Color-Magnetic Permeability of QCD Vacuum

Takesi SAITO and Kazuyasu SHIGEMOTO

Department of Physics, Kyoto Prefectural University of Medicine
Kyoto 603

*Department of Physics, Osaka University, Toyonaka 560

(Received October 27, 1979)

In the very strong background gauge field the QCD true vacuum has been shown to have lower energy than the "perturbative vacuum." The color-magnetic permeability of the QCD true vacuum is then calculated to be $1/2$ within the quark-one-loop approximation.

In a previous paper we have shown that the true ground-state vacuum of QCD (quantum chromodynamics) has lower energy than the "perturbative vacuum." The MIT bag constant may be given by the energy density difference between the perturbative vacuum and the true ground-state vacuum of QCD, because in the MIT bag model the perturbative vacuum is used inside the bag, while on the outside we have for the fields an impenetrable vacuum.

In the present paper we calculate the color-magnetic permeability $\mu_0$ of the QCD true vacuum and show this value to be $1/2$ within the quark-one-loop approximation. A gluon-loop contribution to the QCD vacuum is shown to be neglected when the number of quark flavor is large. If $\mu_0=0$, the vacuum is the perfect diamagnet so that quarks and gluons may be confined inside Abrikosov flux lines by the Meissner effect. However, in view of our result $\mu_0=1/2$ this confinement is still imperfect at this stage.

We begin with a review of our previous work. We adopt the following expression of the $SU(2)$ background gauge field:

$$A_s^a = K \delta_s^a, \quad (a=1,2,3; \mu=0,1,2,3)$$

where $K$ is a constant. This gives the constant field strength of the form

$$F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g \epsilon^{abc} A_i^b A_j^c$$

The energy density $E_0$ of the background gauge field $A_s^a$ is therefore, given by

$$E_0 = gK\epsilon^{aef}, \quad (i,j=1,2,3)$$

Contrary to Savvidy et al., the above choice of $A_s^a$ is apparently non-Abelian like. The energy density $E_0$ of the background gauge field $A_s^a$ is, therefore, given by
where $B^{0} = \sqrt{3} gK^{2}$ is the color-magnetic field strength.

Now we calculate a quark one-loop contribution to $E_{0}$ under the very strong background magnetic field strength $B$. The quark one-loop correction to the Yang-Mills classical action is given by the Feynman path integral

$$\Delta z = \int [d\phi] [d\overline{\phi}] \exp \left\{ i \int d^{4}x \left[ \overline{\phi} i \gamma^{\mu} \left( \partial_{\mu} - ig \frac{2}{g^{2}} A_{\mu}^{a} \right) \phi - m_{q} \overline{\phi} \phi \right] \right\} = \det \left( 1 + i \frac{1}{\gamma^{a} \cdot \partial - m_{q}} g \frac{2}{g^{2}} \gamma^{a} A_{\mu}^{a} \right),$$

apart from the normalization factor, where $A_{\mu}^{a}$ is the strong background field (1). This determinant will be calculated by neglecting 1 in Eq. (4) in the following way: (this approximation is valid when $A \ll gK$, $A$ being a cutoff parameter)

$$\Delta z \equiv \det \left( i \frac{1}{\gamma^{a} \cdot \partial - m_{q}} g \frac{2}{g^{2}} \gamma^{a} A_{\mu}^{a} \right) \quad \text{is given by}$$

$$\Delta z = i \int d^{4}x \left\{ -i \int \frac{d^{4}p}{(2\pi)^{4}} \ln \det \left( \frac{1}{\gamma^{a} \cdot \partial - m_{q}} g \frac{2}{g^{2}} \gamma^{a} A_{\mu}^{a} \right) \right\} = \exp \left[ i \int d^{4}x \left\{ -i \int \frac{d^{4}p}{(2\pi)^{4}} \ln \left[ \frac{9}{(\gamma^{a} \cdot \partial - m_{q})^{4}} gK^{2} \right] \right\} \right] = \exp \left[ i \int d^{4}x \Delta L \right],$$

Hence the $\Delta L$ is given by

$$\Delta L = -i \int \frac{d^{4}p}{(2\pi)^{4}} \sum_{q} \ln \left[ \frac{9}{(\gamma^{a} \cdot \partial - m_{q})^{4}} gK^{2} \right],$$

where the summation over quark flavor $q$ is necessary if one considers many-flavor case. After Wick rotation Eq. (6) can be integrated in the form$^{3}$

\[ \int dx \ln(x^2 - b^2 - a^2) = \frac{1}{2} \left( x^2 - \frac{b^2}{a^2} \right) \ln(x^2 + b^2) - \frac{1}{2} \left( \frac{b}{a} \right)^2 \left( x^2 - \frac{b^2}{a^2} \right). \]

From (4) one can see $\Delta L$ to be zero at $K=0$. However, this behavior does not appear in (7), because (7) is valid only for large $K$. 

$^{3}$ Here we have used the following formulae:
where \( A \) is a cutoff parameter. In terms of \( B \) defined by Eq. (3) the total energy density \( E \) is now

\[
E = E_0 - \Delta L
\]

where \( \Delta L \) is given by

\[
\Delta L = \sum_q \left[ \frac{A^q - m_q^2}{8\pi^2} \ln \frac{\sqrt{3} g^2 K^2}{4 (A^q + m_q^2)} + \frac{m_q^4}{8\pi^2} \ln \frac{\sqrt{3} g^2 K^2}{4 m_q^2} + \frac{1}{8\pi^2} \left( \frac{A_q^2 - m_q^2 A^2}{2} \right) \right].
\]

The minimum for the energy density \( E \) is found when

\[
E = E_0 - \frac{B^2}{2} \sum_q \left[ \frac{A^q - m_q^2}{8\pi^2} \ln \frac{g B}{4 (A^q + m_q^2)} + \frac{m_q^4}{8\pi^2} \ln \frac{g B}{4 m_q^2} + \frac{1}{8\pi^2} \left( \frac{A_q^2 - m_q^2 A^2}{2} \right) \right].
\]

where \( N_f \) is the number of quark flavor. Substituting (9) into (8), we have

\[
E_{\text{min}} = \frac{B^2}{2} \sum_q \left[ x_q - \frac{c}{1 + x_q} \ln \frac{c}{x_q} - x_q \frac{c}{x_q} \ln \frac{c}{x_q} \right],
\]

where

\[
x_q = m_q^2 \sqrt{N_f} / (\sqrt{8\pi} B_0),
\]

\[
c = g \sqrt{N_f} / (8\sqrt{2} \pi).
\]

Since \( B_0 \) is assumed to be large, i.e., \( x_q \ll 1 \), Eq. (10) is reduced to

\[
E_{\text{min}} = -B_0^2 \ln c = -B_{\text{bag}}.
\]

Thus we find that the true ground-state vacuum of QCD has lower energy than the perturbative vacuum in the strong coupling region and therefore the bag constant \( B_{\text{bag}} \) is approximately given by Eq. (11).

So far we have neglected a gluon-loop contribution to \( E \). In the Appendix we shall show that this can be neglected when the number of quark flavor is large, i.e., \( 8N_f \gg 12 \).

It should be noted that our calculation is made in the strong coupling region, so that the usual renormalizable perturbative calculation is not applicable here. This is a reason why we have introduced the cutoff parameter \( A \). In the small coupling region, of course, the renormalizable perturbative calculation is applicable.

The equation (9) means \( \sqrt{N_f} A = g K^2 \). On the other hand, Eq. (5) is valid when \( A < gK \). These two conditions are satisfied if \( 1 < g \sqrt{N_f} \) (see Eq. (12)).
by keeping 1 inside Eq. (4). In this case, however, we never find the negative $E_{\text{min}}$, but $E_{\text{min}}=0$.

We are now in a position to calculate the color-magnetic permeability $\mu_c$ of the QCD true vacuum. For this purpose we set

$$B=B_0+B',$$

where $B_0$ is defined by (9) and $B'$ is regarded as a weak field strength. Substituting (13) into (8) and expanding it with respect to $B'$ up to $B'^2$, we have

$$E=E_{\text{min}}+B'E'(B_0)+\frac{B'^2}{2}E''(B_0),$$

where

$$E'(B_0)=B_0-\frac{N_fA^4}{8\pi^2}\frac{1}{B^4}=0,$$

$$E''(B_0)=1+\frac{N_fA^4}{8\pi^2}\frac{1}{B^6}=2.$$ (16)

We now choose $E_{\text{min}}=0$; that is, the energy density of true vacuum is zero. The permeability $\mu_c$ is then defined by

$$E=\frac{1}{2\mu_c}B'^2,$$

so that

$$\mu_c=1/E''(B_0)=1/2.$$ (18)

This shows that the QCD true vacuum is a colored diamagnet, but not a perfect diamagnet in the present approximation. Therefore, quark confinement is still imperfect at this stage.

Acknowledgements

The authors are grateful to Dr. T. Kugo for valuable discussions. Thanks are also due to Professor H. Masuda for critical reading of the manuscript.

Appendix

The QCD Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}=-\frac{1}{4}(F_{\mu\nu}^a)^2+\bar{\psi}\left(i\gamma^\mu\partial_\mu-m+\frac{e}{2}\gamma^\mu A^\mu\right)\psi.$$ 

Now we set $A_{\mu}^a=Kb_{\mu}^a+b_{\mu}^a$ in $\mathcal{L}$. Expanding $\mathcal{L}$ with respect to $b_{\mu}^a$, we get, for large $K$, 

\[ \mathcal{L} = -\frac{1}{2} B^\nu + \bar{\psi} \left( i \gamma ^\nu \partial _\nu - m + g \frac{x^a}{2} T^a K \partial _\nu \psi \right) \]
\[ -2 g^2 K^a b_i ^a + \frac{1}{2} b_i ^a \mathcal{M}^{ab}_{ij} b_j ^b + O ( \partial ^2, \bar{\psi} \gamma ^\nu \psi ) , \]

where

\[ \mathcal{M}^{ab}_{ij} = ( g^a - \partial ^a ) \partial _i ^j + g^2 K^a \{ (-2 g^a + \partial ^a ) \partial _i ^j - \partial _j ^i \partial _i ^j \} . \]

The second term in \( \mathcal{L} \) has been considered in the text. In this appendix the third and fourth terms will be considered. We take a gauge such that

\[ b_1 ^i = b_2 ^i = b_3 ^i = 0 . \]

Hence the third term in \( \mathcal{L} \) vanishes. The infinitesimal gauge transformation

\[ \delta A_r ^k = -\frac{1}{g} \partial _\nu \theta ^i + \epsilon _{ijk} \theta ^j A_r ^k \]

is now reduced to

\[ \delta b_i ^a = -\frac{1}{g} \partial _\nu \theta ^i + \epsilon _{ijk} \theta ^j ( K \partial _\nu b_i ^a + b_i ^a ) \]

or

\[ \frac{\delta b_i ^a}{\delta b_i ^a} = -\frac{1}{g} \left( \begin{array}{ccc} \partial _1 & -g ( K \partial _1 ^i + b_1 ^i ) & g ( K \partial _2 ^i + b_2 ^i ) \\ -g ( K \partial _1 ^i + b_2 ^i ) & \partial _2 & -g ( K \partial _3 ^i + b_3 ^i ) \\ -g ( K \partial _3 ^i + b_3 ^i ) & \partial _3 & \partial _i \end{array} \right) \left( \begin{array}{c} \theta ^i \\ \theta ^j \\ \theta ^k \end{array} \right) \]

\[ = M \theta . \]

Hence, the Fadeev-Popov determinant which is given by \( J = \det M \) is independent of \( K \). Therefore, for large \( K \) the Fadeev-Popov ghost contribution to the vacuum energy may be neglected.

The gluon-one-loop contribution to the vacuum energy is finally calculated, i.e.,

\[ J_{\text{gluon}} = \int \left[ \frac{1}{2} d^4 x \right] \exp \left\{ i \int d^4 x \left[ \frac{1}{2} ( b_i ^a \mathcal{M}^{ab}_{ij} b_j ^b + b_i ^a \mathcal{M}^{ab}_{ij} b_j ^b + O ( \partial ^2, \bar{\psi} \gamma ^\nu \psi ) \right) \right\} \]

\[ = \exp \left[ \frac{1}{2} \int d^4 x \left\{ -A_i ^a \mathcal{M}^{ab}_{ij} b_j ^b + O ( \partial ^2, \bar{\psi} \gamma ^\nu \psi ) \right\} \right], \]
which should be compared with the fermion-loop contribution (5), i.e.,

$$\Delta z_{\text{fermion}} = \exp \left[ i \int d^4x \sum_{\sigma \nu} \frac{A_\sigma^\nu}{32\pi^2} \ln \left( \frac{gK}{A} \right) \right].$$

Therefore, for the large number of $N_f$, i.e., $8N_f \gg 12$, one can neglect the gluon-loop contribution.

References