Superfluid $^3$He in a Magnetic Field

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The exact self-consistent solutions of the $p$-wave, spin triplet pairing hamiltonian in a magnetic field of arbitrary strength are found by applying the method of the 5-dimensional spin. The solutions are of non-unitary type and reduce to the ABM state and the BW state in the weak limit of the magnetic field. The unitary transformation which diagonalizes the hamiltonian including Zeeman energy is given explicitly. We get the formulae of the magnetization of $^3$He-A and $^3$He-B. The results are valid at any temperature.

The superfluid $^3$He-A and $^3$He-B phases are identified with the ABM state and the BW state which belong to the unitary type. However, this property should be modified in a finite magnetic field.

Ambegaokar and Mermin and Takagi have investigated the transition in a magnetic field relying on the G-L expansion. Carton has studied the general case for the A-B transition by a Green’s functional approach. Engelsberg et al. and Greaves have calculated the deviation from the BW state due to a magnetic field by a perturbational approach. In this letter, making use of the technique of the 5-dimensional spin, we diagonalize the hamiltonian without assuming the gap $\Delta$ and the magnetic field $H$ to be small.

By the notation of the 5-dimensional spin, the pairing hamiltonian in magnetic field along the $z$-axis is written as

$$\mathcal{H} = \sum_p \mathcal{H}_p(p),$$

$$\mathcal{H}_p(p) = 2\epsilon_p J_{15}(p) - 2\Delta_4(p) J_4(p) - 2\Delta_5(p) J_5(p) - iHJ_1 z(p),$$

(1)

in which $\Delta_\alpha(p)$ is determined self-consistently by the gap equation:

$$\Delta_\alpha(p) = 3g \sum_{p'} (p' \cdot \hat{p}) \langle J_\alpha(p') \rangle.$$

(\alpha = 4, 5)

(2)

In the presence of the magnetic field, the self-consistency is satisfied only by taking the non-unitary gap function, which we assume to be the following form. For the generalized ABM state, the gap $\Delta_\alpha(p)$ is assumed as

$$\begin{pmatrix}\Delta_4(p) \\ \Delta_5(p)\end{pmatrix} = \begin{pmatrix}\cos \beta_p \sin \beta_p & (\Delta \sin \phi \hat{d}) \\ -\sin \beta_p \cos \beta_p & (\Delta' \sin \phi \hat{e})\end{pmatrix},$$

(3)

in which $\hat{d}$, $\hat{e}$ and $\hat{h} = H/H = 2$ form a triad of unit vectors in a spin space. For the generalized BW state, $\Delta_\alpha(p)$ is taken to be

$$\begin{pmatrix}\Delta_4(p) \\ \Delta_5(p)\end{pmatrix} = \begin{pmatrix}\Delta \hat{p} + \Delta \hat{e} \times \hat{h} \\ \Delta \hat{p} \times \hat{h} \end{pmatrix}.$$
where
\[ \tan \gamma = \frac{(d + d') \sin \alpha_p}{\epsilon_p - (1/2) \gamma H}, \]
\[ \tan \beta = \frac{(d - d') \sin \alpha_p}{\epsilon_p + (1/2) \gamma H}. \]

The hamiltonian is diagonalized,
\[ U_{nm}U_n^\dagger(p)U_{nm} = (A + B)J_{45} + (A - B)J_{12}, \quad \text{(6)} \]
where
\[ A^2 = \left( \epsilon - \frac{1}{2} \gamma H \right)^2 + (d + d')^2 \sin^2 \alpha_p, \]
\[ B^2 = \left( \epsilon + \frac{1}{2} \gamma H \right)^2 + (d - d')^2 \sin^2 \alpha_p. \]

We have two independent gap equations for \( J + J' \) and \( J - J' \):
\[ J + J' = \frac{3g}{8} \int d\Omega \int_{-\epsilon_c}^{\epsilon_c} d\epsilon \ n(\epsilon) \times \sin^2 \alpha_p \frac{\theta(\beta A/2)}{A}, \quad \text{(7)} \]
\[ J - J' = \frac{3g}{8} \int d\Omega \int_{-\epsilon_c}^{\epsilon_c} d\epsilon \ n(\epsilon) \times \sin^2 \alpha_p \frac{\theta(\beta B/2)}{B}, \quad \text{(8)} \]
where \( n(\epsilon) = (\epsilon^2 / \pi^2) d\epsilon / d\epsilon \). The transition temperature between normal and \( A_1 \) is determined by Eq. (8) and the transition temperature between \( A_1 \) and \( A \) is determined by Eq. (7).\(^a\)

For the generalized BW state with the gap (4), the unitary operator is found to be
\[ U_{BW} = \exp(-i\theta J_x \hat{\mathbf{g}}) \exp(i\varphi J_y \hat{\mathbf{h}}) \times \exp\left\{ i\left( \alpha + \beta \right) J_x \hat{\mathbf{g}} \right\} \times \exp\left\{ -i\left( \alpha - \beta \right) J_y \hat{\mathbf{f}} \right\}. \quad \text{(9)} \]
\( ^a \) The spins antiparallel to the field are favorable because the gyromagnetic ratio \( \gamma \) of \(^4\)He is negative.

\[ \hat{\mathbf{f}} = \frac{\mathbf{p} \times \hat{\mathbf{h}}}{|\mathbf{p} \times \hat{\mathbf{h}}|}, \]
\[ \hat{\mathbf{g}} = \frac{\hat{\mathbf{h}} \times \hat{\mathbf{f}}}{|\hat{\mathbf{h}} \times \hat{\mathbf{f}}|}, \]
\[ \tan \theta = \frac{2d}{\gamma H \sqrt{1 - p_z^2}}, \]
\[ \tan \varphi = \frac{(d + d_1) \rho_z G}{\epsilon (\gamma H/2) - d_2 (1 - p_z^2) + (d + d_1) \rho_z^2 G}, \]
\[ \tan \alpha = \frac{d (\gamma H/2) + d_2 \sqrt{1 - p_z^2}}{\epsilon - G^2}, \]
\[ \tan \beta = \frac{d (\gamma H/2) + d_2 \sqrt{1 - p_z^2}}{\epsilon + G^2}, \]
\[ F^2 = \left\{ \frac{1}{2} \gamma H^2 - d_2 (1 - p_z^2) \right\}^2 + (d + d_1) \rho_z^2 G^2, \]
\[ G^2 = d_2^2 (1 - p_z^2) + \left( \gamma H / 2 \right)^2. \]

The diagonalized form of the hamiltonian is expressed as
\[ U_{BW} U_n^\dagger(p) U_{BW} = (C + D)J_{45} + (C - D)J_{12}, \quad \text{(10)} \]
where
\[ C^2 = \epsilon^2 + d^2 (1 - p_z^2) + (d + d_1) \rho_z^2 \]
\[ + d_2^2 (1 - p_z^2) + \left( \gamma H / 2 \right)^2 - 2F, \]
\[ D^2 = C^2 + 4F. \]

We have three coupled gap equation:
\[ J + J_1 = (J + J_2) \frac{3g}{8} \int d\Omega \int_{-\epsilon_c}^{\epsilon_c} d\epsilon \ n(\epsilon) \rho_z^2 \times \left\{ \frac{\theta(\beta C/2)}{C} + \frac{\theta(\beta D/2)}{D} \right\} \]
\[ \times \left\{ \frac{\theta(\beta C/2)}{C} - \frac{\theta(\beta D/2)}{D} \right\} \times \frac{d_2^2 (1 - p_z^2) + (\gamma H / 2)^2}{F}, \quad \text{(11)} \]
\[ J + J_2 = (J + J_2) \frac{3g}{8} \int d\Omega \int_{-\epsilon_c}^{\epsilon_c} d\epsilon \ n(\epsilon) (1 - p_z^2) \times \int_{-\epsilon_c}^{\epsilon_c} d\epsilon \ n(\epsilon) (1 - p_z^2) \]
\[ \times J_2. \]
\[ m_p(p) = \langle J_{1z} \rangle = \frac{\beta}{2} \mathbf{H} p_z \]

\[ m_q(p) = \frac{\beta}{2} \mathbf{H} p_z \]

In general, \( U^{-1} J_{1z} U \) is the linear combination of \( J_{n3} \):

\[ U^{-1} J_{1z} U = a J_{1z} + b J_{13} + c J_{14} + \cdots \]  

Because the unitary operator \( U \) is chosen to diagonalize the Hamiltonian as (6) or (10), we have from (15) and (16)

\[ m_p(p) = m_p(p) + m_q(p), \]

where

\[ m_p(p) = \frac{\beta}{2} \mathbf{H} p_z \]

\[ m_q(p) = \frac{\beta}{2} \mathbf{H} p_z \]

We have for the ABM state

\[ m_p(p) = \frac{\beta}{2} \mathbf{H} p_z \]

\[ m_q(p) = \frac{\beta}{2} \mathbf{H} p_z \]

and for the BW state

\[ m_p(p) = \frac{\beta}{2} \mathbf{H} p_z \]

\[ m_q(p) = \frac{\beta}{2} \mathbf{H} p_z \]
When we take the limit $\gamma H/J \to 0$, $\Sigma m_p(p)/H$ and $\Sigma m_q(p)/H$ become the "superfluid" and "normal" susceptibilities given by Leggett and Takagi.  

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Note added in proof: After having submitted this short note, a paper by Tewordt and Schopohl (J. Low Temp. Phys. 37 (1979), 421) appeared. They studied the gap and the collective modes for $^3$He-B in a magnetic field by the $4 \times 4$ matrix Green's function. However, their assumed state is restricted only to the unitary one, i.e., the particle-hole asymmetry is neglected.