QCD Vacuum in the Strong External Fields

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In the very strong external gauge field the QCD true vacuum is shown to have lower energy than the "perturbative vacuum." By using the same calculation a possibility of restoration of the spontaneously broken chiral symmetry of the vacuum is pointed out.

§ 1. Introduction

Callan, Dashen and Gross have recently demonstrated how the MIT bag model is derived from QCD (quantum chromodynamics). In their work instantons in singular gauge act like four-dimensional color magnetic dipoles. Here the bag formation essentially depends on the choice of this singular gauge.

Another approach to the bag model from QCD is due to Savvidy and others. They have shown that a ferromagnetic state of QCD has lower energy than the "perturbative vacuum." The MIT bag constant B may be given by energy density difference between the perturbative vacuum and the true ground-state vacuum (ferromagnetic vacuum) of QCD, because in the MIT bag model one uses inside the bag the perturbative vacuum, while on the outside we have for the fields an impenetrable vacuum.

Savvidy and others have calculated the one-loop correction to Yang-Mills classical action due to vacuum polarization. They adopted the following expression of the SU(2) classical field:

$$A_{\mu}^a = -\frac{1}{2} F_{\mu\nu}^{YM} x^a_{\mu\nu},$$

where $F_{\mu\nu}^{YM}$ and $n^a$ do not depend on $x$ and $n^a \cdot n^a = 1$. This type of $A_{\mu}^a$, of course, gives the constant field strength:

$$F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g e^{\alpha\beta} A_{\mu}^\beta A_{\nu}^\alpha = F_{\mu\nu}^{YM} n^a.$$  (2)

However, this choice of $A_{\mu}^a$ is apparently Abelian-like; non-abelian nature of Yang-Mills field does not come out here.

In the present paper we propose another choice of $A_{\mu}^a$ for the $SU(2)$ classical gauge field, i.e.,
where $K$ is a constant. This yields a constant field strength of the form

$$F_{ij}^a = gK^a \epsilon_{ij}, \quad (i,j=1,2,3)$$

Contrary to Savvidy et al., the above choice of $A^a_i$ is apparently non-Abelian like. The energy density $E_0$ of the classical gauge field $A^a_i$ is, therefore, given by

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{3} \langle F_{ij}^a \rangle = \frac{3}{2} gK^a = \frac{1}{2} H^2. \quad (H=\sqrt{3gK^a})$$

In § 2 we calculate a quark one-loop contribution to $E_0$ under the very strong external “magnetic” field strength $H$ defined by Eq. (5). A gluon-loop contribution to $E_0$ is shown to be neglected when the number of quark flavor is large. Then we shall show that the true ground-state vacuum has actually lower energy than the “perturbative vacuum.”

In § 3, by using the above calculation we point out a possibility of restoration of the spontaneously broken chiral symmetry of the vacuum. This possibility will be seen at a critical “magnetic” field strength $H_c$. The notion of critical fields above which spontaneously broken symmetries are restored is well known from the Ginzburg-Landau theory of superconductivity. Many authors have discussed that non zero expectation values for certain scalar fields may make a phase transition to a zero value for certain critical temperatures and also for certain critical magnetic field strengths. Our model in § 3 is one of such examples.

§ 2. QCD vacuum

The quark one-loop correction to the Yang-Mills classical action is given by the

$$\Delta Z = \int [d\psi] [d\bar{\psi}] \exp \left[ i \int d^4x \left\{ \bar{\psi} i \gamma^\mu \left( \partial_\mu - ig A_\mu^a \right) \psi - m_{\text{q}} \bar{\psi} \psi \right\} \right]$$

$$= \det \left( 1 + \frac{1}{i \gamma \partial - m_{\text{q}}} g \frac{\gamma^\mu}{2} \tau^a A_\mu^a \right),$$

apart from the normalization factor, where $A_\mu^a$ is the strong classical field (3). This determinant will be calculated by neglecting $1$ in the following way (this approximation is good when $A \ll gK$, $A$ being a cutoff parameter):

$$\Delta Z \approx \det \left( \frac{1}{i \gamma \partial - m_{\text{q}}} g \frac{\gamma^\mu}{2} \tau^a A_\mu^a \right)$$

$$= \exp \left[ i \int d^4x \left\{ -i \int \frac{d^4p}{(2\pi)^4} \ln \det \left( \frac{1}{i \gamma \partial - m_{\text{q}}} g \frac{\gamma^\mu}{2} \tau^a A_\mu^a \right) \right\} \right]$$
\[= \exp\left[i \int d^4x \left\{-i \int \frac{d^4p}{(2\pi)^4} \ln \left[ \det \left( \frac{1}{\gamma - m_q} \right) \det \left( \frac{g_\nu^2 \gamma^\mu A^\mu}{2} \gamma^\nu \right) \right] \right\} \right] \]
\[= \exp\left[i \int d^4x \left\{-i \int \frac{d^4p}{(2\pi)^4} \ln \left[ \frac{9}{(\rho^2 - m_q^2)^4} \left( \frac{gK}{2} \right)^4 \right] \right\} \right] \]
\[= \exp\left[i \int d^4x \Delta L \right]. \tag{7} \]

Hence the \(\Delta L\) is given by
\[\Delta L = -i \int \frac{d^4p}{(2\pi)^4} \sum_\tau \ln \left[ \frac{9}{(\rho^2 - m_q^2)^4} \left( \frac{gK}{2} \right)^4 \right], \tag{8} \]

where the summation over quark flavor \(q\) is necessary if one considers many flavor case. After Wick rotation Eq. (8) can be integrated in the form of
\[\Delta L = \sum_\tau \left[ \frac{A^i - m_q^4}{8\pi^2} \ln \frac{\sqrt{3} g^2 K^2}{4 (A^i + m_q^4)} + \frac{m_q^4}{8\pi^2} \ln \frac{\sqrt{3} g^2 K^2}{4 m_q^4} + \frac{1}{2} \frac{(A^i - m_q^4)^2}{8\pi^2} \right] \tag{9} \]

where \(A\) is a cutoff parameter. In terms of \(H\) defined by Eq. (5) the total energy density \(E\) is now
\[E = E_0 - \Delta L \]
\[= \frac{H^2}{2} - \sum_\tau \left[ \frac{A^i - m_q^4}{8\pi^2} \ln \frac{gH}{4 (A^i + m_q^4)} + \frac{m_q^4}{8\pi^2} \ln \frac{gH}{4 m_q^4} + \frac{1}{2} \frac{(A^i - m_q^4)^2}{8\pi^2} \right]. \tag{10} \]

The minimum for the energy density \(E\) is found when
\[H = H_0 = \frac{\sqrt{N_f} A^i}{2\sqrt{2} \pi}, \tag{11} \]

where \(N_f\) is the number of quark flavor.** Substituting (11) into (10) gives
\[E_{\text{min}} = \frac{H_0^2}{2} \sum_\tau \left[ x_q - (1 - x_q) \ln \frac{c}{1 + x_q} - x_q \ln \frac{c}{x_q} \right], \tag{12} \]

where
\[x_q = m_q^2 \sqrt{N_f} / (\sqrt{8} \pi H_0), \]
\[c = g \sqrt{N_f} / (8 \sqrt{2} \pi). \]

If we assume \(x_q \ll 1\) for simplicity, Eq. (12) is reduced to

** Here we have used the following formulae: \[\int d^4p f(\gamma^\mu A^\mu) \text{Wick rotation} \int d^4p f(-\gamma^\mu), \]
and
\[\int d^4x \ln(ax + b) = \frac{1}{2} \left( x^2 - \frac{b^2}{a} \right) \ln(ax + b) - \frac{1}{2} \left( x^2 - \frac{b}{a} \right). \]

** Equation (11) means \(\sqrt{N_f} A^i = gK\). On the other hand, Eq. (7) is valid when \(A \ll gK\). These two conditions are satisfied if \(1 < g \sqrt{N_f}\) (See Eq. (14)).
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\[ E_{\text{min}} = -H^2 \ln c = -B. \]  \quad (13)

This \( E_{\text{min}} \) is negative, when \( c > 1 \), i.e.,

\[ \frac{g^2}{4\pi} > \frac{32\pi}{N_f}. \]  \quad (14)

Thus we find that the true ground-state vacuum of QCD has lower energy than the “perturbative vacuum” in the strong coupling region and therefore the bag constant \( B \) is approximately given by Eq. (13).

So far we have neglected a gluon-loop contribution to \( E \). A rough estimation of the gluon-loop correction is

\[ Z_{\text{gluon}} \approx \left[ \prod_{x,s,s'} \det \left( \frac{1}{g^2} \gamma^\alpha A^\alpha_s \right) \right]^{-1/4} \]

\[ = \exp \left[ \frac{i}{\Lambda^4} \int d^4x \left\{ -\frac{A^4}{32\pi^2} \ln \left( \frac{g^2K^2}{A^2} \right) \right\} \right], \]  \quad (15)

which is compared to the fermion-loop correction (7), i.e.,

\[ Z_{\text{fermion}} \approx \prod_{\text{flavor}} \det \left( \frac{1}{g^2} \gamma^\alpha \gamma^5 A^\alpha \right) \]

\[ = \exp \left[ \frac{i}{\Lambda^4} \int d^4x \sum_{\text{flavor}} \left\{ A^4 \ln \left( \frac{g^2K^2}{A^2} \right) \right\} \right]. \]  \quad (16)

Therefore, for the large number of \( N_f \), (i.e., \( 8N_f \gg 12 \)), one can neglect the gluon-loop contribution.

Finally it should be noted that our calculation is the calculation in the strong coupling, so that the usual renormalizable perturbative calculation is not applicable here. This is a reason why we introduced the cutoff parameter \( \Lambda \). In the small coupling region, of course, the renormalizable perturbative calculation is applicable by keeping 1 inside Eq. (6). In this case, however, we never find the negative \( E_{\text{min}} \), but \( E_{\text{min}} = 0 \).

§ 3. Restoration of spontaneously broken chiral symmetry

In this section we start with the following Lagrangian density:

\[ \mathcal{L} = -\frac{1}{4} (F^a)^2 + \bar{\psi} i\Gamma^\alpha \left( \partial_\mu - ig \frac{\sigma^a}{2} A^a_\mu \right) \psi + G \bar{\psi} (\sigma + i\tau_3 \pi) \psi + \mathcal{L}' (\sigma, \pi), \]  \quad (17)

where

\[ \mathcal{L}' (\sigma, \pi) = \frac{1}{2} (\partial \pi)^2 + \frac{1}{2} (\partial \sigma)^2 + \frac{1}{2} \mu (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2. \]  \quad (18)
The vacuum expectation values of $\sigma$ and $\pi$ are chosen such that

$$\langle \sigma \rangle_s = \sqrt{\frac{\mu^2}{\lambda}} \quad \text{and} \quad \langle \pi \rangle_s = 0, \quad (19)$$

This means that the original chiral symmetry is spontaneously broken down.

Now we shall show that there is a possibility of recovering the above spontaneous-symmetry breaking. For this purpose we choose $A_{\mu}^0$ in (17) as the very strong external field of Eq. (3). The $\sigma$ and $\pi$ fields are also chosen to be constants ($\sigma_0 \neq 0$, $\pi = 0$). In the same way as the preceding section the fermion-loop correction to the classical action is quite the same as Eq. (9), if $m_q$ is replaced by $G\sigma$. Hence the total energy density $E$ is now given by

$$E = \frac{H^2}{2} - \sum \left[ \frac{A^\mu - G\sigma^4}{8\pi^2} \ln \left( \frac{gH}{4(A^\mu + G^2\sigma^4)} \right) + \frac{G^2\sigma^4}{8\pi^2} \ln \left( \frac{gH}{4G^2\sigma^4} \right) ight] + \frac{\lambda}{4} \sigma^4 - \frac{\mu^2}{2} \sigma^2. \quad (20)$$

The minimum for $E$ is found when

$$H = H_s = \sqrt{N_f A^4}$$

and

$$\sigma = 0 \quad \text{or} \quad \sigma = \sigma_0 \neq 0, \quad (22)$$

where $\sigma_0$ is satisfied by the equation

$$\ln(1 + x) = x \left(1 - \frac{\mu^2}{cA^2}\right) + \frac{\lambda}{cG^2}, \quad x = \frac{A^4}{G^2\sigma_s^4}, \quad c = \frac{N_f G^2}{2\pi^2}. \quad (23)$$

The condition that $\sigma = \sigma_s$ be an unstable point is the following:

$$\frac{\partial^2 E}{\partial \sigma^2} \bigg|_{\sigma = \sigma_s} < 0,$$

which is equivalent to

$$cA^4 - \mu^2 > \frac{\mu^2}{x}. \quad (24)$$

In this case one can see that $\sigma = 0$ is a stable point, because

$$\frac{\partial^2 E}{\partial \sigma^2} \bigg|_{\sigma = 0} = cA^4 - \mu^2 > 0. \quad (25)$$

Conversely, when $cA^4 - \mu^2 < 0$, $\sigma = 0$ is an unstable point, while $\sigma = \sigma_s$ is a stable point, because $cA^4 - \mu^2 < \frac{\mu^2}{x}$ and hence $(\partial^2 E/\partial \sigma^2)_{\sigma = \sigma_s} > 0$.

* Here we shall regard the "potential" of $\sigma$ and $\pi$ as the "effective potential".
This phenomenon is very exciting for us. When \( cA^2 < \mu^2 \) (or equivalently \( H < H_c = \pi \mu^2 / G^2 \sqrt{2N_f} \)), spontaneous-symmetry breaking actually occurs \( (\delta_i \neq 0, \pi_0 = 0) \), whereas, when \( cA^2 > \mu^2 \) \( (H > H_c) \), the symmetry breaking is recovered \( (\delta_i = 0, \pi_0 = 0) \).

The Lagrangian (17) is phenomenological rather than the QCD lagrangian. Here we have, in addition to the quark \( \langle \phi \rangle \) and gluon \( (A_{\mu}^a) \) fields that live inside the bag, the \( \pi \) and the \( \sigma \) fields that live outside the bag. The Lagrangian (17) should be considered to describe the system outside the bag, where the quark mass \( m_q = G \pi \) seems to be large or infinite when the chiral symmetry is breaking. However, the \( \pi \) and \( \sigma \) fields couple to the quark on the surface of the bag.

In this section we have pointed out the possibility that the chiral symmetry breaking is recovered when \( H > H_c \), and in this case the quark mass outside the bag is zero, whereas the spontaneous-chiral symmetry breaking occurs when \( H < H_c \), where \( H_c \) is the critical “magnetic” field strength given by \( H_c = \pi \mu^2 / G^2 \sqrt{2N_f} \).

References