Scaling Hypothesis for the Rapidity Distributions and Information Theory

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(Received June 29, 1979)

By means of information theory we propose a new scaling hypothesis for the rapidity distributions both in semi-inclusive and inclusive reactions, which indicates the same type of scaling as KNO scaling and scaling in the mean. Such scaling laws are supposed to be mutually in close connection and in addition to contradict Feynman scaling.

§ 1. Introduction

In this decade several scaling hypotheses have been proposed for hadron-hadron reactions; i.e., Feynman scaling, KNO (Koba, Nielsen and Olesen) scaling, scaling in the mean, etc. Among them, KNO scaling and scaling in the mean seem to belong to the same category, which may be called "generalized scaling in the mean (GSIM)."

Dao et al. introduced scaling in the mean for the semi-inclusive momentum distributions. Since then there have been several arguments on this scaling hypothesis. Here we especially pay our attention to the work of Ernst and Schmitt (ES) who used information theory to derive the functional form of scaling in the mean. The theoretical bases of GSIM seem to be founded by generalizing their point of view.

In this paper we shall indicate that the same type of scaling law as scaling in the mean holds also for the rapidity distributions both in semi-inclusive and inclusive reactions. We propose Gaussian scaling function for them by the use of information theory according to ES. The experimental data, which are compared with our theoretical calculations, seem to support our present investigations.

In § 2 we express our fundamental standpoint about GSIM on the basis of information theory. It is shown in § 3 that scaling in the mean for the rapidity distributions holds well in comparison with the experimental data both in semi-inclusive and inclusive reactions. We shall argue in § 4 that the inclusive distribution functions can be derived from the semi-inclusive ones. Section 5 is devoted to the discussion of the relation between GSIM and Feynman scaling.
§ 2. Information theory and GSIM

Let us first consider the distribution \( f(x) \) with respect to a physical quantity \( x \). Information theory provides us a method called maximum entropy estimate \(^5\) which can reproduce the most probable functional form of \( f(x) \) under several subsidiary conditions of the definite information. The most crucial point of this theory is what information we may choose among many possibilities. If we know all the contributions \( f_i(x) \) to \( f(x) \) from all the possible mechanisms \( i \), we can express \( f(x) \) as

\[
f(x) = \sum_i f_i(x).
\]

Actually we usually have only a few pieces of information about these mechanisms, but it is fortunately expected that there is only a few mechanisms which practically determine the functional form of \( f(x) \). Accordingly, if we find the dominant mechanisms, we shall be able to get \( f(x) \) by means of maximum entropy estimate using the reliable information of these dominant (production) mechanisms.

We start out from the assumption that the most dominant mechanism determines the average value \( \langle x \rangle \) of \( x \). And so, we divide all the mechanisms, which yield \( f(x) \), into the following two groups:

(I) The most dominant mechanism, which results from the energy-momentum conservation law, well-known \( p_T \)-cutoff, etc., contributes to \( f(x) \) through \( \langle x \rangle \).

(II) All the mechanisms other than (I), where we need only some dominant mechanisms which come from multiperipheral model, resonance (cluster) production picture, etc.

If we get \( \langle x \rangle \) by means of kinematical and (or) dynamical conditions (mechanism (I)), next we can investigate the functional form of \( f(x) \) by considering model dependent production mechanism (mechanism (II)). Here it should be noted that the exact functional form \( f(x) \) is determined only by (II). Hence, if GSIM is satisfied, i.e., it is confirmed to represent \( f(x) \) in terms of \( \xi = x/\langle x \rangle \), we can avoid the influences of (I) to study the production mechanism.

ES showed\(^6\) that the gross features of the semi-inclusive momentum distributions are obtained by fixing only \( \langle p_L \rangle \) and \( \langle p_T \rangle \) (the average value of the longitudinal and transverse-momentum at fixed multiplicity \( n \)) as follows:

\[
\frac{\langle p_L \rangle \langle p_T \rangle}{n \sigma_n} \frac{d\sigma_n}{dp_L dp_T} = 4 \frac{p_T}{\langle p_T \rangle} \frac{p_L}{\langle p_L \rangle} \exp \left\{ - \left( \frac{2 p_T}{\langle p_T \rangle} + \frac{p_L}{\langle p_L \rangle} \right) \right\}. \tag{2.2}
\]

The major defect of Eq. (2.2) comes from the fact that it does not include the \( p_T p_L \) correlation (seagull effect). As concerns the multiplicity distribution, we cannot give even the gross feature of the distribution function if we specify only \( \langle n \rangle \). Indeed, it turns out from maximum entropy estimate that

\[
\frac{\langle n \rangle \sigma_n}{\sigma} = \exp \left( - \frac{n}{\langle n \rangle} \right), \tag{2.3}
\]
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where \( \sigma = \sum \sigma_n \).

The difference between Eq. (2·2) (or Eq. (2·3)) and the exact distribution is considered to be ascribed to mechanism (II). Therefore, the investigation of these differences leads us to clear knowledge of (II). The most remarkable fact, in this connection, is that mechanism (I) leads to GSIM as in Eqs. (2·2) and (2·3), whereas we cannot obtain the exact form of \( f(x) \) since we disregard mechanism (II). On the other hand, mechanism (II), which is inevitable to get the exact functional form of \( f(x) \), mostly yields the scaling violation term which should disappear at infinite energy. GSIM is, hence, considered to be only an approximate formula under the present accelerator energy region.

§ 3. Scaling in the mean for the semi-inclusive rapidity distributions

In the preceding section we have remarked that the approximate scaling phenomena results from the mechanism which determines the functional form of \( f(x) \) through the average values of the physical quantities. In this context we must search for the variables which make the scaling function be as close to the exact one as possible with the help of the minimum information. We restrict ourselves in this section to the semi-inclusive momentum distributions for the sake of concreteness.

When Dao et al. proposed scaling in the mean, they chose \( p_T \) and \( p_L \) as the variables. The resulting scaling law is

\[
\frac{d\sigma}{dp_L dp_T} = f \left( \frac{p_L}{<p_L>}, \frac{p_T}{<p_T>} \right)
\]

(3·1)

However, their choice of the variables is not convenient in two respects. Firstly, one \( (p_L) \) of them is not the Lorentz invariant quantity for the longitudinal boost, and secondly there exists the complicated \( p_T-p_L \) correlation. Therefore, we will propose a new choice of the variables; the longitudinal rapidity \( y \) and \( p_T \). Our scaling hypothesis requires

\[
\frac{d\sigma}{dy dp_T} = \phi \left( \frac{y}{<y>}, \frac{p_T}{<p_T>} \right)
\]

(3·2)

where \( \phi \) and \( \bar{\phi} \) are considered to be Gaussian distribution (see following discussion) and Hagedorn type distribution respectively;

\[
\phi \left( \frac{y}{<y>}, \frac{p_T}{<p_T>} \right) = \frac{1}{\pi} \exp \left\{ -\frac{1}{\pi} \left( \frac{y}{<y>} \right)^2 \right\}
\]

(3·3)

and

\[
\bar{\phi} \left( \frac{p_T}{<p_T>} \right) = \frac{10}{3\sqrt{\pi}} \left( \frac{5}{2} \frac{p_T}{<p_T>} \right)^{1/2} \exp \left\{ -\frac{5}{2} \left( \frac{p_T}{<p_T>} \right) \right\}
\]

(3·4)

We cannot distinguish Hagedorn type distribution (see Ref. 3) from \( \phi = (4p_T/<p_T>) \exp \left( -2p_T/<p_T> \right) \) (cf. Eq. (3·5)) in comparison with the experimental data in the present situation.
Fig. 1. Plot of $\langle y \rangle_n < p_T >_n$ versus $y/\langle y \rangle_n$ for three energy points, $n=4$; $102 \text{ GeV/c}$, $n=6$; $102 \text{ GeV/c}$, $n=8$; $102 \text{ GeV/c}$, $n=10$; $102 \text{ GeV/c}$, $n=12$. The solid line represents the Gaussian scaling function of Eq. (3.3).

Fig. 2. Comparison of the Gaussian scaling function (Eq. (3.3)) with the data of $\langle y \rangle_n < p_T >_n$ for three energy points, $n=4$; $102 \text{ GeV/c}$, $n=6$; $102 \text{ GeV/c}$, $n=8$; $102 \text{ GeV/c}$, $n=10$. The data are taken from Refs. 7) and 8). The solid line represents the Gaussian scaling function of Eq. (3.3).

As a result of the analysis of the inclusive momentum distributions, we see that the distribution functions concerning $p_T$ and $y$ are probably factorizable. In Fig. 1 we show $\phi(x)$ in terms of $x = y/\langle y \rangle_n$ for three energy points, and in Fig. 2 for several prong numbers at $69 \text{ GeV/c}$ in $pp \rightarrow \pi^- + x$ reactions. The experimental data seem to show the scaling behavior in these figures.

Let us investigate our scaling hypotheses by means of information theory. We obtain a scaling function

$$\frac{\langle y \rangle_n < p_T >_n}{n \sigma_n} \frac{d \sigma_n}{dy dp_T} = 4 \frac{p_T}{\langle p_T \rangle_n} \exp \left\{ -\left( \frac{2}{\langle p_T \rangle_n} + \frac{y}{\langle y \rangle_n} \right) \right\}$$

from maximum entropy estimate in accordance with ES. This $p_T$-distribution function is the same as one of Eq. (2.2). Furthermore the experimental study suggests the factorization property of Eq. (3.5) with respect to $p_T$ and $y$. There is, however, a disagreement between the functional form of the rapidity distribution of Eq. (3.5) and the corresponding data. In a nutshell, Eq. (3.5) cannot reproduce the central plateau of the rapidity distributions. It is probable that the
plateau comes from the mechanism (II) (see § 2). Information theory tells us how to find Gaussian distribution (Eq. (3.3)) which yields the rapidity central plateau. Provided we admit mechanism (II) which gives the constraint equation \( \int_0^\infty x^2 \phi dx = \pi \)** by incorporating it with the normalization condition of \( \phi(x) \) and the definition of \( \langle x \rangle \), we obtain \( \phi(x) = (2/\pi) \exp(-x^2/\pi) \)** by means of maximum Shannon-entropy principle with Lagrange multipliers. Thus, if we assume, besides the mechanism which determines \( \langle p_T \rangle_b \) and \( \langle y \rangle_b \), another mechanism which yields the plateau of the rapidity distributions, we can reproduce the gross feature of the semi-inclusive momentum distributions.

Consequently, our choice of the variables \( \langle p_T \) and \( y \rangle \) is superior to ES’s analysis in the following three points:

(i) Our scaling function is Lorentz invariant for the longitudinal boost.

(ii) It satisfies the factorization of the distribution function with respect to \( p_T \) and \( y \).

(iii) Mechanism (II) is considered in the appearance of the central plateau of the rapidity distribution.

§ 4. The inclusive rapidity distributions

Hitherto we have considered the following scaling functions:

\[
\frac{\langle n \rangle \sigma_b}{\sigma_b} = \phi \left( \frac{n}{\langle n \rangle} \right),
\]

\[
\frac{\langle p_L \rangle_b d \sigma_b}{n \sigma_b d p_L} = f \left( \frac{p_L}{\langle p_L \rangle_b} \right)
\]

and (or)

\[
\frac{\langle y \rangle_b d \sigma_b}{n \sigma_b d y} = \phi \left( \frac{y}{\langle y \rangle_b} \right).
\]

On the analogy of them, we can suppose that the same scaling law holds also for the inclusive longitudinal-momentum distributions

\[
\frac{\langle p_L \rangle}{\langle n \rangle \sigma} d \sigma = F \left( \frac{p_L}{\langle p_L \rangle} \right),
\]

or the rapidity distributions

\[
\frac{\langle y \rangle d \sigma}{\langle n \rangle \sigma d y} = \phi \left( \frac{y}{\langle y \rangle} \right).
\]

** This is a subsidiary condition of maximum entropy estimate in the course of actual calculations. One may refer to multi-peripheral model** as this dynamical picture of mechanism (II).

** It should be noted that we use \( \phi(x) = (1/\pi) \exp(-x^2/\pi) \) in comparison with the experimental data in \( x \geq 0 \) region.
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Fig. 3. Comparison of our Gaussian scaling function of Eq. (4.3b) \( \Phi(x) = \frac{1}{\sigma} \exp\left( -\frac{x^2}{\sigma^2} \right) \) in terms of \( x = \frac{y}{\langle y \rangle} \) with the data of \( \langle y \rangle / \langle n \rangle \sigma \frac{d\sigma}{dy} \) in the c.m. system for \( p\bar{p}\rightarrow\pi^- + x \) at 69 GeV/c, 102 GeV/c and 400 GeV/c (see Refs. 7 and 8). The normalization of the data is given by \( \int dy \frac{d\sigma}{dy} \). The data are from Ref. 12.

Fig. 4. We compare our Gaussian scaling function with the data of \( \langle y \rangle / \langle n \rangle \sigma \frac{d\sigma}{dy} \) in the c.m. system for \( p\bar{p}\rightarrow\pi^+ + x \) at 9.1 GeV/c, 14.8 GeV/c and 100 GeV/c. The data are from Ref. 12.

requiring GSIM hypothesis. We plot the inclusive rapidity distributions for \( p\bar{p}\rightarrow\pi^- + x \) in Fig. 3, and for \( p\bar{p}\rightarrow\pi^+ + x \) in Fig. 4. They show the appropriate scaling behaviors as we expect. It is the purpose of this section to demonstrate that Eq. (4.3) is derived from Eq. (4.2) with the help of KNO scaling.

We obtain the inclusive momentum distribution functions from the semi-inclusive distributions, regarding \( n \) as a continuous quantity, as follows:

\[
\frac{\langle y \rangle}{\langle n \rangle} \frac{d\sigma}{dy} = \int_0^\infty dz \ z \phi(z) \phi\left( \frac{y}{\langle y \rangle} \right) \frac{\langle y \rangle}{\langle y \rangle} = \frac{\langle y \rangle}{\langle y \rangle} = H(z),
\]

where \( z \) denotes \( n / \langle n \rangle \). Since the definitions of \( \langle y \rangle \) and \( \langle y \rangle \) lead to

\[
\int_0^\infty dz \ z \phi(z) \frac{\langle y \rangle}{\langle y \rangle} = 1,
\]

it turns out that

\[
\frac{\langle y \rangle}{\langle y \rangle} = H(z).
\]

Substituting Eq. (4.6) into Eq. (4.4), we get Eq. (4.3b) as
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\[ \frac{\langle y \rangle}{\langle n \rangle} \frac{d\sigma}{dy} = \int_0^\infty dz z H(z) \phi(z) \phi \left( \frac{H(z) \langle y \rangle}{\langle y \rangle} \right) \]

\[ = \Phi \left( \frac{y}{\langle y \rangle} \right). \quad (1.7) \]

In the same manner Eq. (4·3a) is derived from Eqs. (4·1) and (4·2a).

It is worth while noticing that the scaling functions in Figs. 1~4 show almost the same curvature regardless of the type of reactions. Hence, it is presumable that \( \langle y \rangle \sim \langle y \rangle \) except for small \( n \) region.

§ 5. Summary and discussion

On the basis of GSIM we have proposed scaling law for the rapidity distributions both in semi-inclusive and inclusive reactions. Their functional forms have been determined by means of information theory. However, as for the inclusive longitudinal-momentum distributions, one knows Feynman scaling which is different from ours. In this connection we suppose it necessary to comment on the relation between Feynman scaling and GSIM.

Feynman scaling assumes

\[ \frac{E}{\sigma} \frac{d\sigma}{dp_L} = G \left( \frac{2p_L}{\sqrt{s}} \right) \]

or

\[ \frac{1}{\sigma} \frac{d\sigma}{dy_{lab}} = g(y_{lab}) \]

in the laboratory system. Comparing these representations with Eq. (4·3), we notice that they behave contrary to our scaling formulas. Actually, we showed in the preceding paper\(^a\) that one cannot obtain Feynman scaling, even if one admits logarithmic violation at central region, if one assumes Eqs. (4·1) and (4·2a) in full momentum region. In that article we derived Feynman scaling in \( x = 2p_L / \sqrt{s} \neq 0 \) region and the violation of it at \( x = 0 \) by the use of the two-component model. The two-component model tells us that Feynman scaling is attributed to the diffractive mechanism, whereas the violation of scaling and the energy dependent rapidity plateau come from the multiperipheral mechanism. In this sense we understand the rise of the rapidity plateau with energy to reflect the increase of the number of resonances produced in the high energy hadron-hadron collisions. We guess that GSIM is assured by the mechanism which determines \( \langle y \rangle \). The increase of the rapidity plateau should never be identified with the fragmentation mechanism especially at high energies. Thus, the inconsistency between Feynman scaling and GSIM would be ascribed to these two mechanisms. Feynman scaling becomes valid because of the diffractive dissociation being dominant. On the other hand,
GSIM is valid if the mechanisms which determine $\langle n \rangle$, $\langle p_T \rangle$, $\langle y \rangle$, etc. are really efficient. Therefore, GSIM becomes more and more significant as energy increases.

Finally, we would like to give an example of useful applications of information theory. This theory provides us a method for constructing the functional form of the distributions, even if we do not have the complete set of information. For instance, we can investigate the quantities of the semi-inclusive rapidity distributions at $y=0$ as a function of the multiplicity in the following manner.

It turns out from Eq. (4·2b) that

$$\frac{1}{\sigma_s} \frac{d\sigma_n}{dy}_{y=0} = \frac{n_-}{\langle y \rangle_n} \phi(0)$$

(5·2)

for $pp \rightarrow n_-\pi^- + x$, and

$$\frac{1}{\sigma_s} \frac{d\sigma_n}{dy}_{y=0} = \frac{n_+}{\langle y \rangle_n} \phi(0)$$

(5·3)

for $pp \rightarrow n_+\pi^+ + x$. Requiring the law of charge conservation

$$\begin{cases} n_- = (n_z - 2)/2, \\ n_+ = n_- + (2 - n_p), \end{cases}$$

(5·4)

where $c$, $+$, $-$ denote the charge character of the relevant particles and $n_p$ is the number of leading protons, we obtain

$$\frac{1}{\sigma_s} \frac{d\sigma_n}{dy}_{y=0} = \frac{\phi(0)}{2\langle y \rangle_n} (n_z - 2)$$

(5·5)

for $pp \rightarrow \pi^-$, and for $pp \rightarrow \pi^+$

$$\frac{1}{\sigma_s} \frac{d\sigma_n}{dy}_{y=0} = \frac{\phi(0)}{2\langle y \rangle_n} (n_z + 2 - 2n_p).$$

(5·6)

For these expressions we put $n_p = 1.4$ and $\langle y \rangle_n$ being the decreasing function of $n_z$ because of the experimental studies.\(^4\) Equations (5·5) and (5·6) approximately represent the linear behaviors with respect to $n_0$. In Fig. 5 we show the comparison of our analysis with the data. It should be noted that the replacement of $n_0$ by $n_0 - n_0$ ($n_0 = 1$) improves the data fitting. It is the same situation as in the case of KNO scaling, where $n_0$ is the effect of the diffraction dissociation.$^{10}$

In this way, we can use information theory to search for the unknown mechanisms. If the reconstruction of the distribution function reproduces the more reliable features of the distributions because of the new information of the mechanism,

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\(^{4}\) $\langle y \rangle_n \approx 1.15, 1.05, 0.85, 0.80$ for $n_0 = 4, 6, 8, 10$ respectively in 60 GeV/c $pp$ reactions (see Ref. 12).
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we may suppose it to be a significant one. We shall be able to get the abundant information for hadron dynamics by iterating such a procedure.

Acknowledgements

We are particularly indebted to Professor S. Hori for his encouragement and reading the manuscript. We would like to thank Professor E. Yamada, Professor A. Takaya, our colleagues Dr. T. Toiya, Dr. M. Aoki and Dr. K. Kudo for their helpful comments and criticism.

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