Intrinsic Angular Momentum and Mass Current in Superfluid \(^{3}\)He-A

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Expressions for the mass current of \(^{3}\)He-A phase are discussed with an emphasis on its dependence on the intrinsic angular momentum density, both microscopically and phenomenologically. A brief discussion is also given concerning the distorting effect of Bogoliubov excitations on the Cooper pair wavefunction.

§ 1. Introduction

The magnitude of the intrinsic orbital angular momentum in the ABM ground state was discussed by one of us. This problem is still controversial. However it has turned out that the basic point at issue is the structure of the BCS ground state wavefunction. We shall use the function

\[
|\Phi\rangle = \frac{1}{A} \exp Q^* |0\rangle
\]

with

\[
Q^* = \frac{1}{2} \int d^3r_1 d^3r_2 \varphi_{\alpha \beta}^* (r_1, r_2) \varphi_{\alpha \beta}^+ (r_1, r_2), \tag{1.1}
\]

where \(\varphi_{\alpha \beta} (r_1, r_2)\) is the wavefunction of a Cooper pair and \(A\) is a constant for normalization. It was shown in the paper that the magnitude of the total angular momentum \(L\) in the state (1.1) is \(\frac{1}{2} \hbar N\) with \(N\) being the averaged total number of particles in the system. If the system is homogeneous and has a constant order parameter throughout, we will refer to the angular momentum \(L\) as the intrinsic one \(L_0\).

On the other hand Combescot postulated that in the limit \(\delta \to 0\) the resultant state should be the normal state which has zero angular momentum and proposed the state function

\[
|\Phi\rangle = \frac{1}{B} \exp \bar{Q}^* |\Phi_n\rangle,
\]

\(*\) A preliminary report was given at ULT. Hakone.
\[ \tilde{\Phi}(r_1, r_2) = \frac{1}{2} \int \int d^3r_1 d^3r_2 \left( f_{a+b}^{(+)}(r_1, r_2) \phi_a(r_1) \phi_b(r_2) - f_{a+b}^{(-)}(r_1, r_2) \phi_b(r_1) \phi_a(r_2) \right) \]  

where \( |\phi_n\rangle \) is the normal ground state and the function \( f_{a+b}^{(+)}(r_1, r_2) \) consists of the one-particle states outside the normal Fermi sphere while \( f_{a+b}^{(-)}(r_1, r_2) \) consists of those inside. In fact the wavefunction (1.2) gives zero intrinsic angular momentum.

Combescot's state \( \tilde{\Phi}(1.2) \), if it can be constructed with one particle states with discrete quantum numbers, is equivalent to the standard BCS state \( \Phi_{BCS} \) except a phase factor \( \prod_n <\tilde{\psi}_n|\Phi> \): The state \( \Phi_{BCS} \) can be transformed identically as

\[ \Phi_{BCS} = \prod_n \langle u_n + \tilde{\psi}_n c_n^+ c_n^+ \rangle 0 \rangle \]

where

\[ f_{a+b}^{(+)}(r_1, r_2) = \sum_n \theta (\tilde{\psi}_n - E_f) \frac{\tilde{\psi}_n}{u_n} \phi_a(r_1) \phi_b(r_2) \]

and

\[ f_{a+b}^{(-)}(r_1, r_2) = \sum_n \theta (E_f - \tilde{\psi}_n) \frac{\tilde{\psi}_n}{v_n} \phi_a(r_1) \phi_b(r_2) \]

with \( \phi_n \) being the one particle state corresponding to the quantum number \( n \). We have denoted the pairing states by \( n \) and \( \overline{n} \).

On the other hand Brinkman and Cross 10 doubted equivalence of the Ambegaokar state \( \Phi(1.1) \) and \( \Phi_{BCS}(1.1a) \): If these states were equivalent, the function \( \varphi_n \) in \( \Phi \) should be given by \( \varphi_n / u_n \), which will give \( \phi(r_1, r_2) = \sum \varphi_n \phi_n(r_1) \phi_b(r_2) \) not well-defined when some of \( u_n \)'s are zero. The singularity of \( \varphi_n \) due to the node of the gap \( \Delta \) is not harmful because of its vanishing measure. The singularity of \( \varphi_n \) arising from zero gap in the core region of the Fermi sphere in the weak-coupling approximation of BCS may be considered non-existing; as was shown by Bogoliubov 11 in discussing the effect of Coulomb repulsion on superconductivity, the repulsion between the particles near the Fermi surface and the particles in the core region makes up coherence and gives nonvanishing \( \Delta \), which in its turn keep \( u_n \) non-vanishing even in the core. Therefore physically \( \varphi_n \)'s are well-defined everywhere and we conclude \( \Phi(1.1) \) and \( \Phi_{BCS}(1.1a) \) are equivalent as long as it can be constructed with discrete one particle states. However when we take up general \( \varphi \) or \( f^{(+) \dagger} \) the function \( \Phi(1.1) \) and \( \tilde{\Phi}(1.2) \) are not necessarily equivalent and there arises a question as to which is physically realized.

Here we prefer the expression (1.1) because we expect a supercurrent \( \rho \) at
$T=0$, which is naturally obtained by displacing all the atoms by an equal momentum $m\mathbf{v}_c$ without destroying the coherent structure. If we then let $J\rightarrow 0$ we will get the normal ground state flowing with the velocity $\mathbf{v}_c$. This flowing state has nothing to do with superfluidity because it does not correspond to a stationary value of energy. Therefore, our state (1-1) will get unstable at a moment when $J$ attains a certain critical value dependent on $\mathbf{v}_c$. Beyond that we have no reason to claim that our state be correct. This argument shows that Combescot's postulate is not physical and need not be obeyed. Besides in order to get an equivalent current structure from (1-2) one would have to move the normal Fermi core coherently with the Cooper pairs, which coherence will easily be destroyed because it is normal.

In this paper we discuss the relation between the mass current and the intrinsic angular momentum. According to the microscopic calculations by Wolff, Cross and Blount, the supercurrent is given as

$$j_s = \rho_c \cdot \mathbf{v}_c + C \cdot (\mathbf{r} \times \mathbf{l}) + \frac{1}{2} \mathbf{r} \times (\mathbf{L} \cdot \mathbf{d})$$

(1.3)

with $C_{ij} = C_{0ij} - C_{ij}v_c$. The tensor $C$ is of the form $C(T) (1 - 2\mathbf{I})$ in weak coupling approximation and $C(T=0)$ is equal to $\frac{1}{2} \hbar n$. $L_0$ is Cross' intrinsic angular momentum density which is equal to $\hbar n_s (T_c/E_F)^2$ except a numerical factor. However, as we shall show in the following, the BCS ground state (1-1) gives another result

$$j_s = \rho_c \cdot \mathbf{v}_c + \frac{1}{2} \mathbf{r} \times \left( \frac{1}{2} \hbar n \mathbf{l} \right),$$

(1.4)

in contrast with the result obtained by Cross and Blount.

Wolff as well as Cross and Blount made calculation in the plane wave representation indifferently to (1-1) or (1-2). It is therefore necessary to scrutinize the coefficients of the gradient terms in the free energy. We should also clarify the physical meaning of the term $\mathbf{r} \times \mathbf{l}$ in (1.3) in comparison with the third term $\frac{1}{2} \mathbf{r} \times (\mathbf{L} \cdot \mathbf{d})$.

The second point we discuss in this paper is the effect of the Bogoliubov excitations on the value of the orbital angular momentum or of the mass current at finite temperatures. It was pointed out by Leggett and Takagi that one would get $L_{\text{int}} = \frac{1}{2} \hbar N$ at all temperatures if one extend the state vector (1-1) to excited states. However the existence of Bogoliubov excitations will deform the Cooper pair wavefunction so that this is no more the eigenstate $l_z=1$. We should take due account of this situation.

$^*$ Since Wolff and Blount calculated $j_s$ on the basis of particle-hole symmetry, the third term in (1.3) is missing in their expressions. Moreover, Wolff's expression is valid in the region $T \geq T_c$. 
We discuss the mass current and the free energy in § 2 and the effect of Bogoliubov excitations in § 3.

§ 2. Free energy and mass current

We first investigate the structure of the mass current density expected in the inhomogeneous BCS state, Eq. (1·1). We reduce this wavefunction \( \Phi \) to the density matrix for one-body:\(^6\)

\[
\rho(r, r') = \langle \Phi | \phi^+(r) \phi(r') | \Phi \rangle = \int d^3r \phi^*(r, r) \Phi (r, r') = \int d^3r' \Phi^*(r, r') \phi (r, r').
\]

(2·1)

Here we have suppressed the spin quantum number and summation over it because we are going to discuss the properties which are independent of the spin. The function \( \Phi (r, r') \) here is the order parameter defined by

\[
\Phi (r, r') = \langle \Phi | \phi (r) \phi (r') | \Phi \rangle.
\]

(2·2)

Inserting Eq. (2·1) into the defining equation for the mass current density

\[
j(r) = \frac{i \hbar}{2} \left( \frac{\partial}{\partial r} - \frac{\partial}{\partial r'} \right) \rho(r, r') |_{r' \rightarrow r},
\]

(2·3)

we get

\[
j(r) = - \frac{i \hbar}{2} \int d^3r \left\{ \phi^*(r, r) \frac{\partial}{\partial r} \Phi (r, r') - \frac{\partial \Phi^*(r, r')}{\partial r} \phi (r, r) \right\}.
\]

(2·3')

Let us write the Cooper pair wavefunction \( \psi \) and the order parameter \( \Phi \) of the ABM state in the following way:

\[
\psi(r, r') = \psi(\rho; R) = \lambda(\rho \cdot l, \rho \times l; R) \tilde{d}(R) \cdot \tilde{\phi}
\]

and

\[
\Phi (r, r') = \Phi (\rho; R) = \chi(\rho \cdot l, \rho \times l; R) \tilde{d}(R) \cdot \tilde{\phi},
\]

(2·4a)

where \( \tilde{d} \) is a complex vector which satisfies

\[
d \cdot d = 0, \quad d^* \cdot d = 2 \quad \text{and} \quad l = \frac{i}{2} d \times d^*
\]

(2·4b)

and

\[
\rho = r - r', \quad R = (r + r')/2 \quad \text{and} \quad \tilde{\phi} = \rho/|\rho|.
\]

(2·4b')

We assume \( \chi \) to be real, which makes \( \lambda \) also be real.

We consider here a situation in which the order parameter and the Cooper pair wavefunction vary very little over distances shorter than \( \xi_0 \), the coherence length. We may then Taylor-expand the integrand in Eq. (2·3)', considered as
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a function $\mathbf{R} = \frac{(r_1 + r)}{2}$, around the point $\mathbf{R} = r$, up to first order in deviation. The result is

$$j(r) = -\frac{i\hbar}{4} \int d^3\rho \varphi^*(\rho; r) \frac{\partial \mathbf{T}(\rho; r)}{\partial \mathbf{r}}$$

$$+ \frac{i\hbar}{4} \frac{\partial}{\partial r_i} \left( \int d^3\rho \varphi^*(\rho; r) \frac{\partial \mathbf{T}(\rho; r)}{\partial \rho} \right) + \text{c.c.} \quad (2.5)$$

The $j$-th component of the second term on the r.h.s. of Eq. (2.5) can be rewritten as

$$\frac{i\hbar}{4} \frac{\partial}{\partial r_i} \left( \int d^3\rho \varphi^*(\rho; r) \frac{\partial \mathbf{T}(\rho; r)}{\partial \rho} \right)$$

$$= \frac{i\hbar}{4} \frac{\partial}{\partial r_i} \left( \int d^3\rho \varphi^*(\rho; r) \left( \rho_i \frac{\partial}{\partial \rho_j} - \rho_j \frac{\partial}{\partial \rho_i} \right) \mathbf{T}(\rho; r) \right)$$

$$+ \frac{i\hbar}{4} \frac{\partial}{\partial r_i} \left( \int d^3\rho \varphi^*(\rho; r) \left( \rho_i \frac{\partial}{\partial \rho_j} + \rho_j \frac{\partial}{\partial \rho_i} \right) \mathbf{T}(\rho; r) \right),$$

of which the second term is purely imaginary because it can be transformed by utilizing the structure of $\varphi$ and $\mathbf{T}$, Eqs. (2.4a), (2.4b), as follows:

$$\int d^3\rho \varphi^*(\rho; r) \left( \rho_i \frac{\partial}{\partial \rho_j} + \rho_j \frac{\partial}{\partial \rho_i} \right) \mathbf{T}(\rho; r)$$

$$= \int d^3\rho \left\{ \mathbf{d} \cdot \mathbf{\hat{z}} \left[ \lambda \left( \rho_i \frac{\partial}{\partial \rho_j} + \rho_j \frac{\partial}{\partial \rho_i} \right) \mathbf{z} - 2\rho_i \frac{\partial}{\partial \rho_i} \right] \right\}$$

$$+ k_\lambda |\mathbf{\hat{z}} \times \mathbf{1}| \left[ \mathbf{d} \times \mathbf{d} + \mathbf{d} \times \mathbf{d} \right],$$

which clearly shows it is real. Therefore the mass current density (2.5), in the case of ABM state, reduces to

$$j_i(r) = -\frac{i\hbar}{4} \int d^3\rho \varphi^*(\rho; r) \frac{\partial \mathbf{T}(\rho; r)}{\partial r_i} - \text{c.c.}$$

$$+ \frac{1}{2} \frac{\partial}{\partial r_j} \left[ \frac{i\hbar}{2} \int d^3\rho \varphi^*(\rho; r) \left( \rho_j \frac{\partial}{\partial \rho_i} - \rho_i \frac{\partial}{\partial \rho_j} \right) \mathbf{T}(\rho; r) + \text{c.c.} \right]$$

$$= \rho v_n + \frac{1}{2} \left( \mathbf{\Gamma} \times \mathbf{L}_n \right)_i. \quad (2.6)$$

Here we have denoted the intrinsic angular momentum density by $\mathbf{L}_n$:

$$\mathbf{L}_n = -\frac{i\hbar}{4} \int d^3\rho \varphi^*(\rho; r) \left( \mathbf{\rho} \times \frac{\partial}{\partial \rho} \right) \mathbf{T}(\rho; r) + \text{c.c.}$$

$$= \frac{\hbar}{2} \eta(r) \mathbf{I}(r), \quad (2.7)$$
where \( n(r) \) is the number density, while \( \rho \) represents the mass density and \( v_s \), the superfluid velocity.\(^4\)

The supercurrent density in the ABM state is given in general by Eq. (1.3) with \( C_{ij} \) being an axial symmetric tensor (see (2.15)). Comparison of Eq. (2.6) with Eq. (1.3) gives

\[
C_{ij} = C_{00} - C_{0i}, \quad j = 0, 1, 2
\]

\[
L_0 = \frac{1}{2} \hbar n(r), \quad (2.8)
\]

which are valid in the ground state Eq. (1.1). That is, in contrast with the results by Cross\(^0\) and Blount,\(^5\) \( L_0 \) is much larger while \( C = C_0 = 0 \). In other words there is no term proportional to \( \mathbf{F} \times \mathbf{I} \) in the supercurrent density. The situation at finite temperatures would be quite different. However, it seems to be very difficult to do a similar analysis at finite temperatures.

Therefore it is appropriate here to study the structure of the phenomenological expressions for the free energy density and the mass current density. The order parameter of the ABM state which we are concerned with is a bivector \( \mathbf{d}_{a*}(r) \) and the free energy \( \mathbf{F} \) is in general, a functional of a gauge invariant \( \mathbf{d}_{a*}(r_i) \mathbf{d}_{\beta}(r_j) \). The non-local part of the free energy density \( \mathbf{F}(r) \), which we shall call gradient part, may be written as

\[
\mathbf{F}_{\text{grad}}(r) = \sum_{a,b} \int d^3r \mathfrak{d}^a \mathfrak{d}^a \mathbf{K}_{ab}(r; r_1, r_2) \left[ \mathbf{d}_{a*}(r_1) \mathbf{d}_{\beta}(r_2) - \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r) \right] \quad (2.9)
\]

which is correct up to second order in the non-locality. Here the kernel

\[
\mathbf{K}_{ab}(r; r_1, r_2) = \mathbf{K}_{ab}(r; r_2, r_1)
\]

is a functional of \( \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r) \). If the wavelength of the inhomogeneity is larger than the coherence length \( \xi \), we can put

\[
\mathbf{K}_{ab}(r_1, r_2) = \mathbf{K}_{ab}(r_1 - r, r_2 - r) = \mathbf{K}_{ab}(\rho_1, \rho_2).
\]

Substitution of a series expansion of the non-locality

\[
\mathbf{d}_{a*}(r_1) \mathbf{d}_{\beta}(r_2) - \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r) = \rho_1 \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r) + \rho_1 \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r) + \rho_2 \mathbf{d}_{a*}(r) \mathbf{d}_{\beta}(r)
\]

\[
+ \rho_1 \rho_2 \mathbf{d}_{a*} \mathbf{d}_{\beta} + \frac{1}{2} \rho_1 \rho_2 \mathbf{d}_{a*} \mathbf{d}_{\beta} + \frac{1}{2} \rho_1 \rho_2 \mathbf{d}_{a*} \mathbf{d}_{\beta} + \cdots
\]

into the integral expression for \( \mathbf{F}_{\text{grad}}(r) \) gives

\(^4\) The relation between \( L_0 \) here and the total angular momentum density \( \mathbf{L} \) in §4 of Ref. 1 was discussed previously. (See Ref. 13.)
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\[ F_{\text{grad}}(r) = T_{\text{grad}} \partial_i \vec{d}_a \partial_i \vec{d}_b \]
\[ + \frac{1}{2} U_{\text{grad}} (\partial_i \vec{d}_a \partial_i \vec{d}_a^*) \vec{d}_b + \frac{1}{2} U_{\text{grad}} (\partial_i \vec{d}_a \partial_i \vec{d}_b) \]
\[ + \cdots, \quad (2.10) \]

where we have taken into account that

\[ \int \int K_{a \beta} (\rho_1, \rho_2) \rho_1 \rho_2 \partial_1 \partial_2 \rho_1 \rho_2 = 0. \]

This is because the only vector we can construct from $\vec{d}_a(r)$ is $\vec{d}_a \times d^*_a$, which is inversion symmetric while $\rho_i$ is antisymmetric. We also have introduced the notation,

\[ \int \int K_{a \beta} (r; \rho_1, \rho_2) \rho_1 \rho_2 \partial_1 \partial_2 \rho_1 \rho_2 = T_{\text{grad}}. \]

However the quadratic combinations of $d$'s and their derivatives that are invariant under gauge transformations and spatial rotations are the following:

\[ (1) \partial_a \vec{d}_a \partial_b \vec{d}_b, \quad (2) \partial_a \vec{d}_a \partial_b \vec{d}_b, \quad (3) \partial_a \vec{d}_a \partial_b \vec{d}_b, \]
\[ (1)' \vec{d}_a \partial_a \partial_b \vec{d}_b + \text{c.c.}, \quad (2)' \vec{d}_a \partial_a \partial_b \vec{d}_b + \text{c.c.} \quad \text{and} \quad (3)' \vec{d}_a \partial_a \partial_b \vec{d}_b + \text{c.c.} \]

\[ (2.11) \]

The expression (3)' is, as a matter of fact, equivalent to (1)'.

De Gennes assumed $F_{\text{grad}}$ in the temperature region $T \sim T_c$ ("GL region") to be given by a sum of the invariants (1), (2) and (3) with coefficients $\gamma_1$, $\gamma_2$, and $\gamma_3$ respectively, excluding (1)' and (2)'. However, while we have to take the latter invariants also into account as the coefficients $T's$ and $U's$ are dependent on the coordinate $r$, we have to take account of the fact that the symmetry of the ABM state, in general, is not the invariance under arbitrary spatial rotations but the invariance under those rotations around the local $l$-axis. Hence we separate these five expressions into 12 expressions with the axial symmetry. The gradient free energy $F_{\text{grad}}$ can be written in the following form of a linear combination with coefficients $\gamma_1'$ and $\gamma_2'$. The suffix of $\gamma$ and $\kappa$ indicates the original group (2.11) and the superfix represents half of the number of longitudinal-components (parallel to $l$). We need here two independent $\gamma_i$'s, which we denote $\gamma_1'$ and $\gamma_2'$. Considering also the fact that $\vec{d}_a$ as a spatial vector is perpendicular to $l$, we get

\[ F_{\text{grad}} = \gamma_1' l_1 \vec{d}_a \partial_1 \vec{d}_a \partial_2 \vec{d}_2 + \gamma_2' l_2 \vec{d}_a \partial_2 \vec{d}_a \partial_1 \vec{d}_1 + \text{c.c.} \]
As is seen in (2.12), the sum \( \gamma^4 + \gamma^6 + \gamma^8 \) makes an independent parameter so that only eight of \( \gamma^i \)'s are independent.

If divergence terms are neglected, some of the axial invariants derived from (1) and (2) can be transformed into expressions of the same type as derived from (1) to (3). Throwing away these terms and expressing \( d \) in terms of \( I, v_s \)

and \( d \) we get from Eq. (2.12)

\[
F_{\text{grad}}(I, v_s, d) = \left( \frac{2m}{h} \right) \left( \gamma^4 + \gamma^6 + \gamma^8 \right) (v_s \times I)^2 + \left( \frac{2m}{h} \right) \left( \gamma^4 - \gamma^6 \right) (v_s \times I)^2
\]

\[
+ \frac{2m}{h} v_s \cdot [1 - \mathbb{I}] \cdot \{ (\gamma^4 - \gamma^6) (F \times I) + \gamma^8 (F \times v_s \times F) \}
\]

\[
+ \frac{2m}{h} v_s \cdot [1 - \mathbb{I}] \cdot \{ (\gamma^4 - \gamma^6 - \gamma^8 \frac{\kappa^4}{2}) (F \times I) + \gamma^8 (F \times v_s \times F) \}
\]

\[
+ \frac{1}{2} d^2 (\gamma^4 + \gamma^6 + \gamma^8 - \kappa^4) (I \times (F \times I))^2
\]

\[
+ \frac{1}{2} d^2 (\gamma^6 + \kappa^4 - \kappa^6 - \kappa^8) (F \cdot I)^2 + \frac{1}{2} d^2 (\gamma^8 - \kappa^8) (I \cdot F \times I)^2
\]

\[
- \frac{2m}{h} d^2 (\gamma^4 + \kappa^4 - \kappa^6 - \kappa^8) (F \times v_s \cdot I)
\]

\[
+ (\gamma^4 + \gamma^6) (I \times F \cdot d)^2 + \gamma^4 (I \cdot F \cdot d)^2
\]

\[
- \frac{1}{2} (\gamma^4 + \kappa^4 - \kappa^6 - \kappa^8) d l \cdot F \cdot F \cdot d
\]

\[
- \frac{1}{2} d^2 (\kappa^4 \cdot I \times (F \times I)) + (\kappa^4 - \kappa^6) d l \cdot F \cdot F \cdot d
\]

\[
+ \kappa^6 d l \cdot F \cdot d + \frac{m}{h} d^2 \kappa^6 \cdot (v_s \times I) - \frac{1}{4} \kappa^4 [1 - \mathbb{I}] \cdot F \cdot d^2.
\]
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Here we have neglected spatial variation of the spin part of \( \overrightarrow{d} \) and we also have utilized the following relations:

\[
\overrightarrow{d}_{ab} = \Delta d_a V_t \\
\delta d_a = -I_a \delta I_d + i \delta \phi d_a
\]

and

\[
v_{oa} = \frac{\hbar}{2m} \overrightarrow{d}_b \cdot \overrightarrow{r}_a d_b.
\] (2.14)

We have also introduced the abbreviations \( r_s = (r_1^s + r_3^s)/2 \).

To get the gradient free energy in the case of normal fluid also flowing, we only have to replace \( v_s \) in Eq. (2.13), except that in \( F \times v_s \) by \( v_s - v_n \) and subtract \( \rho v_n^2 \). This is because of Galilean relativity. The reason why the term \( F \times v_s \) makes an exception is that it can be written solely in terms of \( I \) and \( \overrightarrow{r} I \) according to Mermin and Ho's relation.\(^{10}\) We shall refrain from writing down the expression of \( F_{\text{grad}}(I, v_s, v_n) \) because it is a lengthy expression analogous to Eq. (2.13).

The mass current density \( j \), which is equal to \(-\delta F_{\text{grad}}/\delta v_n\), is therefore given by

\[
j = 8 \left( \frac{m}{\hbar} \right)^2 \overrightarrow{d} \left\{ (r_1^s + r_3^s - \kappa_s^1) [I - II] + (r_1^s - \kappa_s^1) II \right\} \cdot \overrightarrow{v}_s
\]

\[
+ \frac{2m}{\hbar} \overrightarrow{d} \left\{ (r_1^s - r_3^s) [I - II] - (r_1^s + r_3^s - \kappa_s^1 - 2\kappa_s^1) II \right\} \cdot (F \times I)
\]

\[
+ \frac{2m}{\hbar} \gamma \cdot F \times (\overrightarrow{d} I) + \frac{m}{\hbar} \overrightarrow{d} (I \times \gamma \kappa_s^1)
\]

\[
+ \left[ \rho - 8 \left( \frac{m}{\hbar} \right)^2 \overrightarrow{d} \left\{ (r_1^s + r_3^s - \kappa_s^1) [I - II] + (r_1^s - \kappa_s^1) II \right\} \cdot \overrightarrow{v}_s \right].
\] (2.15)

This expression for the current density \( j \), which is derived from the gradient energy (2.14) of which the structure has been determined so as to conform to the symmetry of the ABM state, reproduces the various results reported thus far\(^{6,12}\) if we give appropriate values to \( \gamma_i^s \) and \( \kappa_i^s \).

However, if we assume further that there is distributed intrinsic angular momentum of density \( L_i^s \) we get an additional condition to be satisfied by the coefficients. In this case, we shall have a contribution \( \frac{1}{2} F \times (L_i^s) \) in the current density,\(^{12}\) which, in its turn, will give rise to a term of the form \( \frac{1}{2} (F L_o) \times I \). Picking out terms of such a form from (2.15) we get the following equation:

\[
\frac{1}{2} F L_o = \frac{m}{\hbar} \left( 2T - \delta^s \overrightarrow{d} \cdot \overrightarrow{d} \gamma^s \right).
\] (2.16)
Solution of this differential equation tells that

\[ 2\gamma_+^{0} = -\kappa_+^{0} + f(\mathcal{J}) , \] (2.17)

where \( f \) is some function of \( \mathcal{J} \) only. Then Eq. (2.15) reduces to (1.3) with

\[
\begin{align*}
L_0 &= \frac{2m}{\hbar} \left[ 2\gamma_+^{0} \mathcal{J} - \mathcal{J} f(\mathcal{J}) + \int_0^{\mathcal{J}} f(x) \, dx \right], \\
C &= \frac{2m}{\hbar} \mathcal{J} \gamma_+^{1} - L_0/2
\end{align*}
\]
and

\[
C_0 = -\frac{m}{\hbar} \mathcal{J}^2 (4\gamma_+^{1} + \kappa_+^{0} - \kappa_+^{1}) .
\] (2.18)

We may put the lower limit of integration in the expression of \( L_0 \) equal to zero if we assume that \( L_0 \) vanishes at \( T = T_c \) and \( f(x) \) is regular at \( x=0 \).

Our expression (2.6) for \( j \) at \( T=0 \), in particular, corresponds to the following values of \( \gamma \) and \( \kappa \):

\[
\begin{align*}
\gamma_+^{1} &= \gamma_-^{0} - \frac{1}{2 \mathcal{J}^2} \int_0^{\mathcal{J}} x f'(x) \, dx = \gamma_+^{1} - \kappa_+^{1} \\
&= \gamma_+^{0} - \gamma_+^{1} - \kappa_+^{1} = \hbar^2 n/8m \mathcal{J}^2
\end{align*}
\]
and

\[ 4\gamma_+^{1} + \kappa_+^{0} - \kappa_+^{1} = 0 . \]

(2.19)

As the temperature approaches \( T_c \) the coefficients should recover spherical symmetry. Therefore we can take \( \gamma_+^{1} = \gamma_+^{0} \) and \( \kappa_+^{1} = \kappa_+^{0} \) in the GL region and Eq. (2.15) reduces to

\[
j = 8 \left( \frac{m}{\hbar} \right)^2 \mathcal{J}^2 \{ (\gamma_+^{+} + \gamma_+^{-}) \mathbf{l} - \gamma_+^{-} \mathbf{l} \} \cdot \mathbf{v}_n \\
+ 2 \frac{m}{\hbar} \mathcal{J} \gamma_+^{1} (\mathbf{l} - 2\mathbf{l}) \cdot (\mathbf{f} \times \mathbf{I}) + \frac{1}{2} \mathbf{f} \times \left( \frac{4m}{\hbar} \gamma_+^{-} \mathcal{J} \mathbf{l} \right) \\
+ \left[ \rho - 8 \left( \frac{m}{\hbar} \right)^2 \mathcal{J}^2 \{ (\gamma_+^{+} + \gamma_+^{-}) \mathbf{l} - \gamma_+^{-} \mathbf{l} \} \right] \cdot \mathbf{v}_n ,
\]

(2.20)

where we have thrown away the term coming from \( f(\mathcal{J}) \) because it is of higher order in \( \mathcal{J} \). It is to be noted that \( C_0 = -2C = -4m\gamma_+^{-} \mathcal{J}^2/\hbar \) and \( L_0 = 4m\gamma_+^{-} \mathcal{J}^2/\hbar \).

Now we have microscopic calculations of the current density in this region done by Wölfle, Cross and Blount. So long as we calculate the value of \( \rho_0 = \rho - \rho_a \) on the basis of excitation quanta in accordance with Landau’s formulation, the result will certainly be reliable even if we calculate in the plane wave
representation.\textsuperscript{47} Therefore we take the superfluid density tensor obtained by these people as guaranteed. It is

\[
\mathbf{\rho}_s = \frac{7}{5\pi^2} \zeta(3) \frac{\mathcal{J}(T)}{\mathcal{T}_c} \mathbf{\rho} \left(1 - \frac{1}{2} \mathbf{\Pi}\right).
\]

Comparison of this expression with (2·20) gives

\[
\mathbf{\tau}_1 + \mathbf{\tau}_1 = 2\mathbf{\tau}_1 = 2\mathbf{\tau} = \frac{7\zeta(3)}{10\pi^2} \left(\frac{\hbar}{2m}\right)^2 \mathbf{\rho} \mathcal{T}_c,
\]

relations valid between the coefficients \(\mathbf{\tau}_1, \mathbf{\tau}_1\) and \(\mathbf{\tau}^*\).\textsuperscript{48,49}

\section{Discussion}

In this section we discuss the physical meaning of the term \(\mathbf{C} \cdot (\mathcal{F} \times \mathbf{I})\) in the expression (1·3) of supercurrent and the change of angular momentum due to excitations. This we do from the point of view that the internal structure of the pair wavefunction is distorted.

Cross regarded \(\mathbf{C} \cdot (\mathcal{F} \times \mathbf{I})\) as another term, like \(\mathbf{p}_s \cdot \mathbf{v}_s\), due to rotation of the order parameter.\textsuperscript{47,49} In order to see whether this viewpoint is appropriate, let us consider the current due to pair condensate \(\mathcal{F}(r, r')\)

\[
j_c(r) = \int d^3r_1 d^3r_2 \mathcal{F}^*(r_1, r_2) \frac{1}{4} \sum_{n_1, n_2} \{ p_n \delta (r - r_1) \} \mathcal{F}(r_1, r_2)
\]

\[
= -\frac{i\hbar}{2} \int d^3r_1 \{ \mathcal{F}^*(r_1, r) \frac{\partial}{\partial r} \mathcal{F}(r_1, r) - \text{c.c.} \}.
\]

A similar procedure to that in deriving (2·6) enables us to transform this to the following:

\[
j_c(r) = \mathbf{p}_s \mathbf{v}_s + \frac{1}{2} \mathbf{r} \times \left(\frac{1}{2} \hbar n_s \mathbf{I}\right),
\]

where \(n_s(r)\) represents the particle number density of the pair condensate:

\[
n_s(r) = \int d^3p |\mathcal{F}(\mathbf{p}; r)|^2, \quad \mathbf{p}_s = m n_s.
\]

\textsuperscript{47} We have calculated the current density in a rotating cylinder with an infinite radius by adding a term \(-\mathbf{\omega} \cdot \mathbf{L}\) to the Hamiltonian. This infinite radius justifies our assumption that the \(\mathbf{L}\) vector points everywhere along the cylindrical axis. We do the calculation in the cylindrical wave representation. Identifying \(p_s\) as the coefficient of \(\mathbf{r} \times \mathbf{\omega}\) we get a result for \(p_s = p - p_s\) in agreement with Eq. (2·21).

\textsuperscript{48,49} Leggett\textsuperscript{47} put \(\mathbf{\tau}_1 = \mathbf{\tau}_2 = \mathbf{\tau}_3 = \mathbf{\tau}\) on the basis of the superfluid density tensor microscopically calculated, which is not justifiable.
We see in Eq. (3.2) there exists no term involving \( \mathbf{F} \times \mathbf{L} \). What relation, then, will be between the condensate current \( j_c(r) \) and the supercurrent \( j_s(r) \)? By analogy to the similar situation in Bose liquid we may suppose that \( \rho_0 e_0 \) will become \( \rho_0 e_0 \) by renormalization due to depletion. Here the distribution of excitation quanta is anisotropic because of the gap anisotropy, and therefore the superfluid density is endowed with tensor character. What about the second term on r.h.s. of Eq. (3.2), then? Such a simple scheme of renormalization as above will be invalid, except for the ground state (1.1), because the internal structure of the Cooper pair wavefunction \( \varphi \) is distorted by excitation.

In order to clarify this situation, we consider the state with one Bogoliubov quantum excited above the ground state (1.1). We should represent this Bogoliubov excitation in terms of single particle states that satisfy the correct boundary condition. Here, however, let us take up the plane wave states for simplicity's sake. The state vector is, therefore, given by

\[
|\tilde{\Theta}_k\rangle = \alpha_k^+ |\Theta\rangle = \frac{1}{A} c_k^+ \exp \tilde{Q}_k^+ |0\rangle ,
\]

where \( \alpha_k^+ \) and \( c_k^+ \) are creation operators of the Bogoliubov quantum and the normal particle, respectively. The pair creation operator \( \tilde{Q}_k^+ \) is given by

\[
\tilde{Q}_k^+ = \frac{1}{2} \int d^3r d^3r' \varphi(r, r') \psi^+(r', \varphi^+(r) ,
\]

where the Cooper pair wavefunction \( \varphi \) in the excited state is

\[
\varphi(r, r') = \varphi(r, r') e^{ik \cdot (r-r')},
\]

In the case of A-phase liquid in a cylinder of infinite radius with its \( l \) vector along the cylindrical axis \( z \), the corresponding expression for the wavefunction is

\[
\varphi(r, r') = \varphi(r, r') + \mathcal{A} f_m (r, r') e^{im \varphi(r-r')},
\]

Here \( m \) is the azimuthal quantum number of the Bogoliubov excitation.

We can derive the one-particle reduced density matrix \( \tilde{\rho}(r, r') \) from Eq. (3.4) in the same way as in the case of the ground state:

\[
\tilde{\rho}(r, r') = e^{-i \mathbf{k} \cdot (r-r')} + \int d^3r \tilde{\rho}^*(r, r) \varphi^+(r, r'),
\]

where

\[40\] When we discuss the total angular momentum of the system it is mandatory to take account of the boundary condition at the container wall. In the following discussion, however, the boundary condition has no critical effect.
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$$\vec{\mathcal{J}}(r, r') = \langle \vec{\mathcal{N}} | \psi(r) \psi(r') | \vec{\mathcal{N}} \rangle$$

\begin{equation}
= \vec{\mathcal{N}}(r, r') - \int d^3r'' \vec{\mathcal{N}}(r'', r') \delta(r'', r).
\end{equation} \tag{3.8}

The first term on r.h.s. of Eq. (3.7) originates in the Bogoliubov excitation and will take part in the normal fluid, while the superfluid part is included in the second term.$^1$

It is important here to note that the function $\vec{\mathcal{N}}(r_1, r_2)$ is not an eigenstate with $l_z=1$ even when $\varphi(r_1, r_2) = \varphi(r_1 - r_2; (r_1 + r_2)/2)$ is an eigenstate of $l_z=1$ with respect to the relative coordinate $r_1 - r_2$. Therefore the function $\vec{\mathcal{N}}(r_1, r_2)$ is not an eigenstate with $l_z=1$ either. At finite temperatures we have to get expressions similar to Eq. (3.7), representing excited states with various Bogoliubov excitations, and to take a statistical average. Of course we have to take the value of $\mathcal{J}$, the gap amplitude implicit in the functions $\varphi$ and $\vec{\mathcal{N}}$, to be that appropriate to the temperature $\mathcal{J}(T)$. Then the order parameter is given by the average of Eq. (3.8) and it revives as an eigenstate with $l_z=1$. Thus the relation between the one-particle reduced density matrix $\rho$ and the order parameter $\mathcal{N}$ gets complex at finite temperatures in contrast with the ground state (cf. (3.7) against (2.1)) and we are forced to abandon the simple picture of quasi-molecules with respect to the superfluid.$^{**}$

Generalizing this argument to the case of slight inhomogeneity, we can get an expression for mass current density at finite temperatures by replacing $\varphi$ and $\vec{\mathcal{N}}$ in (2.6) by $\varphi$ and $\vec{\mathcal{N}}$ in each excited state and averaging them with the appropriate statistical weight:

\begin{equation}
\begin{aligned}
j_{\mu} &= -\frac{i\hbar}{4} \int d^3\rho \left( \bar{\varphi}(\rho; r) \frac{\partial}{\partial r_{\mu}} \mathcal{N}(\rho; r) - \text{c.c.} \right) \\
&\quad + \frac{i\hbar}{4} \frac{\partial}{\partial r_{\mu}} \int d^3\rho \left( \bar{\varphi}(\rho; r) \left[ \rho_{\mu} \frac{\partial}{\partial \rho_{\mu}} + \rho_{\nu} \frac{\partial}{\partial \rho_{\nu}} \right] \mathcal{N}(\rho; r) - \text{c.c.} \right) \\
&\quad + \frac{i\hbar}{4} \frac{\partial}{\partial r_{\mu}} \int d^3\rho \left( \bar{\varphi}(\rho; r) \left[ \rho_{\mu} \frac{\partial}{\partial \rho_{\mu}} - \rho_{\nu} \frac{\partial}{\partial \rho_{\nu}} \right] \mathcal{N}(\rho; r) - \text{c.c.} \right).
\end{aligned}
\end{equation} \tag{3.9}

In the first term on r.h.s. non-vanishing contribution comes only from the spatial derivative of $\mathcal{J}$. Hence the result has the form

\begin{equation}
\rho_{\mu} \vec{r}^{\mu} + C_{\mu \nu} \mathcal{N} j_{\nu},
\end{equation} \tag{3.10}

where

$^1$ The second term also contributes to the normal part.
$^{**}$ This point of view of ours seems to go together with the comment of Mermin.$^{11}$ In discussing supercurrent in the GL region phenomenologically, Mermin pointed out one should not identify the order parameter with the pair wavefunction.
\[ \rho_{ij} = \rho_i \delta_{ij} - \rho_{ij} l_j \]

and

\[ C_{\mu\nu} = C_{\mu\nu} - C_{\mu} l_\nu l_\mu. \]

The integral in the second term is a gauge-invariant symmetric tensor which does not involve any derivative with respect to the center of gravity. Therefore it must be expressed in terms of \( d \) and \( l \), hence it must be of the form \( \alpha \xi_{ij} + \beta l_i l_j \) with \( \alpha \) and \( \beta \) independent of \( d \) and \( l \). However this expression is even under time reversal. This shows that this term makes no contribution to the current density \( j_s \), which is odd with respect to time reversal.

The last term can be written as \( \frac{1}{2} \mathbf{P} \times \mathbf{L}_n \), with

\[ L_n = \frac{-i\hbar}{4} \int d^3 \rho \left\langle \hat{\mathbf{\Psi}}^\dagger (\rho; r) \mathbf{\rho} \times \frac{\partial}{\partial \mathbf{\rho}} \hat{\mathbf{\Psi}} (\rho; r) \right\rangle + \text{c.c.} \quad (3.11) \]

Adding up all the contributions we finally have

\[ j_s = \rho \cdot v_s + C \cdot (\mathbf{P} \times \mathbf{L}) + \frac{1}{2} \mathbf{P} \times \mathbf{L}_n. \quad (3.12) \]

From the discussion above we conclude that the term \( C \cdot (\mathbf{P} \times \mathbf{L}) \) in the expression (1·3) of the supercurrent originates in the distortion of the Cooper pair wavefunction due to Bogoliubov excitations. Therefore the only superfluid state that can be visualized in terms of the “quasi-molecular” picture is the “ground state” represented by Eq. (1·1). In fact when use is made of the expression (1·1) we only obtain the term \( \frac{1}{2} \mathbf{P} \times (\frac{1}{2} \hbar \mathbf{L}) \) through a simple renormalization of \( \frac{1}{2} \mathbf{P} \times (\frac{1}{2} \hbar \mathbf{L}) \) by the depletion and never such a term \( C \cdot (\mathbf{P} \times \mathbf{L}) \).

At finite temperatures we have the term \( C \cdot (\mathbf{P} \times \mathbf{L}) \) as well as \( \frac{1}{2} \mathbf{P} \times \mathbf{L}_n \).

The expression (3·11) is evidently a natural generalization of Eq. (2·7) and, therefore, can be identified with the intrinsic angular momentum density. Assuming uniform \( l \)-vectors one should be able to calculate the total angular momentum at finite temperatures. However the straightforward calculation would be certainly not so simple as in the ground state.

References

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