sentence on p. 4; it may very well be a valid one, but the discussion preceding it is irrelevant to it.

L. TREFETHEN.[4] Would the authors be willing to comment on whether the vortices are parallel to the yawed cylinder, or normal to the flow, or perhaps at an in-between angle? Also, if the cylinder is free to move, would its vibration affect the angle?

Bubble Trajectories and Equilibrium Levels in Vibrated Liquid Columns[1]

FRANKLIN T. DODGE.[2] The authors have derived a partly nonlinear theory to describe bubble motions in vertically vibrated tanks. In Bleich's original analysis (Reference [1] of the paper), as well as in the other extensions to this theory mentioned by the authors (references [2, 3, 4, 5, 6, and 7], only those nonlinear terms in the equations were retained that were absolutely necessary to show that stationary bubbles existed; even so, the comparison of theory and experiment was reasonably close. The authors, however, have attempted to improve on this theory by retaining certain other nonlinear terms, although various other nonlinearities were discarded. Some other important effects, such as the influence of the tank walls (i.e., finite volumes of liquid), were not considered. Nevertheless, their analysis does indicate that nonlinear effects do have a noticeable role in the bubble behavior since their theory compares slightly better to experimental results than do the previous, more highly linearized theories.

It is not clear, however, what justification there is for retaining the particular nonlinear terms retained in the theory while neglecting others. Perhaps the authors have order-of-magnitude arguments to justify those terms they have retained.

It is worth noting that in a recent study at SwRI[3] which completed our work reported in reference [6] of the paper, the influence of finite bubble size was determined, both by experiment and by a partially linearized analysis similar to Bleich's. It was found that finite-size effects became apparent for bubbles whose diameters were 5 percent or more of the tank diameter. Presumably, the bubbles used in the experiments reported by the authors were smaller than this.

R. J. SCHOENHALS.[4] The authors are to be congratulated on their fine contribution in an area of considerable importance, both scientifically and technically. There are two items which are deserving of further comment in the opinion of this discusser. First, it seems that the hydrodynamic force has been formulated for a spherical bubble which is simultaneously accelerating and undergoing a change in volume, while the cited reference gives the resultant hydrodynamic force only for the case of a sphere of constant volume. Could the authors show further detail, or possibly indicate an additional reference, which would illustrate the principles on which their expression is based? Second, λ is apparently one of the independent dimensionless parameters, among several listed below equation (12), which has an influence on the resulting bubble motion as described by $\bar{Z}$. The experimental measurements of equilibrium levels also show that a variation in λ does have some influence on the observed results. However, the integration method used for analytical prediction of the equilibrium levels yields values which do not depend on λ, as indicated by equation (16) for example. Is this due to the fact that ϵ has been assumed to be small in the derivation of equation (16)? Further comment by authors on this second point would also be appreciated.

Authors' Closure

R. J. Schoenhals

Additional detail on the formulation of the hydrodynamic force may be found in reference [1] of the paper.

With regard to the vibrational amplitude parameter, λ, equation (16) simply indicates that the motion of a bubble, undergoing *isothermal* pulsations, about an equilibrium level is independent of λ. This is not true for other thermodynamic behavior of the bubble.

Franklin T. Dodge

In this paper, the authors attempted to develop equations which describe the bubble trajectory as well as the location of equilibrium levels. In the development, it was necessary to discard certain nonlinearities in order to obtain a solution. As shown in the paper, the assumption of small Δ/A is not only incorrect but unnecessary as well. Luckily, this assumption has little effect on the location of equilibrium levels; however, the bubble trajectory and its motion about an equilibrium level are significantly affected by such an assumption. Admittedly, at present, the bubble trajectory is of academic interest only.

Bubble size was not rigidly controlled in the experimental work reported in the paper. An attempt was made, however, to use only "small" bubbles. Perhaps the "finite-size" effect mentioned accounts for part of the discrepancy between the data and theory.

Effects of Gravity and Surface Tension Upon Liquid Jets Leaving Poiseuille Tubes[1]

C. P. HUANG.[2] The neatness of the author's attack on the linearization of two dimensional boundary layer equations is very valuable. This paper contributes to the understanding of the development of velocity profile, exit contraction and gravitational effect of free jets; these aspects of flow behavior are crucial to certain applications.[3] In appraising the results, there are two interesting points for discussion.

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[4] Tuft University, College of Engineering, Medford, Mass.

[1] By J. M. Foster, et al., published in the March, 1968, issue of the JOURNAL OF BASIC ENGINEERING, pp. 125–132.

[2] Senior Research Engineer, Department of Mechanical Sciences, Southwest Research Institute, San Antonio, Texas. Mem. ASME.

[3] Kana, D. D., and Chu, W.-H., "Bubble Dynamics in Vibrated Liquids Under Normal and Simulated Low Gravity Environments," Tech. Rept. No. 8, Contract NAS8-11045, Southwest Research Institute, San Antonio, Texas, Feb. 1967.

[4] Associate Professor, School of Mechanical Engineering, Purdue University, Lafayette, Ind. Assoc. Mem. ASME.

[1] By J. H. Lienhard, published in the June issue of the JOURNAL OF BASIC ENGINEERING, pp. 262–268.

[2] Senior Mechanical Engineer, Engineering Research Department, Minnesota Mining and Manufacturing Company, St. Paul, Minn. Assoc. Mem. ASME.

[3] For example, Hansen, R. S., Purchase, M. E., Wallace, T. C., and Woody, R. W., "Extension of the Vibrating Jet Method for Surface Tension Measurement to Jets of Nonuniform Velocity Profiles," *Journal of Physics Chemistry*, Vol. 62, Feb. 1958, pp. 210–214.

1 The flow development in the inlet region of tubes and parallel plate channels is a reverse physical model as compared to that in the present study. If we define the entrance length of flow in a tube, x_{en}, as the distance between the inlet and the point where the pressure drop reaches a constant value, Sparrow, Lin and Lundgren[4] in their different approaches to the linearization of the momentum equation on the flow in the entrance region of tube show that x_{en} is approximately equal to 0.2. If we define the exit length of a free jet, x_{ex}, as the distance between the exit and the point where the contraction of the jet reaches a constant value, the author's work gives $x_{ex} = 0.2$. Interestingly, x_{en} and x_{ex} are of the same value.

2 The linearization given in equation (2*a*) may not be true at the very first portion of the free jet. It can be easily verified that at this portion, the two terms uu_x and vu_y are in the same order of magnitude. Wang and Longwell[5] in their numerical solution of the complete two-dimensional Navier-Stokes equations for laminar flow in the inlet section of parallel plates indicate that velocity profile at the very first portion of the inlet section is slightly concave at the center (i.e., the maximum in the velocity profile does not occur on the axis). Such a concavity on the velocity profile does not reveal in Sparrow, Lin, and Lundgren's linearized result. Accordingly, I speculate that a concave or a flat segment at the center of the velocity profile might also exist in the early portion of the free jet.

Author's Closure

I am grateful to Dr. Huang for his interest in this work. In particular, I was remiss in failing to emphasize that there is some relationship between the pipe entrance problem and the present problem. The analogous behavior is definitely worth noting.

However, there is a basic difference between the two problems that causes me to hesitate in consenting to Dr. Huang's suggestion that the behavior might be analogous at low x. In the entrance problem, a developing boundary displaces and shapes the flow of an almost inviscid core rather as a nozzle would. The streamlines pile together at the outer edge of this core resulting briefly in a slightly higher velocity just outside of the boundary layer.[1] The approximate treatments of the entry problem generally lose this rather minor feature of the flow.

My analysis of the departing flow is really a kind of refined integral treatment of a flow that is completely viscous. No doubt Bruns' and my velocity data form a flatter profile at small x and r than is predicted. And no doubt the predicted velocity profile achieves its maximum error in this region (about 4 percent for $x = 0.0185$ and $r = 0$). The error does indeed result from the causes Dr. Huang suggests, but the basic direction of the error is an underestimation of v. This in turn results in a predicted profile that spreads out a little more slowly than it should.

I am disinclined to expect that any additional inflections will be found in the velocity profiles of departing jets. Of course, if such anomalous behavior occurs it will only be revealed by very careful experiments at very small x's and/or by a computer solution of the full Navier-Stokes equations as Dr. Huang suggests.

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[4] Sparrow, E. M., Lin, S. H., and Lundgren, T. S., "Flow Development in the Hydrodynamic Entrance Region of Tubes and Ducts," *The Physics of Fluids*, Vol. 7, No. 3, March 1964, pp. 338–347.

[5] Wang, Y. L., and Longwell, P. A., "Laminar Flow in the Inlet Section of Parallel Plates," *AIChE Journal*, Vol. 10, No. 3, May 1964, pp. 323–329.

[1] I am grateful to Prof. R. Eichhorn for calling my attention to this interpretation of the concavity in an early developing entry flow.

Secondary Flow Effects in a Bounded Rectangular Jet[1]

G. HESKESTAD.[2] The authors are to be commended for having begun a line of inquiry of interest not only to the fluid device investigator, but also to the experimenter in turbulent shear flows who is concerned with establishing reliable data on "simple" shear flows, such as the two-dimensional jet, to aid a theoretical approach to the turbulent shear-flow problem. As evident in the paper (Fig. 4) and apparent from further comparisons among results obtained by the investigators quoted [1, 5, 6], data recorded in different *nominal* two-dimensional jets by different people with different instruments[3] are disappointingly inconsistent. The mechanism suggested and substantiated by the authors, and previously hinted at by Newman [17],[4] may be at least partially responsible for the disagreements. A great share of the blame goes to the instruments employed, each being highly inaccurate in the kind of flow considered [5, 18, 19, 20, 21] where turbulence intensities over extensive regions are of order unity. Other causes for the disagreements may be traced to the boundary conditions of the potential entrainment flow and to conditions at the nozzle, including degree of flow-homogeneity along the nozzle length. Discrepancies among static pressure measurements often reflect more on instrument characteristics than actual differences among the flows; Fiedler [21] has demonstrated that mean pressures relative to ambient sensed by a particular probe may contain an error greater than the mean pressure difference to be measured, even on the center line of the jet.

It is an interesting fact that references [1, 5, 6] have reported measurements only to a distance downstream of the nozzle which corresponds roughly to the length of the nozzle. Thus Miller and Comings [1] report measurements to $x/a = 40$, their nozzle aspect ratio being 40; Heskestad [5] reports measurements to $x/a = 155$, nozzle aspect ratio being 120; Bradbury [6] measured to $x/a = 70$ with a nozzle aspect ratio of 48. In the first case (Miller and Comings) it appears that regions further downstream than $x/a = 40$ were avoided because variations in center-line mean velocity became inconsistent with self-preservation. The discusser's measurements [5] were stopped at $x/a = 155$ because somewhat further downstream the center-line mean velocity began decreasing faster with x/a than was consistent with the preceding self-preserving range (as in the case of Miller and Comings) and also because the intermittency factor on the center line began decreasing from unity[5] (not reported in either of references [4, 5]). Bradbury evidently gives no explicit reason for stopping measurements at a particular downstream location, nor do his data suggest any reason. Coupled with the findings of the present paper, the foregoing observations seem to imply that secondary flows generated by the bounding walls may have influenced the flow in the midplane (where the measurements were made) at all locations somewhat further downstream than one nozzle length. But the possibility that even regions upstream wereaffected, per haps in a subtle way, cannot be ruled out.

Unfortunately the experiments reported in this paper do not settle whether effects of secondary flows are indeed confined to downstream regions or are felt throughout the flow. They might

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[1] By J. F. Foss and J. B. Jones, published in the June 1968, issue of Journal of Basic Engineering, Trans. ASME, pp. 241–249.

[2] American Standard Inc., Research and Development Division, New Brunswick, N. J.

[3] Miller and Comings [1] used a constant-temperature hot wire, Heskestad [4, 5] used a constant-temperature hot wire with linearized output, and Bradbury [6] used a constant-current hot wire for turbulence measurements and combination of pitot and static tubes for mean velocity field.

[4] Numbers 17–22 in brackets designate Additional References at end of discussion.

[5] At the time it was believed that uncontrollable drafts in the laboratory were to blame.