frac{\partial}{\partial \beta} \langle \sigma_i \sigma_k \rangle_{\beta} \right\}
\]

\[
+ \frac{J}{4} \left( \text{ th } 4\beta J - 2 \text{ th } 2\beta J \right) \frac{\partial}{\partial \beta} \langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle_{\beta},
\]

in which less divergent terms such as $\partial^2 X/\partial \beta^2 \langle \sigma_i \sigma_j \rangle_{\beta}$ are neglected and $\langle \sigma_i \sigma_j \rangle_{\beta}$, $\langle \sigma_i \sigma_k \rangle_{\beta}$ and $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle_{\beta}$ denote the averages of the nearest-neighbour, second-neighbour and four-body correlations in Model A. The left-hand side of Eq. (11) is proportional to the specific heat of VLO model which behaves as $|T - T_c|^{-\alpha_{\text{VLO}}}$ near the critical temperature $T_c$. Because the most singular terms on the right-hand side are surely $\partial \langle \sigma_i \sigma_j \rangle_{\beta}/\partial \beta$ and $\partial \langle \sigma_i \sigma_k \rangle_{\beta}/\partial \beta$ and they do not cancel out each other, we conclude the relation.
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\[ \partial \langle \sigma_i \sigma_j \rangle_\beta / \partial \beta \sim |T - T_c|^{-\alpha_{\text{VLO}}} . \]  \hspace{1cm} (12)

Now, \( \langle \sigma_i \sigma_j \rangle_\beta \) is given by

\[ \langle \sigma_i \sigma_j \rangle_\beta = \frac{\text{Tr} \sigma_i \sigma_j \exp \left\{ (2X + K') \sum \sigma_m \sigma_n + X \sum \sigma_m \sigma_p + Y \sum \sigma_m \sigma_q \sigma_r \right\}}{\text{Tr} \exp \left\{ (2X + K') \sum \sigma_m \sigma_n + X \sum \sigma_m \sigma_p + Y \sum \sigma_m \sigma_q \sigma_r \right\}} . \hspace{1cm} (13) \]

By assuming the values of parameters \( J_1, J_2, \) and \( J_4 \) to differ from the respective values at the critical point by \( \Delta J_1, \Delta J_2, \) and \( \Delta J_4, \) respectively, we obtain up to first orders of \( \Delta J_1, \Delta J_2, \) and \( \Delta J_4 \)

\[ \langle \sigma_i \sigma_j \rangle_\beta = \langle \sigma_i \sigma_j \rangle_\beta + J_1 \sum_{(m,n)} \langle \langle \sigma_i \sigma_j \sigma_m \sigma_n \rangle_\beta - \langle \sigma_i \sigma_j \rangle_\beta \langle \sigma_m \sigma_n \rangle_\beta \rangle \]

\[ + J_2 \sum_{(m,p)} \langle \langle \sigma_i \sigma_j \sigma_m \sigma_p \rangle_\beta - \langle \sigma_i \sigma_j \rangle_\beta \langle \sigma_m \sigma_p \rangle_\beta \rangle \]

\[ + J_4 \sum_{(m,n,p,q)} \langle \langle \sigma_i \sigma_j \sigma_m \sigma_n \sigma_p \sigma_q \rangle_\beta - \langle \sigma_i \sigma_j \rangle_\beta \langle \sigma_m \sigma_n \sigma_p \sigma_q \rangle_\beta \rangle , \hspace{1cm} (14) \]

where \( \langle \cdots \rangle_\beta \) denotes the average in Model A with constant values of \( J_1, J_2, \) and \( J_4 \) and \( \Delta J_1, \Delta J_2, \) and \( \Delta J_4 \) are of order of \( |T - T_c| \). The most divergent terms in the derivatives with \( \beta \) which are obtained by substituting Eq. (14) into (12) is surely \( \partial \langle \sigma_i \sigma_j \rangle_\beta / \partial \beta, \) and accordingly by using the relation \( T - T_c \sim \hat{T} - \hat{T}_c \) we obtain

\[ \partial \langle \sigma_i \sigma_j \rangle_\beta / \partial \beta \sim |\hat{T} - \hat{T}_c|^{-\alpha_{\text{VLO}}} . \hspace{1cm} (15) \]

The specific heat \( c_\beta \) of Model A is expressed as

\[ c_\beta = \frac{1}{k} \frac{\partial}{\partial \beta} \left( 2J_1 \langle \sigma_i \sigma_j \rangle_\beta + 2J_2 \langle \sigma_i \sigma_k \rangle_\beta + J_4 \langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle_\beta \right) . \hspace{1cm} (16) \]

By the reason similar to for Eq. (11), the most divergent term on the right-hand side of (16) is \( \partial \langle \sigma_i \sigma_j \rangle_\beta / \partial \beta, \) and thus we obtain from (15)

\[ c_\beta \sim |\hat{T} - \hat{T}_c|^{-\alpha_{\text{VLO}}} . \hspace{1cm} (17) \]

In the case of finite magnetic field \( H \), there appear three-body interaction terms by the cross-square transformation. Similarly to Eq. (11), we can obtain after some calculation

\[ \frac{1}{N} \frac{\partial \ln Z_{\text{VLO}}}{\partial H} \bigg|_{H=0} = \frac{1}{16} \{ (8 + 4\beta J + 2 \text{th} 2\beta J) \langle \sigma_i \rangle_\beta \}

+ (4\beta J - 2 \text{th} 2\beta J) \langle \sigma_i \sigma_j \rangle_\beta \}. \hspace{1cm} (18) \]

The left-hand side of (18) behaves as \( (T_c - T)^{\beta_{\text{VLO}}} \) where \( \beta_{\text{VLO}} \) is the critical index of the magnetic moment of VLO model. The dominant term on the right-hand side of Eq. (18) is \( \langle \sigma_i \rangle_\beta \) because \( \langle \sigma_i \sigma_j \sigma_k \rangle_\beta \sim \langle \sigma_i \rangle_\beta \), and accordingly we conclude \( \langle \sigma_i \rangle_\beta \sim (T_c - T)^{\beta_{\text{VLO}}} \). By the deduction similar to that of (17), we find
In this way, we can conclude that the critical indices $\gamma$ and $\delta$ for Model A are the same as those of VLO model, respectively. According to Vaks et al.,

$$\langle \sigma_i \rangle_A \sim (\bar{T} - \bar{T})^{\gamma_{VLO}}.$$  \hspace{1cm} (19)

We can thus conclude that the indices $\alpha$, $\beta$ and $\gamma$ for Model A are not affected by second-neighbour and four-body interactions of values $J_2$ and $J_4$ shown in Fig. 3.

§ 3. Application to Fisher's model

Fisher discussed antiferromagnetic properties of such a decorated square lattice as shown in Fig. 4(a), where only the decorative sites shown by open circles are magnetic and the sites of matrix square lattice shown by closed circles are non-magnetic. The Hamiltonian for this system in the presence of a uniform and staggered magnetic field, $H$ and $H_{st}$, is expressed as

$$\mathcal{H} = -J \sum s_i \sigma_j + J \sum s_i \sigma_k - (H + H_{st}) \sum \sigma_j - (H - H_{st}) \sum \sigma_k,$$

where $s_i$ and $\sigma_j$ are the Ising variables for the non-magnetic and magnetic sites, respectively, and the summations are taken over all horizontal and all vertical pairs in the first and second summands, respectively. Magnetic sites compose a square lattice, which can be divided into two sublattices $a$ and $b$, and the summations in the third and fourth summands on the right-hand side of (20) are taken over the $a$ and $b$ sublattices, respectively. In case of vanishing staggered field $H_{st}$, by performing the summation over all states of magnetic spins, we find that the partition function, $Z_{FA}(K, L)$, of Fisher's model is related to that of the ordinary Ising model, $Z_I(K)$, as

$$Z_{FA}(K, L) = C' \times Z_I(K'),$$

where $L = \beta H$, $N$ denotes the total number of non-magnetic spins and $K'$ is given by

$$K' = \ln \{ \text{ch} (2K + L) / \text{ch} L \} / 2.$$  \hspace{1cm} (21)

By making use of the critical value $T_c'$ which is determined by $\text{sh } K_c' = 1$, we can find the critical temperature $T_c$ of Fisher's model as a function of $H$ given by

$$\text{ch} \{ (2J + H) / kT_c \} / \text{ch} H / kT_c = \sqrt{2} + 1.$$  \hspace{1cm} (22)
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In such a way, Fisher studied the thermal behaviour of his antiferromagnetic model in the presence of finite magnetic field. In the presence of nonvanishing staggered field $H_a$, Fisher's model is reduced to the usual ferromagnetic Ising model with finite magnetic field, in which no phase transition occurs.\(^\text{11}\)

By applying the cross-square transformation to Fisher's model, in which the summations with nonmagnetic sites are carried out, we obtain the Ising system with second-neighbour and four-body interactions (referred to as Model B, hereafter), whose partition function $Z_B(K, J_1, J_2, L)$ with the reduced field $\tilde{L}(=\beta H)$ is related to $Z_{FA}(K, L)$ by

$$C^\text{v}Z_B(-X, X, Y, L) = Z_{FA}(K, L), \quad (24)$$

where $X$ and $Y$ are determined from Eq. (2). As shown in Fig. 4(b), the second-neighbour and four-body interactions exist only in one half of the alternate square faces. It is noticed that the nearest-neighbour interaction $J_1$ is negative because of the opposite signs of vertical and horizontal couplings. The second-neighbour one $J_2$ is positive and $J_4 = |J_1|$. By making use of Eqs. (2) and (24), the critical point for Model B is related to that of Fisher's model given by (23), where the critical temperature $kT_c/|J_1|$ and values $J_4/|J_1|$ and $\tilde{H}/|J_1|$ are determined by the critical values of $X^{-1}$, $Y/X$ and $L/X$, respectively. For example, we obtain $kT_c/|J_1| = 3.3809$ for the case $J_2/|J_1| = 1$, $J_4/|J_1| = -0.4899$ and $H = \tilde{H} = 0$. In Fig. 5, we show $T_c$ for some pair of values of $\tilde{H}$ and $J_1$ by omitting...
the tilde, where the broken curve in Fig. 5 represents the $T_c-H$ relation for Fisher’s model in a suitable scale.

Here, we note the phase diagram on $H-T$ plane. Both curves for the critical temperature $T_c$ in Fig. 5 go down with the increase of the field $H$. However, some approximate methods applied to the ordinary antiferromagnetic Ising model with nearest-neighbour interaction give such a result as shown by the curve (i) in Fig. 6. That is, the mean-field theory does for any type of lattice and does the Bethe approximation\(^\text{19}\) for the lattice with coordination number $z$ larger than $z_c$ (≈ 5.14) which is determined by $(z - 2 z^* = (z - 1) z^*$. The computation by the Monte Carlo method\(^\text{13}\) for b.c.c. lattice also gives such a curve. The critical temperature $T_c$ shown by (i) in Fig. 6 increases with increasing field $H$ in the vicinity of the critical field $H_c$ at which $T_c = 0$. For Model B the ground state of the antiferromagnet is stabilized by the second-neighbour and four-body interactions by the amount of $-(J_2 + J_4 / 2)$, which is surely negative for the cases shown in Fig. 5. Moreover, both of these interactions give no effect on the critical field $H_c$. Accordingly, for a magnetic field smaller than $H_c$, the critical temperature of Model B is higher than that of the nearest-neighbour antiferromagnet such as shown by (ii) and (iii) in Fig. 6 schematically. Thus, it has been proved for the usual antiferromagnetic Ising model with only nearest-neighbour interaction on the square lattice that $\frac{d T_c(H)}{d H}_{H_-H_c}$ is negative.

It is noticed that for a nonvanishing $\tilde{H}_st$, no phase transition occurs in Model B as well as in Fisher’s model, whereas two phases coexist on the lines $\tilde{H} = \pm \tilde{H}_st + 4J_t > 0$ and $\tilde{H} = \pm \tilde{H}_st - 4J_t < 0$ on the zero temperature plane. This is in contrast with the result of mean field approximation to the same model with $J_4$ vanishing, where the phase boundaries extend to finite temperatures and end at the critical lines.\(^\text{16}\) If the mean field result can be believed qualitatively, our result indicates that the four-body interaction suppresses the phase transition in Model B to occur at finite temperature and staggered field.

The critical indices of Model B can be proved to be the same as those of the usual Ising model; $\alpha$ is found by a deduction similar to that in the preceding section, and $\beta$, $\gamma$ and $\delta$ are more straightforwardly. They are all the same as those of the ordinary Ising system.

§ 4. Discussion and concluding remarks

In the preceding sections, we have shown that VLO model and Fisher’s superexchange antiferromagnet are reduced to the Ising systems on the square lattice with nearest-neighbour, second-neighbour and four-body interactions, and with a finite magnetic field in the case of Fisher’s model. In the reduced systems, the energy parameters, $J_2$, $J_4$ and $\tilde{H}$, are not arbitrary but related to one another. As a result, the transition temperatures for the reduced systems are found exactly. On the basis of the phase diagram for Model B, the derivative $\frac{d T_c(H)}{d H}$ at the
critical field $H_c$ is proved to be negative for the usual Ising antiferromagnet with only the nearest-neighbour interaction on the square lattice. Critical indices for our models are proved to be the same as those of the usual Ising model. Thus, it seems possible quite in general to conclude that the second-neighbour and four-body interactions and the uniform applied field in antiferromagnet are irrelevant. However, it is not certain, because the dependence of the critical indices on the energy parameters, $J_2$, $J_4$ and $\vec{H}$, are discussed in our models only for the cases of some special relations between those parameters existing. Only as a consequence of such relations the indices may accidentally reduce to those of the usual Ising model. More believable results on the universality hypothesis are obtained by the following discussion.

The critical index, say, $\alpha$ is assumed to depend on a certain field $\gamma$ generally in either of three different ways; (i) $\alpha$ is independent of $\gamma$, (ii) $\alpha$ only depends on whether $\gamma$ is vanishing or nonvanishing, (iii) $\alpha$ depends on $\gamma$ literally. In the case (iii), we can expect that $\alpha$ is a monotonic function of $\gamma$ as in the case of the eight-vertex model. According to the results in the preceding section, we have

$$\alpha_A(J_2/J_1, J_4/J_1) = \alpha_1, \quad (25)$$

where $\alpha_A(J_2/J_1, J_4/J_1)$, $\alpha_B(J_2/|J_1|, J_4/|J_1|, \vec{H}/|J_1|)$ and $\alpha_1$ represent the indices of the specific heat of Model A, of Model B and of the usual Ising model, respectively. The parameters $J_2$ and $J_4$ in the arguments in (25) are related to each other in such a way as shown in Fig. 3 and $J_1$ and $\vec{H}$ in (26) are in such a way as shown in Fig. 5. As for the uniform magnetic field $\vec{H}$, which can be regarded as a staggered field for the ferromagnetic case in our model, Griffiths asserted that the magnetic field does not influence the critical singularity of the antiferromagnet (smoothness hypothesis), which was also ascertained numerically by Rapaport and Domb. On the basis of that hypothesis we obtain from (26) by using (2)

$$\alpha_B(1, J_4/|J_1|, 0) = \alpha_1, \quad (0.4899 \leq J_4/|J_1| < 1) \quad (27)$$

The relation (27) indicates that $J_4$ is irrelevant and $J_2$ is also irrelevant because $\alpha_B(1, 0, 0) = \alpha_B(0, 0, 0) (= \alpha_1)$. Thus, it has been proved that $\alpha_B$ is independent of both $J_2$ and $J_4$ in Model B. The partition function, $Z_A(K_1, K_2, K_4)$, of Model A is related to that of Model B, $Z_B(K_1, K_2, K_4, \vec{L})$, by Kasai as

$$C_A(K_1) Z_A(K_1, K_2/2, K_4/2) = C_B(K_1^*) Z_B(K_1^*, K_2^*, K_4^*, 0), \quad (28)$$

where $C_A(K_1)$ and $C_B(K_1^*)$ are some smooth functions of respective arguments and the inverse temperature $K_1^* (= \beta^* J_1)$ is related to $K_1$ by
\{\exp (8K_1) - 1\} \{\exp (8K'_1) - 1\} = 8. \tag{29}

We obtain from (28) and (29)

\[ a_A (1/2, 1/2) = a_B' (1, 1, 0), \tag{30} \]

where \( a_B' \) represents the exponent of the ordered phase. As \( a_B' (1, 1, 0) = a_t \), Eq. (30) shows

\[ a_A (1/2, 1/2) = a_1. \tag{31} \]

Equation (25) in the case \( J_2 / J_1 = 1/2 \) reduces to

\[ a_A (1/2, -0.1309) = a_1. \tag{32} \]

By combining Eqs. (31) and (32), we can see that \( J_4 \) is irrelevant. Because we have from Eq. (25)

\[ a_A (J_2 / J_1, 0) = a_1, \quad (0 \leq J_2 / J_1 < \infty) \tag{33} \]

\( J_2 \) is also found irrelevant.

Thus, on the basis of the smoothness hypothesis on the antiferromagnet, we have proved that the critical singularity of the specific heat is influenced neither by the second-neighbour interaction nor by the four-body one in both cases of Model A and Model B. By assuming that one of the other fields, \( J_2 \) or \( J_4 \), is irrelevant, which is also ascertained on the basis of the approximate methods,\(^{8,10,15,16,18,21}\) we can prove the universality hypothesis from the consideration similar to the above one.

References

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