On the Solution of the Renormalization Group Equation of Quantum Electrodynamics

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The renormalization group equations of QED are investigated. They are solved without recourse to the Lie equation of the renormalization group. Various integral representations and series expansions for the invariant charge function are derived.

§ 1. Introduction

The renormalization group equation (RGE) is now one of the central tools to analyze the asymptotic behaviours of Green's functions. Since the idea of the renormalization group was proposed by Stueckelberg and Petermann, a vast amount of analysis has been made on the RGE. The most popular procedure to attack the RGE is to write down the Lie equation of the renormalization group. The Lie equation is a partial differential equation whose general solution contains arbitrary functions. Gell-Mann and Low, through the investigation of the Lie equation, pointed out an interesting but perplexing possibility that the bare coupling constant might be independent of the physical coupling constant. Jouvet made use of the integrated form of the Lie equation and clarified the meaning of the RGE. In his language, the RGE is nothing but the $\alpha$-invertibility of the intermediate charge.

The Lie equation, however, will not be the sole tactics to get the solution of the RGE. The purpose of the present paper is to discuss how to solve the RGE without reckoning upon the Lie equation. We show that the RGE for the invariant charge of QED can be solved by introducing an arbitrary function and its inverse function. The dependence of the solution on the introduced arbitrary function seems to be more transparent than those described in standard textbooks. The RGE's for the electron propagator and the vertex function are also solved without recourse to the Lie equation. Whole the procedure is simple enough.

In § 2, we solve the RGE for the invariant charge function. The solution obtained can be expressed as a contour integral in the complex coupling constant plane, which we discuss in § 3. We give arguments, in § 4, on the coefficients of the perturbative expansion of the invariant charge function. Sum rules for these coefficients are derived. In § 5 we solve the RGE's for the electron propagator and the vertex function. Section 6 is devoted to discussions. An appendix is attached to show the details of some calculations.
§ 2. The solution of the RGE for the invariant charge function

We consider the photon propagator in QED. We work, throughout this paper, in the Landau gauge because, in this gauge, the gauge parameter can be fixed to zero under the renormalization group operation. The renormalized photon propagator $D_{\mu}(k)$ normalized at $k^2 = 0$ is given by

$$D_{\mu}(k) = -\frac{1}{k^2} \left( g_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right) \epsilon_{k} \epsilon_{k}^* d \left( \frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e \right), \quad (2.1)$$

where $e$ and $m$ are the mass and the charge of an electron, respectively. The quantity $\epsilon_{k} d \left( \frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, e \right)$ is called the invariant charge and is invariant under the change of the normalization momentum $\lambda$. If we define the invariant charge function $\tilde{d}(\xi, \eta, \zeta)$ by

$$\tilde{d}(\xi, \eta, \zeta) = \xi d \left( \frac{\eta}{\xi}, \eta, \zeta \right), \quad (2.2)$$

the RGE and the normalization condition for $\tilde{d}(\xi, \eta, \zeta)$ are given by

$$\tilde{d}(\xi, \eta, \zeta) = \tilde{d}(\xi, \sigma, \tilde{d}(\sigma, \eta, \zeta)) \quad (2.3)$$

and

$$\tilde{d}(\eta, \eta, \zeta) = \zeta, \quad (2.4)$$

respectively, where $\sigma$ is an arbitrary parameter. We shall obtain the general form of $\tilde{d}$ determined by the functional equations (2.3) and (2.4). Equation (2.3) means that $\tilde{d}(\xi, \sigma, \tilde{d}(\sigma, \eta, \zeta))$ is independent of $\sigma$ so that the second variable $\sigma$ and the third variable $\tilde{d}(\sigma, \eta, \zeta)$ of $\tilde{d}(\xi, \sigma, \tilde{d}(\sigma, \eta, \zeta))$ should be put together to cancel the $\sigma$-dependence. From this observation, we have

$$\tilde{d}(\xi, \eta, \zeta) = f(\xi, g(\eta, \zeta)), \quad (2.5)$$

where $f$ and $g$ are, at this stage, arbitrary functions of two variables. A restriction on $f$ and $g$ is obtained from (2.4):

$$f(\eta, g(\eta, \zeta)) = \zeta. \quad (2.6)$$

This equation determines $f$ in terms of $g$. Functions $f(\eta, \zeta)$ and $g(\eta, \zeta)$ are inverse in $\zeta$ to each other. We obtain from (2.6) that

$$g(\eta, f(\eta, \zeta)) = \zeta. \quad (2.7)$$

Conversely if $\tilde{d}(\xi, \eta, \zeta)$ is given by (2.5) with $f$ and $g$ satisfying (2.7), we have

$$\tilde{d}(\xi, \sigma, \tilde{d}(\sigma, \eta, \zeta)) = f(\xi, g(\sigma, f(\sigma, g(\eta, \zeta))))$$

$$= f(\xi, g(\eta, \zeta))$$

$$= \tilde{d}(\xi, \eta, \zeta).$$
So (2·5) with (2·6) gives the general solution of (2·3) and (2·4).

The above procedure can be stated differently in the following manner. Introduce an arbitrary function $g(\eta, \zeta)$ of two variables. Set up an equation

$$g(\xi, z) = g(\eta, \zeta).$$

Obtain the solution $z$ of (2·8) such that

$$\lim_{t \to \infty} z = \zeta.$$ (2·9)

Then $z = \bar{d}(\xi, \eta, \zeta)$ solves the RGE.

In closing this section, we make several comments.

(i) The first comment is that the correspondence between $\bar{d}$ and $g$ is not one-to-one at all. For example, the two choices $g(\eta, \zeta) = \eta k(\zeta)$ and $g(\eta, \zeta) = \ln \eta + 1(\zeta)$ lead to the same form of $\bar{d}$:

$$\bar{d}(\xi, \eta, \zeta) = F^{-1}\left(\ln \left(\frac{\xi}{\eta}\right) + F(\zeta)\right)$$ (2·10)

which is the solution obtained by Gell-Mann and Low.\(^9\)

(ii) Solutions of RGE’s similar to (2·5) and (2·6) were formerly discussed by Astaud.\(^7\) He discussed several subgroups of the extended renormalization group of QED. He solved the RGE’s of subgroups $G_{\text{auto}}(Z_0)$, $G_{\text{auto}}(Z_0 Z_1 Z_2)$, $G^{*}(Z_0, \eta)$ and $G_{\text{auto, a}}(\eta)$, while we, in this paper, have solved the RGE of $G_{\text{re}}(Z_0)$.

(iii) The procedure discussed around (2·8) and (2·9) might be regarded as the manifestation of the relation between the RGE and Jouvet’s $\alpha$-invertibility.\(^8\)

(iv) The method presented in standard textbooks\(^3\) is described in the following way. One converts the RGE (2·3) to the Lie equation

$$\frac{\partial}{\partial x} z d(x, y, z) = zd(x, y, z) \varphi\left(\frac{y}{x}, zd(x, y, z)\right),$$ (2·11)

where $\varphi$ is related to $d$ by

$$\varphi(y, z) = \left[\frac{\partial}{\partial x} d(x, y, z)\right]_{x=1}$$ (2·12)

and can be chosen arbitrarily. By specifying the function $\varphi(y, z)$ and solving the partial differential equation (2·12), one arrives at the solution of the RGE. The relation between the choice of $\varphi$ and the resulting form of $d$ is not always transparent. We hope that in some cases our procedure (2·5)~(2·7), (2·8) and (2·9) or integral representations and series expansions for $\bar{d}(\xi, \eta, \zeta)$ given in the following sections will be more tractable and convenient than the Lie equation.

§ 3. Integral representations of the invariant charge function

The solution of the RGE for the invariant charge function was given in § 2.
by \( \tilde{d}(\xi, \eta, \zeta) = f(\xi, g(\eta, \zeta)) \) with \( g \) being arbitrary and \( f \) being determined by \( f(\eta, g(\eta, \zeta)) = \zeta \). In this section, we shall seek a method to construct \( \tilde{d}(\xi, \eta, \zeta) \) from given \( g(\eta, \zeta) \) without explicitly obtaining \( f(\eta, \zeta) \).

We assume that \( g(\xi, z) \), \( \xi \) being fixed, is analytic and single-valued in \( z \) in a simply connected domain \( D \) of the complex \( z \) plane. We also assume that the equation \( s(\xi, z) = g(\eta, \zeta) \) for \( z \) has a unique solution \( z_0(\xi, \eta, \zeta) \) in \( D \) and that \( z_0(\xi, \eta, \zeta) \) tends to \( \zeta \) as \( \xi \) tends to \( \eta \). The closed contour \( C \) is assumed to be in \( D \) and to encircle \( z_0 \) anti-clockwise.\(^1\)

Then we can easily realize that \( d(\xi, \eta, \zeta) \) defined by

\[
\tilde{d}(\xi, \eta, \zeta) = \frac{1}{2\pi i} \oint_C g(\xi, z) \frac{\partial g(\eta, \zeta)}{\partial \zeta} \frac{dz}{z} \quad (3.1)
\]

is equal to that given by (2.5) and (2.6). Taking the derivative of \( \tilde{d}(\xi, \eta, \zeta) \) with respect to \( \eta \) and integrating by part in \( z \), we have

\[
\partial \tilde{d}(\xi, \eta, \zeta) \over \partial \eta = \frac{1}{2\pi i} \oint_C g(\xi, z) - g(\eta, \zeta) \frac{\partial g(\eta, \zeta)}{\partial \eta} \frac{dz}{z} \quad (3.2)
\]

If we integrate the above equation in \( \eta \) and take account of the normalization condition \( \tilde{d}(\eta, \eta, \zeta) = \zeta \), we obtain an integral representation

\[
\tilde{d}(\xi, \eta, \zeta) = \zeta + \frac{1}{2\pi i} \oint_{C_1} \ln \left| \frac{g(\xi, z) - g(\xi, \zeta)}{g(\xi, z) - g(\eta, \zeta)} \right| \frac{dz}{z}, \quad (3.3)
\]

where the contour \( C_1 \) encircles two points \( \zeta \) and \( z_0 \). In writing down (3.3), we have assumed that \( g(\xi, z) \) is analytic inside and on \( C_1 \) and that \( g(\xi, z) - g(\xi, \zeta) \) and \( g(\xi, z) - g(\eta, \zeta) \) vanish only at \( z = \zeta \) and \( z = z_0 \) inside \( C_1 \), respectively. From (3.3), we have

\[
\tilde{d}(\xi, \eta, \zeta) = \zeta + \int_{\gamma(\xi, \zeta)} dt K(\xi, t), \quad (3.4)
\]

where \( K(\xi, t) \) is given by

\[
K(\xi, t) = \frac{1}{2\pi i} \oint_{C_2} \frac{dz}{z-t-g(\xi, z')} . \quad (3.5)
\]

The contour \( C_2 \) in (3.5) encircles \( z' \) where \( z' \) is a solution of \( t-g(\xi, z')=0 \) and

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\(^1\) Although many discussions have been given on the analytic properties of Green’s functions around the origin of the complex coupling constant plane, no definite and rigorous conclusion has been yet attained (F. J. Dyson, Phys. Rev. 85 (1952), 631. P. J. Redmond, Phys. Rev. 112 (1958), 1404. G. Parisi, Nuovo Cim. 21A (1974), 179, etc.). The investigation of the analyticity apart from the origin will be much more complicated. It is therefore beyond the present stage of knowledge to specify \( D \). We expect and assume the existence of the analytic domain \( D \). The rapidly developing analysis on Borel functions might give some lights on this problem (G. ’t Hooft, Erice Lectures (1977). G. Parisi, Phys. Letters 76B (1978), 68. N. N. Khuri, Phys. Letters 82B (1979), 83, etc.).
The function $g(\xi, z)$ is assumed to be analytic in $z$ in the domain considered. The integrands of (3.1), (3.3) and (3.5) are assumed to be analytic except at $z_0$, $z_0'$ and $z'$ in the respective domains.

(a) The contour $C$ encircles the zero $z_0(\xi, \eta, \zeta)$ of $g(\xi, z) - g(\eta, \zeta)$ anti-clockwise.

(b) The contour $C_1$ encircles two points $z_0$ and $\zeta$ anti-clockwise.

(c) The contour $C_2$ encircles the zero $z'$ of $t - g(\xi, z)$ anti-clockwise.

Contours $C$, $C_1$, and $C_2$ depend on $\xi$, $\eta$, and $\zeta$ in that they encircle $z_0$, $\zeta$, and $z'$, respectively. The integrals in (3.1), (3.3) and (3.5), however, do not change under continuous deformations of these contours unless they cross other singularities of integrands than the above ones. If we evaluate these integrals by making use of the residue theorem around $z = z_0$, the procedures become meaningless because we have to obtain $z_0(\xi, \eta, \zeta) = d(\xi, \eta, \zeta)$ in order to get itself. The integral representations (3.1), (3.3) and (3.5) are meaningful when we are able to evaluate these integrals by other methods than the residue theorem around $z = z_0$. Which of (3.1), (3.3) and (3.4) is most convenient depends on the form of given $g(\eta, \zeta)$.

As an application of (3.3), we shall derive an expression of $\tilde{d}(\xi, \eta, \zeta)$ which is convenient when $\xi$ is close to $\eta$. We have

$$d(\xi, \eta, \zeta) = \xi - \frac{1}{2\pi i} \oint_{C_1} \ln \left[ 1 + \frac{g(\xi, \zeta) - g(\eta, \zeta)}{g(\xi, z) - g(\xi, \zeta)} \right] dz$$

$$= \xi - \frac{1}{2\pi i} \oint_{C_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ g(\xi, \zeta) - g(\eta, \zeta) \right]^n dz,$$

and hence

$$d(\xi, \eta, \zeta) = \xi + \sum_{n=1}^{\infty} \frac{1}{n} \left[ g(\eta, \zeta) - g(\xi, \zeta) \right]^{n} \text{Res}_{\xi} \{ g(\xi, z) - g(\xi, \zeta) \}^{-n}, \quad (3.6)$$

where $\text{Res}_{\xi} f(z)$ denotes the residue of $f(z)$ at $z = \xi$. Note that we have made use of the residue theorem not around $z = z_0$ but around $z = \xi$. In Gell-Mann and Low's approximation where we put $g(\eta, \zeta) = \eta k(\zeta)$ or $g(\eta, \zeta) = \ln \eta + 1(\zeta)$, we have

$$d(\xi, \eta, \zeta) = \xi + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\eta}{\xi} - 1 \right)^n \text{Res}_{\xi} \left[ \frac{k(\zeta)}{k(\zeta) - 1} \right]^{-n}.$$
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or

\[ \bar{d} (\xi, \eta, \zeta) = \zeta + \sum_{n=1}^{\infty} \frac{1}{n} \left[ \ln \left( \frac{\eta}{\zeta} \right) \right]^n \text{Res} \{1(\zeta) - 1(\zeta)\}^{-n}, \]

respectively.

§ 4. The perturbative expansion of the invariant charge function

The perturbative calculation gives that

\[ \bar{d} (\xi, \eta, \epsilon) = \epsilon + \epsilon^4 \{h(\eta) - h(\xi)\} + \cdots, \]

where \( h(\xi) \) is given by

\[ h(\xi) = -\frac{1}{2\pi^2} \int_{\epsilon}^{1} \frac{\alpha(1-\alpha)}{\alpha(1-\alpha) - \xi} d\alpha. \]

The higher order calculation is terribly tedious. We shall investigate how the RGE helps us in the calculation of higher order radiative corrections. We expand \( \bar{d} (\xi, \eta, \zeta) \) as

\[ \bar{d} (\xi, \eta, \zeta) = \sum_{\delta=1}^{\infty} C_\delta (\xi, \eta) \zeta^\delta. \]

Functions \( g \) and \( f \) which give \( \bar{d} \) through \( \bar{d} (\xi, \eta, \zeta) = f(\xi, g(\eta, \zeta)) \) are expanded in the following way:

\[ g(\eta, \zeta) = \sum_{x=b}^{\infty} a_x(\eta) \zeta^x \]

and

\[ f(\eta, \zeta) = \sum_{x=b}^{\infty} b_x(\eta) \zeta^x. \]

At first we have to determine \( b_x(\eta) \)’s from \( a_x(\eta) \)’s. This is done by making use of the relation \( f(\eta, g(\eta, \zeta)) = \zeta \). Recalling that the addition of a constant to \( g(\eta, \zeta) \) induces a subtraction of the same constant from \( f(\eta, \zeta) \) and that the multiplication of \( g(\eta, \zeta) \) by a constant induces the division of \( f(\eta, \zeta) \) by that constant, we can, without loss of generality, put

\[ a_0(\eta) = b_0(\xi) = 0 \]

and

\[ a_1(\eta) = b_1(\xi) = 1. \]

Note that the property \( C_1(\xi, \eta) = 1 \) cannot be ensured unless we put \( a_1 \) and \( b_1 \) to be constants such that \( a_0 b_1 = 1 \). With the aid of (4·6), (4·7) and the multinomial
we obtain the expression

$$b_n(q) = \frac{1}{n!} \sum_{m_j = n-1} (-1)^{\sum_j m_j} (n + \sum_j m_j - 1)! \prod_j m_j! \prod_j a_{j+1}(q)^{m_j}, \quad (4.9)$$

where the summation is taken over all the possible configurations of non-negative integers $m_1, m_2, m_3, \ldots$ such that $\sum_j m_j = n - 1$. Coefficients $b_n(q)$'s are given explicitly by

$$b_0(q) = -a_0(q), \quad b_1(q) = 2\{a_2(q)\} - a_0(q), \quad b_2(q) = -5\{a_2(q)\} + 5a_2(q)a_0(q) - a_1(q),$$

and so on. Now we have

$$C_N(\xi, \eta) = \sum_{\sum_i m_i = N-1} \frac{(\sum_i (j+1) m_i)!}{(N-\sum_i (i+1) l_i)!} \prod_i \frac{a_{i+1}(q)^{m_i}}{l_i!} \prod_j \{ -a_{j+1}(\xi) \}^{m_j}, \quad (4.11)$$

from which we obtain a compact expression

$$C_N(\xi, \eta) = \sum_{\sum_i m_i = N-1} \left[ \left( \frac{\partial}{\partial x} \right)^{\sum_i -1} \prod_i \frac{a_{i+1}(q) - x^{i+1} a_{i+1}(\xi) \eta^{x_i}}{l_i!} \right]_{x=1} \quad (4.12)$$

for $N = 2, 3, 4, \ldots$. See Appendix A for details of the derivation of (4.11) and (4.12). Thus $C_N(\xi, \eta)$ is explicitly given through functions $a_0, a_1, a_2, \ldots$ by

$$C_1(\xi, \eta) = 1,$$
$$C_2(\xi, \eta) = a_1(\eta) - a_2(\xi),$$
$$C_3(\xi, \eta) = -2a_2(\xi) \{ a_2(\eta) - a_2(\xi) \} + a_0(\eta) - a_1(\xi), \quad (4.13)$$
$$C_4(\xi, \eta) = -a_2(\xi) \{ a_3(\eta) - a_3(\xi) \} + a_2(\xi) \{ a_2(\eta) - 5a_2(\xi) \}$$
$$-2a_2(\eta) \{ a_2(\eta) - a_2(\xi) \} - 3a_2(\xi) \{ a_2(\eta) - a_2(\xi) \} + a_4(\eta) - a_4(\xi),$$

and so on. They satisfy the sum rules

$$C_N(\xi, \eta) = \sum_{0 \leq M < N} C_{N-M}(\xi, \sigma) \sum_{\sum_i m_i = M} \frac{(N-M)!}{(N-M-\sum_i m_i)!} \prod_i \{ a_{i+1}(\sigma) \}^{m_i}, \quad (4.14)$$

for $N = 2, 3, 4, \ldots$ and the condition $C_1(\xi, \eta) = 1$. Sum rules (4.14) are derived from (2.3) and
After some manipulations, we obtain from (4.3), (4.6), (4.7) and (4.12) the following expression for \( \hat{d} (\xi, \eta, \zeta) \):

\[
\hat{d} (\xi, \eta, \zeta) = \xi + \sum_{n=0}^{\infty} \frac{1}{(m+1)!} \left[ \left( \frac{\partial}{\partial z} \right)^n \left\{ a_n (\eta) \xi^n - \sum_{n=1}^{\infty} a_n (\zeta) z^n \right\} \right]_{z=\xi}
\]

or

\[
\hat{d} (\xi, \eta, \zeta) = \xi + \sum_{n=0}^{\infty} \frac{1}{(m+1)!} \left[ \left( \frac{\partial}{\partial z} \right)^n \{ g (\eta, \zeta) - g (\xi, z) + z - \zeta \} \right]_{z=\xi}
\]

The formula (4.16) gives us a bridge between the perturbative expansion and the integral representation of \( \hat{d} (\xi, \eta, \zeta) \) since we obtain from (4.16) that

\[
\frac{\partial \hat{d} (\xi, \eta, \zeta)}{\partial \eta} \frac{\partial g (\eta, \zeta)}{\partial \eta} = \sum_{n=0}^{\infty} \left[ \frac{1}{m!} \left( \frac{\partial}{\partial z} \right)^n \{ g (\eta, \zeta) - g (\xi, z) + z - \zeta \} \right]_{z=\xi}
\]

\[
= \sum_{n=0}^{\infty} 2\pi i \int_{C_i} \frac{g (\eta, \zeta) - g (\xi, z) + z - \zeta}{(z - \zeta)^{n+1}} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{C_i} \frac{dz}{(z - \zeta) - g (\eta, \zeta) - g (\xi, z) + z - \zeta}
\]

which is equivalent to (3.2).

\[\text{§ 5. The solution of the RGE's for the electron propagator and the vertex function}\]

We now turn our attention to the electron propagator and the vertex function of QED. The RGE's for these quantities are typically given by

\[
S (x, y, e^2) = S (x/t, y/t, e^2 t (t, y, e^2))
\]

\[
S (t, y, e^2) = S (1, y/t, e^2 t (t, y, e^2))
\]

and

\[
\Gamma (x, y, z, u, e^2) = \Gamma (x/t, y/t, z/t, u/t, e^2 t (t, u, e^2))
\]

\[
\Gamma (t, y, z, u, e^2) = \Gamma (1, y/t, z/t, u/t, e^2 t (t, u, e^2))
\]

where \( S \) and \( \Gamma \) are quantities related to the electron propagator and the vertex function, respectively. We are not worried about the dependence of \( S \) and \( \Gamma \) on
the gauge parameter since we are working in the Landau gauge. It can be readily read off that the structure of (5·2) is quite similar to that of (5·1). If we define \( \hat{S} \) and \( \hat{F} \) by

\[
\hat{S}(\xi, \eta, \zeta) = S\left(\frac{\eta}{\xi}, \eta, \zeta\right)
\]

and

\[
\hat{F}(\xi, \eta, \zeta; \alpha, \beta) = F\left(\frac{\eta}{\xi}, \frac{\eta}{\alpha}, \eta, \xi, \zeta\right),
\]

we obtain

\[
\hat{S}(\xi, \eta, \zeta) = S(\xi, \sigma, \tilde{d}(\sigma, \eta, \zeta))
\]

and

\[
\hat{F}(\xi, \eta, \zeta; \alpha, \beta) = F(\sigma, \sigma, \tilde{d}(\sigma, \eta, \zeta; \alpha, \beta)).
\]

Except for the dependence on the extra variables \( \alpha \) and \( \beta \), (5·6) is identical to (5·5). Recalling the result of previous sections that \( \tilde{d}(\sigma, \eta, \zeta) \) be of the form \( f(\sigma, g(\eta, \zeta)) \), we obtain from (5·5) that

\[
\hat{S}(\xi, \eta, \zeta) = P(\xi, \sigma, g(\eta, \zeta))
\]

with

\[
P(\sigma, \sigma, g(\eta, \zeta)) = 1.
\]

where \( P \) is an unknown function of three variables. If we define \( \hat{S} \) by

\[
\hat{S}(\xi, \eta, g(\eta, \zeta)) = S(\xi, \eta, \zeta),
\]

(5·7) becomes

\[
\hat{S}(\xi, \eta, g) = P(\xi, \sigma, g).
\]

From (5·10), we see that \( \hat{S} \) should be of the form

\[
\hat{S}(\xi, \eta, g) = h(\eta) R(\xi, g).
\]

Thus we get the general solution of the RGE (5·5):

\[
\hat{S}(\xi, \eta, \zeta) = h(\eta) R(\xi, g(\eta, \zeta)).
\]

where \( h \) and \( R \) are arbitrary functions of one and two variables, respectively. Conversely if we define \( \hat{S} \) by (5·12), we have
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\[ S(\xi, \sigma, \tilde{d}(\sigma, \gamma, \zeta)) = h(\sigma) R(\xi, g(\sigma, f(\sigma, g(\gamma, \zeta)))) \]
\[ \tilde{S}(\sigma, \sigma, \tilde{d}(\sigma, \gamma, \zeta)) = h(\sigma) R(\sigma, g(\sigma, f(\sigma, g(\gamma, \zeta)))) \]
\[ = \frac{R(\xi, g(\gamma, \zeta))}{R(\sigma, g(\gamma, \zeta))} \]
\[ = \frac{h(\gamma) R(\xi, g(\gamma, \zeta))}{h(\gamma) R(\sigma, g(\gamma, \zeta))} \]
\[ = \frac{\tilde{S}(\xi, \gamma, \zeta)}{\tilde{S}(\sigma, \gamma, \zeta)} \]

showing that (5·12) is actually the solution of (5·5). If we made use of

\[ \tilde{S}(\xi, \gamma, \zeta) = S(\xi, \gamma, \zeta) \] (5·13)

instead of \( \tilde{S}(\xi, \gamma, g(\gamma, \zeta)) \), we should have obtained the solution of the form of

\[ \tilde{S}(\xi, \gamma, \zeta) = j(\zeta) T(\xi, g(\gamma, \zeta)) \] (5·14)

with \( j \) and \( T \) being arbitrary functions. Of course, we can regard both of (5·12) and (5·14) as the solution of the RGE (5·5).

Now it is clear that the solution of the RGE (5·6) is given by

\[ \tilde{F}(\xi, \gamma, \zeta; \alpha, \beta) = a(\gamma; \alpha, \beta) C(\zeta, g(\gamma, \zeta); \alpha, \beta) \] (5·15)

or

\[ \tilde{F}(\xi, \gamma, \zeta; \alpha, \beta) = b(\zeta; \alpha, \beta) D(\xi, g(\gamma, \zeta); \alpha, \beta) \] (5·16)

where \( a, b, C \) and \( D \) are arbitrary functions. We have thus solved all the RGE's of QED in the Landau gauge.

§ 6. Discussion

We have mainly investigated the invariant charge function \( d(\xi, \gamma, \zeta) \) of QED. The RGE for \( d(\xi, \gamma, \zeta) \) was solved by introducing an arbitrary function and its inverse function. Our procedure to solve the RGE of Landau gauge QED applies for other gauges because the RGE for the transverse part of the photon propagator is preserved. Our method works also for other field theories than QED. When the theory contains more than one coupling constants, we have to generalize the definition of an inverse function. Two sets of functions \( f_1, f_2, \ldots, f_n \) and \( g_1, g_2, \ldots, g_n \) of \( n \) variables are defined to be inverse to each other if they satisfy

\[ f_i(g_1(x_1, x_2, \ldots, x_n), g_2(x_1, x_2, \ldots, x_n), \ldots, g_n(x_1, x_2, \ldots, x_n)) = x_i \]
\[ i = 1, 2, \ldots, n \] (6·1)

The RGE of a field theory with \( n \) coupling constants would be solved by introduc-
ing an arbitrary set of functions \( g_1, g_2, \ldots, g_n \) and its inverse set \( f_1, f_2, \ldots, f_n \). As an example, we briefly consider the two charge meson-nucleon theory which involves \( g_1^T \phi \) (invariant nuclear charge function) and \( h_1 \phi \) (invariant meson charge function). The RGE’s for \( \phi \) and \( \psi \) are given by:

\[
\psi(x, y, Y, g^2, h) = \psi(t, x, Y, g^2, h) \psi \left( \frac{x}{t}, \frac{Y}{t}, g_1^2, h_1 \right),
\]

\[
\phi(x, y, Y, g_2^2, h) = \phi(t, x, Y, g_2^2, h) \phi \left( \frac{x}{t}, \frac{Y}{t}, g_1^2, h_1 \right)
\]

with

\[
g_1^2 = g_1^T \phi(t, y, Y, g^2, h) \quad \text{and} \quad h_1 = h_1 \phi(t, y, Y, g^2, h).
\]

If we define \( \tilde{\phi} \) and \( \tilde{\psi} \) by

\[
g_2^T \phi(x, y, Y, g^2, h) = \tilde{\phi} \left( \frac{Y}{x}, \frac{Y}{y}, Y; g^2, h \right)
\]

and

\[
h_2 \psi(x, y, Y, g_2^2, h) = \tilde{\psi} \left( \frac{Y}{x}, \frac{Y}{y}, Y; g_2^2, h \right),
\]

we have

\[
\tilde{\phi}(\xi, \eta, \zeta; g^2, h) = \tilde{\phi}(\xi, \eta, \sigma; \tilde{\psi}(\sigma, \eta, \zeta; g_1^2, h), \tilde{\phi}(\sigma, \eta, \zeta; g_2^2, h))
\]

and

\[
\tilde{\psi}(\xi, \eta, \zeta; g_1^2, h) = \tilde{\psi}(\xi, \eta, \sigma; \tilde{\psi}(\sigma, \eta, \zeta; g^2, h), \tilde{\psi}(\sigma, \eta, \zeta; g^2, h)).
\]

It is easy to understand that

\[
\tilde{\phi}(\xi, \eta, \zeta; g^2, h) = f_1(\xi, \eta; g_1(\zeta, \eta; g^2, h), g_2(\zeta, \eta; g^2, h))
\]

and

\[
\tilde{\psi}(\xi, \eta, \zeta; g_1^2, h) = f_2(\xi, \eta; g_1(\zeta, \eta; g_2^2, h), g_2(\zeta, \eta; g_2^2, h))
\]

are solutions of (6.4), where \( \{g_i(\zeta, \eta; g^2, h), i = 1, 2\} \) is a set of two arbitrary functions and \( \{f_i(\zeta, \eta; g^2, h), i = 1, 2\} \) is its inverse set in \( g^2 \) and \( h \).

The integral representation (3.3) for \( \tilde{\alpha}(\xi, \eta, \zeta) \) is a consequence of the renormalizability of QED. It alone might not be so helpful to make physical predictions unless we have some principles to restrict \( g(\eta, \zeta) \). The situation could be compared with that of dispersion relations for Green’s functions or scattering amplitudes. Dispersion relations are consequences of very general principles and contain unknown spectral functions. They are not so powerful unless they are supplemented by the unitarity condition. The unitarity condition determines spectral func-
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tions. In our problem to extract physical predictions from (3.3), we have yet to seek a principle like the unitarity condition in the dispersion approach.

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Appendix A

We derive here the expressions (4.11) and (4.12) for $C_N(\xi, \eta)$. From (2.5), (4.4) and (4.5), we have

$$d(\xi, \eta, \zeta) = \sum_{n=0}^{\infty} b_n(\xi) \left\{ \sum_{m=0}^{n} a_m(\eta) \zeta^m \right\}^*.$$  \hspace{1cm} (A.1)

By taking (4.6) \sim (4.9) into account, we obtain

$$d(\xi, \eta, \zeta) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l_1, l_2, \ldots, l_n} (-1)^{\sum_{j=1}^{n} m_j} \prod_{j=1}^{n} \frac{(n + \sum_{j=1}^{n} m_j - 1)!}{m_j!} \left\{ a_{1,1}(\xi) \right\}^{n} \times \prod_{j=1}^{n} \left\{ a_{1,1}(\eta) \right\}^{l_j} \left\{ a_{1,1}(\zeta) \right\}^{m_j}.$$  \hspace{1cm} (A.2)

Putting $N = n + \sum_i l_i$ in (A.2), we get

$$d(\xi, \eta, \zeta) = \sum_{N=0}^{\infty} \left[ \sum_{l_1, l_2, \ldots, l_N} \frac{(N - \sum_{i=1}^{N} (i - 1) l_i)!}{l_i!} \prod_{i=1}^{N} \left\{ a_{1,1}(\xi) \right\}^{l_i} \left\{ a_{1,1}(\eta) \right\}^{m_i} \left\{ a_{1,1}(\zeta) \right\}^{m_i} \right]^{*}.$$  \hspace{1cm} (A.3)

Comparing (A.3) with (4.3), we obtain (4.11). To convert (4.11) to (4.12), we put $n = l_i + m_i$ in (A.11). Then we have

$$C_N(\xi, \eta) = \sum_{N=0}^{\infty} \left[ \sum_{l_1, l_2, \ldots, l_N} \frac{(N - \sum_{i=1}^{N} (i - 1) l_i)!}{l_i!} \prod_{i=1}^{N} \left\{ a_{1,1}(\xi) \right\}^{l_i} \left\{ a_{1,1}(\eta) \right\}^{m_i} \left\{ a_{1,1}(\zeta) \right\}^{m_i} \right]^{*}.$$  \hspace{1cm} (A.4)

If we make use of the identity

$$\left[ \left( \frac{d}{dx} \right)^{p-q} x^p (1 - \delta x^n)^* \right]_{x=1} = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{m!} \frac{(p+km)!}{(q+km)!} \delta^n,$$  \hspace{1cm} (A.5)

$p, q, k, n$; non-negative integers,

$p \geq q$.
we have

\[ C_N (\xi, \eta) = \sum_{i=0}^{N-1} \prod_{i} \left\{ \frac{a_{i+1} (\xi)}{n_i !} \right\} \prod_{r=1}^{n_i} \left( \frac{-a_{i+1} (\xi)}{a_i (\eta)} \right) \left[ \frac{(\partial x)}{\partial \xi} \right]^{n_i - 1} \prod_{s=1}^{n_i} \left( 1 - \frac{a_{i+1} (\xi)}{a_i (\eta)} \right) x^s \right]_{s=1}^{n_i}.
\]

Note that (A·6) is not valid for \( N=1 \). Equation (4·12) is obtained from (A·6). The formula (A·6) is convenient to check

\[ C_N (\eta, \eta) = 0, \quad N=2, 3, 4, \cdots. \]

Indeed we have

\[ C_N (\eta, \eta) = \sum_{i=0}^{N-1} \prod_{i} \left\{ \frac{a_{i+1} (\eta)}{n_i !} \right\} \left[ \frac{(\partial x)}{\partial \xi} \right]^{n_i - 1} \prod_{s=1}^{n_i} \left( 1 - \frac{a_{i+1} (\xi)}{a_i (\eta)} \right) x^s \right]_{s=1}^{n_i} = 0.
\]

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