Tidally induced residual flows around an island due to both frictional and rotational effects

R. D. Pingree  Institute of Oceanographic Sciences, Wormley, Godalming, Surrey
Linda Maddock  Marine Biological Association of the United Kingdom, Citadel Hill, Plymouth

Received 1980 March 20; in original form 1979 August 22

Summary. A hydrodynamic numerical model developed in polar coordinates derives the tidal characteristics around an island. Due to the frictional stress over the sloping topography four residual eddies are generated. The effect of the Earth's rotation deflects the tidal flow in the shallow waters near the island and marginally adjusts the residual flows. Relationships for the $M_2$ tidal vorticity are derived for these effects which can be used to estimate the rectified vorticity distribution associated with tidal flow around islands. The corresponding bottom stress distributions, which are important in sediment transport studies, are briefly discussed.

Introduction

In order to examine the tidal physics of headland flows a hydrodynamic numerical model was developed in plane polar coordinates (Pingree & Maddock 1979a). This model uses an irrotational boundary condition at the coastline and avoids the numerical difficulties associated with corners in rectangular grid meshes using Cartesian coordinates. The polar model was then developed to derive the tidal flow around an idealized circular island surrounded by a uniformly sloping beach gradient, and some numerical results showing hourly tidal streams around an island are illustrated in Pingree & Maddock (1979b). Although lateral viscosity and flow separation may cause eddies, the model showed that with advection the frictional stress with a beach gradient could generate four residual 'eddies' around the island. Furthermore, it was shown that the effect of the Earth's rotation was to marginally adjust this flow pattern in the shoaling regions near the island.

In this paper the more theoretical aspects of $M_2$ vorticity generation, due to the frictional effects and due to the use of a rotating reference frame, and the dissipation of this vorticity by bottom friction are considered. Non-linear effects generate residual currents and $M_4$ currents which are important in sediment transport studies (Huthnance 1973), and the bottom stress distributions resulting from these residual flows and $M_4$ currents are also briefly discussed.
In the model, an island, 9 km in diameter, was centred at the origin of a polar coordinate system. The bottom topography was set so that the depth increased radially from 5 to 35 m over a distance of 12 km and was then held constant. The far field $M_2$ tidal flow was specified as an east to west oscillating current with an amplitude of 50 cm s$^{-1}$. To illustrate the friction and rotational effects the same geometry has been used each time although the scale parameters can be varied before numerical instability with advection becomes too severe. The scales were chosen so that vorticity estimates could be made even though the model was non-linear. Essentially similar results were derived using both depth-averaged velocities and water transport vectors. Some differences between the numerical schemes were detected near the coastline where the boundary conditions for the two schemes are different.

1 The equations of momentum, continuity and vorticity

The horizontal momentum equation can be vertically integrated to give an equation for the vertically averaged horizontal velocity, $u$, thus

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + 2\Omega \times u = -g \nabla E - \frac{\tau}{\rho D}$$

(1)

where the symbols are defined in the Appendix. The corresponding vertically integrated equation expressing the conservation of volume becomes

$$\frac{\partial E}{\partial t} + \nabla \cdot (h + E)u = 0.$$  

(2)

The vertical component of vorticity, $\omega$, defined as,

$$\omega = \omega_k = \nabla \times u$$

(3)

can be derived by taking the curl of equation (1), thus,

$$\frac{\partial \omega}{\partial t} + \nabla \cdot u\omega + f \nabla \cdot u = -\left(\nabla \times \frac{\tau}{\rho D}\right) \cdot k.$$  

(4)

These equations are used to define the tidal flow characteristics around an island surrounded by a uniformly sloping bottom topography (Fig. 1). In the effects considered here, vorticity generation associated with sea level ($E$) changes can be neglected in comparison with effects resulting from changes in water depth, $h$. The vorticity equation (4) then takes the more convenient form

$$\frac{\partial \omega}{\partial t} + \nabla \cdot u\omega - \frac{fu}{h} \cdot \nabla h = -\left(\nabla \times \frac{\tau}{\rho h}\right) \cdot k.$$  

(5)

which is used in the following sections.

2 The rate of change of vorticity due to the frictional term $- \nabla \times (\tau/\rho D)$

The rate of change of vorticity due to frictional effects can be found by expanding the term $- \nabla \times \tau/\rho D$, in equation (4), where

$$\tau = \rho C_D u |u|$$

with $C_D = 0.0025$.  

(6)
Tidally induced residual flows

Figure 1. Grid of the polar model showing island centred at the origin of the coordinate system. The island and the sloping area around it are shaded.

Thus,

\[ - \nabla \times \left( \frac{\tau}{\rho D} \right) = - \frac{C_D |\mathbf{u}| \omega}{D} + C_D \mathbf{u} \times \nabla \left( \frac{|\mathbf{u}|}{D} \right) \]

= dissipation + generation of vorticity.

The frictional term generates vorticity due to both spatial changes in tidal stream $|\mathbf{u}|$ amplitude and water depth, $D$. In very shallow water ($D < 10 \text{ m}$) the generation of vorticity is approximately balanced by dissipation provided advection effects are not too large and so equation (7) becomes

\[ \nabla \times \frac{\tau}{\rho D} \sim 0, \]

giving

\[ \omega_k \sim \frac{D}{|\mathbf{u}|} \times \nabla \left( \frac{|\mathbf{u}|}{D} \right). \]

If spatial changes in tidal stream amplitude are small compared with water depth changes (as generally they are for the scales considered here) or if linear friction is assumed, equation (9) simplifies to

\[ \omega_k \sim \frac{D}{D} \times \nabla D. \]

So that the numerical model results were more realistic for later application to sediment transport studies bottom friction was employed in the quadratic form (equation 6). However, essentially similar residual flows were obtained using linear friction and so for
analytical convenience bottom friction is adopted in the linear form

$$\tau = \rho \frac{C_L u}{h} = \rho K u$$  \(\text{(11)}\)

in the following derivations, where the magnitude of the linear drag coefficient, $C_L \sim (8/3\pi)C_D U$ (for rectilinear tidal stream with maximum speed $U$, now assumed spatially uniform) and $K = C_L/h = \Delta T^{-1}$ where $\Delta T$ is the dissipation time-scale.

An equation describing the $M_2$ tidal vorticity, $\omega_2$, resulting from frictional effects, in the absence of contributions due to the Earth's rotation ($f \cdot \nabla u$) and additional contributions from non-linear effects ($\nabla \cdot uu$) is obtained by omitting these terms from equation (5)

$$\frac{\partial \omega_2}{\partial t} = -K\omega_2 - \left( \frac{K}{h} u_2 \times \nabla h \right) \cdot \hat{\mathbf{k}}. \quad (12)$$

Then solving equation (12) for the amplitude of the tidal vorticity, $\omega_{02}$, and the phase difference, $\delta$, between the time of maximum tidal streaming, with amplitude $u_\theta$, resolved in a direction along water depth contours, and the time of maximum tidal vorticity gives

$$|\omega_2| = \frac{K/|u_2 \times \nabla h|}{c^2 + K^2}^{1/2}$$

or

$$\omega_{02} = \frac{u_\theta}{h} |\nabla h| \cos \delta \sim \Delta T^{-1} \left( \frac{\nabla h}{C_D} \right) \cos \delta \quad (13)$$

where

$$\delta = \tan^{-1} \left( \frac{\sigma}{K} \right) \quad (14)$$

$$\omega_2 = \omega_{02} \cos (\sigma t - \delta) \quad (15)$$

$$u_2 \cdot \mathbf{\hat{\theta}} = u_\theta \cos \sigma t \quad (16)$$

and $\mathbf{\hat{\theta}}$ is a unit vector in a direction of constant water depth (positive in an anti-clockwise sense) and $\sigma = 2\pi/T$ where $T$ is the $M_2$ tidal period.

The analytical results expressed by equations (13) and (14) were used to check the numerical output from the polar model in the absence of advection and rotation and with linear friction such that $C_D = 0.0025$ and $C_L = C_D$ 50 cm s$^{-1}$ with a far field oscillating flow of amplitude, $U = 50$ cm s$^{-1}$ and agreement was obtained. The model showed that advection modified these results. In the regions where the $M_2$ vorticity was large, $\omega_{02}$ was smaller (\sim 15 per cent) and $\delta$ was smaller (\sim 5$^\circ$). Advecive effects on $M_2$ vorticity diminish as the ratio $U/\alpha L_0$ becomes smaller (where $L_0$ is the scale of the island's beach-gradient geometry (Fig. 1)) or in very shallow regions where the generation of vorticity is balanced by dissipation (see equation 8).

The numerically derived results with advection and quadratic friction are shown in Figs 2 and 3. In the shallow high velocity regions north and south of the island the production of $M_2$ tidal vorticity is nearly balanced by dissipation with the time of maximum tidal vorticity occurring about $20^\circ$ after the time of maximum tidal streaming (see Figs 2d and 3b).

Through the non-linear advection term $(u \cdot \nabla u)$, the frictional term generates $M_4$ tidal vorticity, $\omega_4$, and mean vorticity, $\bar{\omega}$. As a first approximation to these effects, consider only the generation resulting from $M_2$ tidal oscillations neglecting contributions from residuals.
Figure 2. Amplitude and phase of radial ($u_r$) (a and b) and tangential ($u_\theta$) components (c and d) of the tidal current for the model with quadratic friction and advection but without effects due to the Earth's rotation. The dotted line indicates the limit of the beach gradient, $\nabla h = 2.5 \times 10^{-3}$.

and higher harmonics. Then equations describing the $M_4$ tidal vorticity and the residual vorticity using linearized friction become,

$$\frac{\partial \omega_4}{\partial t} + M_4 \nabla \cdot (u_2 \omega_2) = -K \omega_4$$

(17)

and

$$\nabla \cdot u_2 \omega_2 = -K \omega_3.$$  

(18)

Both the $M_4$ tidal vorticity and the residual vorticity are generated in a similar manner by the divergence of the product of $M_2$ tidal streaming and $M_2$ tidal vorticity and result in similar amplitude distributions (compare Fig. 3c with 3e).

Estimates for the $M_4$ tidal vorticity and the residual vorticity can be made from equations (17) and (18) using derived values of $u_2$ and $\omega_2$ (see equations 13 and 14). For example the residual vorticity, $\omega_3$, can be estimated by considering a vorticity budget in a quadrant bounded by north—south and west—east lines. Under steady state conditions the vorticity transported across the south line will be dissipated in the south-east quadrant. An approximation to the mean vorticity, in a narrow segment of width $l$ (see Fig. 4), can be
made by neglecting contributions from the radial component of the current $u_r$. The integral form of equation (18) is then

$$
\bar{u}_\theta \cos \sigma t \omega_2 l \sim K \bar{\omega} \frac{2\pi r}{4} l
$$

and substituting from equation (13) with $K = C_L/h = \Delta T^{-1}$

$$
\bar{\omega} \sim \left( \frac{\nabla h}{\pi C_L r} \right) u_\theta 2\cos^2 \delta \sim \frac{\omega_0^2 \Delta T}{\pi}
$$
and further approximating $C_L \sim C_D |u| \sim C_D u \theta$ gives

$$\bar{\omega} \sim \frac{1}{\pi} \left( \frac{\nabla h}{C_D} \right) \left( \frac{u \theta}{r} \right) \cos^2 \delta.$$  \hspace{1cm} (20)

Taking $r \sim 10^6$ cm; $\nabla h = 2.5 \times 10^{-3}$, $u \theta \sim 100$ cm s$^{-1}$, and with $\delta \sim 40^\circ$ in about 15 m water depth, gives an estimate for the mean vorticity, $\bar{\omega} \sim 2 \times 10^{-5}$ s$^{-1}$ which approximates to the average value in this region (see Fig. 3e).

In the absence of rotation, all the distributions patterns possess two-fold symmetry. The asymmetry introduced by squashmg and stretching columns of water with planetary vorticity as the tide oscillates over the sloping depth contours in the vicinity of the island is considered in the next section.

3 Rate of change of vorticity due to stretching planetary vorticity, $-f \nabla \cdot u$

Coriolis effects produce additional vorticity where the flow moves across water depth contours. The vorticity distributions due to both the frictional and the rotational effect are shown in Fig. 5 (using quadratic friction as before) and marked asymmetry in the vorticity distribution is now evident. The effects of using the rotating reference frame can be visualized by subtracting from these results (i.e. Fig. 5) the results of the non-rotating case already considered (Fig. 3). These distributions are illustrated in Fig. 6.

An equation describing the generation of $M_2$ tidal vorticity $\omega_{2f}$ due solely to Coriolis effects can be estimated from equation (5) using linear friction and neglecting advection. The criterion for neglecting advection has the additional restriction that $\omega \ll f$.

$$\frac{\partial \omega_{2f}}{\partial t} - \frac{fu}{h} \cdot \nabla h = -K \omega_{2f}$$  \hspace{1cm} (21)

giving

$$|\omega_{2f}| = \frac{fu h}{[\sigma^2 + K^2]^{1/2}} \cdot \nabla h.$$  \hspace{1cm} (22)

$$\omega_{02f} = \frac{fu \nabla h}{h K} \cos \delta_f \sim f \left( \frac{\nabla h}{C_D} \right) \cos \delta_f,$$

$$\delta_f = \tan^{-1} \left( \frac{\sigma}{K} \right).$$  \hspace{1cm} (23)
Where \( \delta_r \) now refers to the phase difference between the maximum current, \( u_r \), resolved in the direction of \( \nabla h \) and the tidal vorticity due to Coriolis effects.

The distributions illustrated in Fig. 6 do not, of course, isolate the vorticity effects and residuals (also Fig. 8b) associated with Earth's rotation. This is because as the flow pattern is deflected by Coriolis effects (compare Figs 3 and 5) frictional effects are also slightly adjusted. The linear model confirmed that the linear results expressed by equations (22) and (23) could be added to those of equations (13) and (14) to obtain the \( M_2 \) vorticity generated by the combined effects of rotation and friction from \( u_r \) and \( u_\theta \). \( M_2 \) tidal vorticity generated in this linear manner due to rotational effects with frictional dissipation (equation 21)
deflects the basic $M_2$ tidal flow clockwise from its east-west symmetry in the shoaling regions near the island so that the pattern of $M_2$ tidal vorticity is correspondingly marginally rotated (Fig. 6a).

The ratio of the friction tidal vorticity to the Coriolis $M_2$ tidal vorticity at a position is

$$\frac{Ku_\theta}{fu_r} \sim \frac{C_D |u| u_\theta}{C_D |u|} \sim \frac{C_D |u|}{fu_r h} \sim \left(\frac{f \Delta T}{h}\right)^{-1}$$

(24)

which is effectively the Ekman Number for the flow.

Frictional and Coriolis $M_2$ tidal vorticity reach maximum values in alternate quadrants around the island where $u_\theta$ and $u_r$ attain maximum amplitudes. In the special case of linear friction which is also spatially uniform, then the ratio $Ku_\theta/fu_r$ may be used for comparing frictional effects with Coriolis effects at different positions around the island with the same water depth since under these conditions $\delta = \delta_F$.

As a first approximation, an equation similar to equation (18) can be used to derive the residual vorticity due to the effects of Earth's rotation with $\omega_2$ replaced by $\omega_{2f}$, neglecting as before higher harmonics and the redistribution of vorticity by the residual flows due to stretching and advecting relative vorticity, i.e. $\nabla \cdot \bar{u} \bar{\omega}$. The equation describing the vorticity balance for residual vorticity due to the effects of Earth's rotation, $\bar{\omega}_f$ becomes

$$\nabla \cdot \bar{u}_2 \omega_{2f} + f \nabla \cdot \bar{u} = -K \bar{\omega}_f.$$
An estimate for the residual vorticity due to the Earth's rotation, \( \bar{\omega}_f \) can be made by considering a vorticity balance in the region inscribed by a circle of radius, \( R \), (see Fig. 7) then

\[
\int_{0}^{2\pi R} \bar{u}_r \cos \theta \omega_{2f} \; dL = - \langle K \bar{\omega}_f \rangle \frac{2\pi R}{2}
\]

where the integral is evaluated along the boundary \( L \) with radius \( R \) and where \( \langle \; \rangle \) denotes the average value in the region bounded by \( R \). Since the water depth is constant along the boundary the term \( f \nabla \cdot \bar{u} \) makes no contribution to this vorticity budget.

Then for simplicity taking \( K \) now as constant and substituting for \( \omega_{2f} \) (see equations 22 and 23) with

\[
\omega_{2f} = \frac{fu_r}{Kh} \nabla h \cos \delta_f \cos (\theta t - \delta_f)
\]

gives

\[
\bar{\omega}_f \sim - \langle u_r^2 \rangle_R \frac{f |\nabla h| R \cos^2 \delta_f}{2K^2 hCR}
\]

where \( \langle \; \rangle_R \) now denotes the average value at a radius \( R \), and this estimate was approximated by a numerical model subject to the same conditions. Further, roughly approximating \( h/l \sim \nabla h, R \sim r \) gives

\[
\bar{\omega}_f \sim - \frac{\langle \omega_{2f} \rangle_R}{2f}.
\]

The identity expressed by equation (28) does not of course imply that the residual vorticity is inversely proportional to \( f \) since \( \omega_{2f} \) is itself proportional to \( f \) (see equation 22). For illustrative convenience the effect of doubling the Coriolis parameter, and neglecting reality, on the residual flows for the model with quadratic friction, advection and rotation is illustrated in Fig. 8.

The form of equation (28) shows that on average in the shallow regions near the island the mean vorticity is negative.
Tidally induced residual flows

Figure 8. (a) Residual currents as for Fig. 5(f) but with the Coriolis parameter doubled. (b) Residual currents as for Fig. 6(d) but with rotational effects doubled.

4 Stress distributions

The bottom stress distributions resulting from the flow around the island are illustrated in Fig. 9. The mean stress distribution (Fig. 9a) reflects the residual flows with smaller contributions from the $M_4$ tidal currents. If the residual flows and the $M_4$ currents are much

Figure 9. Stress distributions resulting from the distribution shown in Fig. 5. (a) Residual stress. (b) Maximum stress. (c) Maximum stress due to $M_2$ and $M_4$ currents only (below each distribution a scale in dyn cm$^{-2}$ is shown).
smaller than the $M_2$ tidal currents then the mean stress, using quadratic friction, can be expressed analytically (Hunter 1979). For example in the simple case where the $M_2$ tidal oscillation is in the $i$ direction and both the $M_2$ and $M_4$ tidal ellipses are rectilinear then:

$$\begin{align*}
[\bar{\tau}_x] &= \frac{2 \rho C_D u_{02}}{\pi} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \dot{\bar{u}} \end{bmatrix} + \frac{2 \rho C_D u_{02}}{3\pi} \cos 2\phi \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{04} \\ v_{04} \end{bmatrix}
\end{align*}$$

(29)

where

$$\tau = \bar{\tau}_x i + \bar{\tau}_y j$$
$$\bar{u} = \bar{u} i + \bar{v} j$$
$$u_2 = u_{02} \cos (\sigma t - \phi)i$$
$$u_4 = u_{04} (\cos 2\sigma t)i + v_{04} (\cos 2\sigma t)j.$$

In sediment transport studies the maximum stress distributions, as well as the mean stress distributions, are likely to be important in determining the direction of sand transport paths and sand wave asymmetry (Caston & Stride 1970; Pingree & Griffiths 1979). The maximum stress distributions (Fig. 9b) in the shallow regions near the island are determined mainly by the residual flows (Fig. 5) and are directed around the island towards the region where $M_2$ tidal streams reach their maximum amplitudes.

As a result of $M_4$ currents the maximum stress vectors in the absence of residual flows are generally directed out of the high current amplitude $M_2$ regions (Fig. 9c). This effect results from the $M_4$ component of tidal vorticity due to the $u \cdot \nabla u$ term. When linear friction is employed there is still in general a direction of maximum stress but the time averaged value of bottom friction is clearly zero. Where quadratic friction is used there is also a small residual stress due to $M_2, M_4$ tidal stress interaction (see equation 29).

$$\bar{\tau} = \rho C_D \bar{u} |\bar{u}|$$

where $u = u_2 + u_4$.

5 Discussion and conclusions

A hydrodynamic numerical model is used to derive the tidal flow around an island with a regularly sloping bottom topography. For an east—west oscillating tidal flow the frictional term $- \nabla \times (\tau / \rho h)$ generates $M_2$ tidal vorticity north and south of the island. Estimates for the $M_2$ tidal vorticity are derived using a linear form for the bottom frictional stress. Introduction of non-linear advection effects ($u \cdot \nabla u$) results in $M_4$ vorticity and residual vorticity. The production of mean vorticity resulting from the effects associated with the advection and stretching of the $M_2$ tidal vorticity by the tidal flow, i.e. $(\nabla \cdot u_2 \omega_2)$ tends to balance the curl of the mean bottom stress (divided by the water depth) and results in residual ‘eddies’ of opposite sign in alternate quadrants bounded by north—south and east—west lines through the centre of the island (Fig. 3f).

In a very qualitative way this pattern of residual flows can be understood by realizing that a region of reduced flow tends to develop behind the island in the lee of the flow due to the advection of the frictionally generated vorticity. In steady flow with bottom friction but in the absence of advection the sea level falls in the direction of the flow. With west to east flow around the island advection results in a further local lowering of sea level in the high vorticity regions north and south of the island. For the scales considered here the adverse
pressure gradient due to advection is comparable with the pressure gradient required to drive the flow against bottom friction. So if frictional effects increase towards the shore due to decreasing water depth then a stagnation region tends to develop behind the island. Thus the east-going currents are reduced on the east side of the island. Similarly the west-going currents are reduced on the west side of the island. When an oscillating flow is considered and the tidal average is taken it is clear that there is a residual flow towards the island from both east and west directions which by continuity must flow along shore and return to deeper water as an offshore flow north and south of the island. Such different flood and ebb patterns of tidal flow will also produce offshore residual flows off headlands and promontories. This tidal rectification of the $M_2$ currents in the lee of the flow is equivalent to the generation of a residual current and higher harmonics in particular $M_4$ currents. Equivalently the $M_2$ tidal vorticity also tends to become halfwave rectified.

The production of $M_2$ tidal vorticity generated east and west of the island by the linear effects due to squashing and stretching water columns with planetary vorticity $(-f \nabla \cdot u_2)$, with large frictional damping, rotates the residual flows leaving the island marginally clockwise from their north–south directions (compare Fig. 3f and Fig. 5f). This can be understood in an approximate qualitative way by realizing that as a flow moves from west to east up the beach gradient the flow acquires negative vorticity as the water columns are squashed. Flow that is directed around the island to the north face deflects towards the shore whereas flow that is starting to pass towards the south face deflects offshore. At the shore the north–south line ceases to determine the region where the strongest flows parallel with the shore occur. These positions are now displaced further around the island in a clockwise sense. Close to the shore the far field flow appears to have come from a direction north of west. Since the description so far is linear similar effects are observed when the flow moves from east to west. This deflected flow pattern for the tidal streams caused by Coriolis effects rotates the residual flows which are moving offshore due to frictional effects, clockwise from the north–south directions. Further adjustments from non-linear effects are also derived.

The results described here have been derived in the absence of horizontal diffusion. Clearly a need exists to examine the effects of horizontal diffusion and other lateral effects and to determine the importance of the vertical structure of the flow with a three-dimensional approach.

Acknowledgments

It is a pleasure to thank Mr G. A. W. Battin for his help with the figures.

References


Appendix: symbols used in text

- $t$: time coordinate
- $x, y$: horizontal spatial coordinate, $x$ - east, $y$ - north
- $r, \theta$: polar coordinates
- $E$: surface elevation
- $h$: water depth
- $D$: total water depth, $D = h + E$
- $u$: vertical integrated horizontal velocity with components $u, v$
- $|u|$: magnitude of the vertically integrated velocity
- $u_2$: $M_2$ tidal velocity with components of amplitude $u_{02}, v_{02}$
- $u_\theta$: amplitude of the tidal current in a direction $\theta$
- $u_\tau$: amplitude of the tidal current in a direction $\tau$
- $u_4$: $M_4$ tidal velocity with components of amplitude $u_{04}, v_{04}$
- $\bar{u}$: residual flow
- $\omega$: vorticity, where $\nabla \times u = \omega = \omega k$
- $\omega_2$: $M_2$ vorticity with maximum amplitude $\omega_{02}$
- $\omega_4$: $M_4$ vorticity
- $\omega_\delta$: residual vorticity
- $C_D$: drag coefficient $= 0.0025$
- $C_L$: linear drag coefficient
- $\mathbf{r}$: bottom drag
- $\bar{r}_x, \bar{r}_y$: average bottom drag with components
- $\rho$: density of sea-water
- $f$: Coriolis parameter $= 1.1 \times 10^{-4}$ at $50^\circ$ north
- $g$: acceleration due to gravity
- $\delta$: phase difference between the time when $u_\tau$ or $u_\theta$ reaches its maximum value and the time of maximum tidal vorticity
- $T$: $M_2$ tidal period
- $\sigma$: angular frequency of the $M_2$ tide
- $\Omega$: angular rotation rate of the Earth
- $\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y}$
- $\nabla \times \frac{\partial}{\partial x} \frac{\partial}{\partial y} x$
- $\langle \rangle$: time average
- $\langle \rangle$: spatial average
- $\mathbf{i}$: unit vector in the $x$ direction, east
- $\mathbf{j}$: unit vector in the $y$ direction, north
- $\mathbf{k}$: unit vector in the vertical direction
- $\hat{\mathbf{r}}$: unit vector in a direction of increasing $\rho$
- $\mathbf{r}$: unit vector in a direction of increasing $r$
- $\cdot$: scalar product
- $\times$: vector product