Optimal Investment with Costly Reversibility

ANDREW B. ABEL and JANICE C. EBERLY

The Wharton School of the University of Pennsylvania and NBER

First version received August 1995; final version accepted March 1996 (Eds.)

Investment is characterized by costly reversibility when a firm can purchase capital at a given price and sell capital at a lower price. We solve for the optimal investment of a firm that faces costly reversibility under uncertainty and we extend the Jorgensonian concept of the user cost of capital to this case. We define and calculate $c_u$ and $c_s$ as the user costs of capital associated with the purchase and sale of capital, respectively. Optimality requires the firm to purchase and sell capital as needed to keep the marginal revenue product of capital in the closed interval $[c_u, c_s]$. This prescription encompasses the case of irreversible investment as well as the standard neoclassical case of costlessly reversible investment.

In traditional capital theory, investment is costlessly reversible and the optimal investment policy of a firm maintains the marginal revenue product of capital equal to the Jorgensonian user cost of capital. Recent literature has shown that if investment is completely irreversible, the nature of the optimal investment policy is different: optimal investment can be characterized as a trigger policy in which capital is purchased to prevent the marginal revenue product of capital from rising above an optimally derived trigger. Values of the marginal revenue product of capital that are lower than the trigger value constitute a range of inaction in which the optimal rate of investment is zero. The boundary of this range of inaction, which is the trigger value for the marginal revenue product of capital, is higher than the Jorgensonian user cost of capital.

Costlessly reversible investment and irreversible investment are opposite ends of a spectrum in which there is costly reversibility. In this paper, we model costly reversibility by introducing a difference between the price at which the firm can purchase capital and the price at which it can sell capital. This wedge between the purchase and sale prices of capital could arise from transactions costs or from the firm-specific nature of capital. When the wedge is zero, we have the traditional case of costlessly reversible investment, and when the sale price of capital is zero (so that the wedge is 100% of the purchase price of capital) we have the case of irreversible investment. The intermediate case, however, motivated Arrow's (1968) seminal paper on irreversible investment, where he states

From a realistic point of view, there will be many situations in which the sale of capital goods cannot be accomplished at the same price as their purchase. There are installation costs, which are added to the purchase price but cannot be recovered on sale; indeed, there may on the contrary be additional costs of detaching and moving machinery. Again sufficiently specialized machinery and plant may have little value to others. So resale prices may be substantially below replacement costs. For simplicity, we will make the extreme assumption that resale of capital goods is impossible, so that gross investment is constrained to be non-negative. (Arrow (1968, pp. 2–3))
In this paper we abandon the "simplicity" achieved by the "extreme assumption" of irreversibility, yet we are able to derive a tractable solution to the more realistic case of costly reversibility.

In Section I, we parametrically specify a continuous-time infinite-horizon investment problem, which is formally identical to the employment model of Bentolila and Bertola (1990) with hiring and firing costs. The solution to this problem is naturally characterized in terms of the shadow price of capital which we call $q$, and we solve for $q$ in Section II.

Optimal behaviour is a two-trigger policy in which the firm purchases capital to prevent the marginal revenue product of capital from rising above the upper trigger value and sells capital to prevent the marginal revenue product of capital from falling below the lower trigger value. Values of the marginal revenue product of capital strictly between the two trigger values define a range of inaction in which it is optimal to neither purchase nor sell capital. While the general form of the solution has been pointed out in a model of investment by Bertola (1988), of employment by Bentolila and Bertola (1990), and in a model of entry and exit by Dixit (1989), our solution focuses attention on the size of the range of inaction, which is measured by the ratio of the upper trigger to the lower trigger. We show in Section III that the entire problem can be solved in terms of this ratio, which is implicitly defined by a single nonlinear equation solvable with a simple algorithm. Even a tiny wedge between the purchase price and the sale price of capital produces a substantial range of inaction. (Formally, the derivative of the trigger ratio with respect to the ratio of the purchase price to the sale price of capital is infinite when the capital price ratio is one.) We derive an approximation to this ratio in Section IV and use the approximation to illustrate the determinants of the trigger ratio.

In Section V we generalize the concept of the user cost of capital introduced in the case of costlessly reversible investment by Jorgenson (1963). When there is a wedge between the purchase and sale prices of capital, there are two notions of the user cost of capital: (1) a user cost, $c_U$, which is relevant for purchasing capital; and (2) a user cost, $c_L$, which is relevant for selling capital. The user costs $c_U$ and $c_L$ are the triggers on the marginal revenue product of capital and they help us to state the optimal two-trigger policy as a simple rule: Purchase capital as needed to prevent the marginal revenue product of capital from rising above the user cost $c_U$, sell capital as needed to prevent the marginal revenue product of capital from falling below the user cost $c_L$, and neither purchase nor sell capital if the marginal revenue product of capital is strictly between $c_L$ and $c_U$. This rule encompasses the prescription for optimal investment in the case of costlessly reversible investment discussed by Jorgenson (1963) as well as the prescription for optimal investment in the burgeoning literature on irreversible investment (Bertola (1988), Pindyck (1988, 1991), Dixit (1989), Bertola and Caballero (1994), and Dixit and Pindyck (1994)).

I. THE FIRM'S OPTIMIZATION PROBLEM

Consider a firm that produces output at time $t$ using capital $K_t$ and variable factors of production. It faces a demand curve that depends on the random variable $X$, which evolves exogenously according to a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dz, \quad \sigma_X > 0 \quad (1)$$

where $dz$ is an increment to a standard Wiener process, with $E\{dz\} = 0$ and $(dz)^2 = dt$. 
Assume that the operating profit of the firm, i.e. revenue minus the cost of the variable factors of production, is given by
\[
\pi(K_t, X_t) = \frac{h}{1-\gamma} X^\gamma K^{1-\gamma} \quad \text{where } h > 0 \quad \text{and} \quad 0 < \gamma < 1.
\] (2)

The specification in equation (2) can be derived for a firm with a constant-returns-to-scale Cobb–Douglas production function facing an iso-elastic demand curve.

The firm can purchase capital at a constant price \( b^u > 0 \), and it can sell capital at a constant price \( b^l = b^u \). Let \( V_t \) be a non-decreasing function of time representing the cumulation of all purchases of capital up to time \( t \), and let \( L_t \) be a non-increasing function of time representing the cumulation of all sales of capital (i.e. negative gross investment) up to time \( t \). The capital stock depreciates at a constant proportional rate \( \delta \geq 0 \), so that the net change in the capital stock at time \( t \) is given by \( dK_t = dU_t + dL_t - (\delta K_t)dt \).

Assume that the firm is risk-neutral and maximizes the expected present value of cash flows discounted at a constant positive rate \( r > \mu_X \). The value of the firm is
\[
V(K_t, X_t) = \max_{dU_t, dL_t} \mathbb{E}_t \left\{ \int_0^\infty e^{-rt} \left[ \pi(K_{t+s}, X_{t+s})ds - b^u dU_{t+s} - b^l dL_{t+s} \right] \right\}. \tag{3}
\]

Since \( U_t \) and \( L_t \) are not differentiable, the last two terms in equation (3) are to be interpreted as Stieltjes integrals.

The Bellman equation associated with the optimization problem in equation (3) is
\[
r V(K, X) = \frac{h}{1-\gamma} X^\gamma K^{1-\gamma} - \delta K V_K(K, X) + \mu_X X V_X(K, X) + \frac{1}{2} \sigma_X^2 X^2 V_{XX}(K, X). \tag{4}
\]
The left-hand side of equation (4) is the required return on the firm. The right-hand side of this equation is the expected return consisting of the cash flow plus the expected change in the value of the firm.

Equation (4) holds identically in \( K \). Thus, the partial derivative of the left-hand side with respect to \( K \) equals the partial derivative of the right-hand side with respect to \( K \) so that
\[
r V_K(K, X) = h \left( \frac{X}{K} \right)^\gamma - \delta V_X(K, X) - \delta K V_{KX}(K, X) + \mu_X X V_{XX}(K, X)
\]
\[+ \frac{1}{2} \sigma_X^2 X^2 V_{XXX}(K, X). \tag{5}
\]

1. Under the optimal policy, the marginal revenue product of capital, \( hX^\gamma K^{1-\gamma} \), satisfies \( c_L \leq hX^\gamma K^{1-\gamma} \leq c_U \) where \( c_L \) and \( c_U \) are constants given in equations (26) and (25) respectively. Since \( (\gamma - 1)/\gamma < 0 \) and \( \pi(K, X) = (h/(1-\gamma))X^\gamma K^{1-\gamma} \), we obtain \( (h^{1/\gamma}/(1-\gamma))c_L \leq h \left( \frac{X}{K} \right)^\gamma X \leq (h^{1/\gamma}/(1-\gamma))c_U \). Since the expected growth rate of \( X \) is \( \mu_X \), the operating profit of the firm is bounded above and bounded below by processes with expected growth rates equal to \( \mu_X \). Therefore, \( \lim_{t \to \infty} e^{-rt} \mathbb{E}_t \left\{ \pi(K_{t+1}, X_{t+1})ds \right\} \) is finite if and only if \( \mu_X < r \). Thus the expected present value of operating profits, \( \mathbb{E}_t \left\{ \int_0^\infty e^{-rt} \pi(K_{t+s}, X_{t+s})ds \right\} \), is finite if and only if \( \mu_X < r \).

2. The right-hand side of equation (4) does not contain any terms reflecting the cash flows associated with purchasing or selling capital. Each unit of capital purchased by the firm increases the value of the firm by \( V_d(K, X) \) and costs \( b^u \), so the contribution of positive investment to the maximand in equation (3) is \( dU_t V_K(K, X) - b^u \). As shown later, under the optimal policy \( dU_t \) is non-zero only when the marginal valuation of capital \( V_d(K, X) \) equals the purchase price of capital \( b^u \), so that \( dU_t V_d(K, X) - b^u \) is always zero. Similarly, each unit of capital sold by the firm increases the value of the firm by \( b^l - V_s(K, X) \), so the contribution of negative investment to the maximand in equation (3) is \( -dL_t (b^l - V_s(K, X)) \). As shown later, under the optimal policy \( dL_t \) is non-zero only when the marginal valuation of capital \( V_s(K, X) \) equals the sale price of capital \( b^l \), so that \( -dL_t (b^l - V_s(K, X)) \) is always zero. Thus, the Bellman equation in (4) holds for non-zero as well as zero optimal investment.
Equation (5) characterizes the marginal valuation of capital $V_K(K, X)$. To simplify the expression for the marginal valuation of capital, define $y = X/K$. When gross investment is zero, $y$ evolves as a geometric Brownian motion with instantaneous drift $\mu_y = \mu_x + \delta$ and standard deviation $\sigma_y = \sigma_x$. Because $V(K, X)$ is homogeneous of degree one in $X$ and $K$, $V_x(K, X)$ is homogeneous of degree zero in $X$ and $K$. Defining $q(y)$ as the marginal valuation of capital, $V_K(K, X)$, and substituting this definition into equation (5) yields a second-order ordinary differential equation for the marginal valuation of capital $q(y)$,

$$(r + \delta)q(y) = hy^r + \mu_y q'(y) + \frac{1}{2} \sigma^2_y y^2 q''(y).$$

(6)

In addition to satisfying the differential equation (6), $q(y)$ must satisfy the boundary conditions. Optimal investment is zero when the marginal valuation of capital $q(y)$ is in the interior of the interval $[b_L, b_U]$. The firm will undertake non-zero gross investment only if $q(y)$ reaches one of the boundaries $b_L$ or $b_U$. The values of $y$ at these boundaries, $y_L$ and $y_U$, are given by the smooth-pasting conditions

$$q(y_L) = b_L \quad \text{and} \quad q(y_U) = b_U. \quad (7a, b)$$

In addition to satisfying the smooth-pasting conditions, $q(y)$, $y_L$, and $y_U$ must satisfy the high-contact conditions

$$q'(y_L) = 0 \quad \text{and} \quad q'(y_U) = 0. \quad (8a, b)$$

The high-contact conditions guarantee that if the upper boundary is reached at time $t$, the value of an additional unit of capital, $q$, will equal its cost, $b_U$, at both $K_t$ and $K_t^+$. Similarly, if the lower boundary is reached at time $t$, $q$ must equal the resale price of capital, $b_L$, both at $K_t$ and $K_t^-$. The high-contact conditions therefore ensure that the marginal valuation of capital, $q$, does not change when investment is non-zero.

II. THE SOLUTION FOR $q(y)$

The solution to the differential equation (6) involves the roots of the following quadratic equation (see, for example, Dixit and Pindyck (1994))

$$\rho(\eta) = -\frac{1}{2} \sigma^2_y \eta^2 - (\mu_y - \frac{1}{2} \sigma^2_y) \eta + r + \delta = 0.$$  

(9)

Note that $\rho(\eta)$ is strictly concave, $\rho(0) = r + \delta > 0$, and $\rho(1) = r + \delta - \mu_y > 0$. Thus, $\rho(y) > 0$, and $\rho(\eta) = 0$ has two distinct roots, $\alpha_P > 0$ and $\alpha_N < 0$, which satisfy

$$\alpha_N < 0 < \eta < 1 < \alpha_P. \quad (10)$$

The roots also satisfy the following equation

$$\rho(\eta) = \frac{(\eta - \alpha_P)(\eta - \alpha_N)}{\alpha_P \alpha_N} (r + \delta). \quad (11)$$

In order to simplify the notation in the subsequent analysis we define the function $\theta(x)$ for $x \geq 0$ as

$$\theta(x) = \frac{x^{\alpha_P} - x^{\alpha_N}}{x^{\alpha_P} - x^{\alpha_N}}. \quad (12)$$

3. See Dumas (1991) for a clear presentation of the smooth-pasting and high-contact conditions used below.
It is obvious that $0 \leq \theta(x) \leq 1$. The Appendix shows that $\theta'(x) > 0$ for $x > 0$ and that

$$\theta(0) = 0 < \theta(1) = \frac{\alpha_p - \gamma}{\alpha_p - \alpha_N} < 1 = \theta(\infty).$$ (13)

We will verify (Proposition 4) that the solution to the differential equation (6) that satisfies the boundary conditions is

$$q(y) = Hy^\gamma - \frac{\gamma H}{\alpha_N} \theta(G)y_L^{-\sigma_N} - \frac{\gamma H}{\alpha_p} [1 - \theta(G)]y_L^{1-\sigma_p}y_p^\sigma,$$ (14)

where

$$H \equiv \frac{h}{\rho(y)} > 0 \quad \text{and} \quad G \equiv y_U/y_L.$$  

is the ratio of the value of $y$ that triggers purchases of capital ($y_U$) to the value of $y$ that triggers sales of capital ($y_L$).

The marginal valuation of capital $q(y)$ equals the present value of expected marginal revenue products of capital under optimal behaviour, and equation (14) expresses this expected present value as the sum of three components: (1) $Hy^\gamma$ is the present value of expected marginal revenue products of capital if the firm were prevented from ever purchasing or selling capital; (2) If the firm has the option to purchase capital, this opportunity to increase the capital stock in the future decreases expected future marginal revenue products and reduces the present value of expected marginal revenue products by $(-\gamma H/\alpha_p) [1 - \theta(G)]y_L^{-\sigma_p}y_p^\sigma$; and (3) If the firm has the option to sell capital, this opportunity to reduce the future capital stock increases expected future marginal revenue products of capital and increases the present value of expected future marginal revenue products by $-(-\gamma H/\alpha_N) \theta(G)y_L^{-\sigma_N}y_p^\sigma > 0$. In the general case of costly reversibility the firm can both purchase and sell capital, so the marginal value of capital equals $Hy^\gamma$ plus the effects of the call option to purchase capital and the put option to sell capital.\footnote{Bertola (1988) uses contingent claims pricing to derive the general forms of these option terms but does not derive explicit expressions for the constants. Abel, Dixit, Eberly and Pindyck (1995) discuss the put and call options in a simple two-period model with costly reversibility and costly expandability.}

III. THE SOLUTION FOR G

The expression for $q(y)$ in equation (14) depends on the ratio $G$. To solve for $G$, it is convenient to define the function $\phi(x)$ as

$$\phi(x) \equiv \frac{H}{h} \left\{ 1 - \frac{\gamma}{\alpha_N} \theta(x) - \frac{\gamma}{\alpha_p} [1 - \theta(x)] \right\}. \quad (15)$$

It follows from equations (13) and (11) and the definition of $H$ that

$$0 < \phi(0) = \frac{\alpha_N}{\alpha_N - \gamma} \frac{1}{r + \delta} < \phi(1) = \frac{1}{r + \delta} < \frac{\alpha_p}{\alpha_p - \gamma} \frac{1}{r + \delta} = \phi(\infty).$$ (16)

Inspection of equation (15) together with the fact that $\theta(x)$ is strictly increasing for $x > 0$ implies that $\phi(x)$ is strictly increasing for $x > 0$. The Appendix shows that

$$\phi'(1) = \frac{\gamma}{2} \phi(1).$$ (17)
We now use the function \( q(x) \) to determine \( G \) for a given \( R \equiv b_U/b_L \geq 1 \). In Proposition 4 we verify that the optimal value of \( G \) satisfies

\[
J(R, G) = R \phi(G) - G' \phi(G^{-1}) = 0.
\]  

(18)

**Proposition 1.** For any finite \( R \geq 1 \), there exists a \( G \geq 1 \) such that \( J(R, G) = 0 \).

**Proof.** \( J(R, G) \) is continuous in \( G \) for \( G \geq 1 \) and \( \lim_{G \to \infty} J(R, G) = -\infty \). If \( R > 1 \), \( J(R, 1) = (R - 1)\phi(1) > 0 \) so there exists a \( G > 1 \) for which \( J(R, G) = 0 \). For \( R = 1 \), we have \( J(1, 1) = 0 \).

Proposition 1 states that there is a value of \( G \geq 1 \) that satisfies equation (18). Elsewhere (in Abel and Eberly (1995)) we have proved that this value is unique and is strictly increasing in \( R \). However, these proofs are too lengthy to include here.

**Proposition 2.** Let \( G(R) \) denote the value of \( G \geq 1 \) that satisfies equation (18) for given \( R \). \( G'(1) = \infty \).

**Proof.** \( J_G(R, G) = R \phi'(G) - \gamma G^{-1} \phi(G^{-1}) + G^{-2} \phi'(G^{-1}) \). Therefore, \( J_G(1, 1) = 2\phi'(1) - \gamma \phi(1) = 0 \) where the second equality follows from equation (17). \( J_K(1, 1) = \phi(1) > 0 \). Application of the implicit function theorem implies that \( G'(1) = \infty \).

Interpreting \( R - 1 \) as a transaction cost, Proposition 2 states that the derivative of the (geometric) distance between the upper and lower triggers with respect to the transaction cost is infinite when evaluated at zero transaction cost.\(^5\)

Observe that \( G_R = (hy_U)/(hy_L) \) is the ratio of the trigger values of the marginal revenue product of capital. The following proposition shows that if \( R \), the ratio of the purchase and sale prices of capital faced by the firm, is greater than one, then \( G_R \) is larger than \( R \).

**Proposition 3.** For any finite \( R > 1 \), \( R < G < \{[(1 - \gamma/\alpha_N)/(1 - \gamma/\alpha_F)]R \} \).

**Proof.** \( R > 1 \) implies \( G > 1 \). Equation (18) implies that \( G' = \{\phi(G)/\phi(G^{-1})\}R \). Since \( \phi(G)/\phi(G^{-1}) \) is strictly increasing in \( G \), \( 1 < \phi(G)/\phi(G^{-1}) < \phi(\infty)/\phi(0) = (1 - \gamma/\alpha_N)/(1 - \gamma/\alpha_F) \) for \( G > 1 \).

The existence of a closed interval containing \( G_R \) suggests a simple iterative algorithm to determine \( G \): divide the interval in half, choose the half-interval for which \( J(R, G) \) is of opposite sign at the endpoints, and repeat this procedure until the desired degree of precision is achieved.

Now that we have a solution for \( G \), we can express the values of the triggers \( y_U \) and \( y_L \) using the function \( \phi(x) \) and the ratio \( G \). It is most convenient to express these triggers in terms of the triggers for the marginal revenue product of capital, \( hy^* \),

\[
hy_U^* = \frac{b_U}{\phi(G^{-1})} \quad \text{(19a)}
\]

\[
hy_L^* = \frac{b_L}{\phi(G)} \quad \text{(19b)}
\]

5. A similar finding is also reported by Dixit (1989, p. 630) in a model of entry and exit.
Proposition 4. The expression for \( q(y) \) in equation (14) with \( G \) given by equation (18), \( y_U \) given by equation (19a), and \( y_L \) given by equation (19b) satisfies the differential equation (6) and the boundary conditions (7a, b; 8a, b).

Proof. To verify that equation (14) satisfies the differential equation (6), differentiate equation (14) twice with respect to \( y \), and use the definition of \( H \) along with the facts that \( \rho(a_N) = \rho(a_F) = 0 \). Evaluate equation (14) at \( y = y_L \), using the definition of \( \phi(x) \) and equation (19b) to obtain \( q(y_L) = hy_L\phi(G) = b_L \) which satisfies equation (7a). Similarly, evaluate equation (14) at \( y = y_U \), using the definition of \( \phi(x) \), equation (19a), and the facts that \( y_U = G y_L \), \( \theta(G^{-1}) = \theta(G)G^{a_N - r} \), and \( 1 - \theta(G^{-1}) = [1 - \theta(G)]G^{a_P - r} \) to obtain \( q(y_U) = hy_U\phi(G^{-1}) = b_U \) which satisfies equation (7b). Differentiating equation (14) with respect to \( y \), and evaluating the derivative at \( y_L \) easily yields \( q'(y_L) = 0 \) as in equation (8a). Evaluating this derivative at \( y_U \) and using the facts that \( \theta(G^{-1}) = \theta(G)G^{a_N - r} \) and \( 1 - \theta(G^{-1}) = [1 - \theta(G)]G^{a_P - r} \) yields \( q'(y_U) = 0 \) as in equation (8b). Finally, divide each side of \( hy_L\phi(G^{-1}) = b_U \) by the corresponding side of \( hy_L\phi(G) = b_L \) to obtain \( G\phi(G^{-1})/\phi(G) = R \) which is equivalent to equation (18).

IV. A LOCAL APPROXIMATION FOR THE OPTIMAL VALUE OF \( G \)

Although we have presented a simple algorithm to compute the optimal value of \( G \), in this section we present an analytic approximation to the optimal value of \( G \). This approximation clearly illustrates the effects on \( G \) of various parameters of the problem facing the firm.

In the Appendix we show that taking a Taylor-series approximation of \( J(R, G) \) around the point \( R = G = 1 \), and setting \( J(R, G) \) equal to zero, as in equation (18), yields

\[
G \approx 1 + \left[ \frac{6(\gamma)^2}{\gamma(r + \delta)} \right]^{1/3} (R - 1)^{1/3}. \tag{20}
\]

According to equation (20) the wedge between the upper and lower boundary values of \( y \), \( G - 1 \), is proportional to the cube root of the wedge between the purchase and sale prices of capital, \( R - 1 \). This cubic function displays the infinite value of \( \frac{dG}{dR} \) at \( R = 1 \) as presented in Proposition 2.

For a given value of \( \gamma \), the wedge \( G_\gamma \) between the upper and lower trigger values of the marginal revenue product of capital is an increasing function of \( \sigma_\gamma \). If the boundaries \( y_L \) and \( y_U \) were to remain fixed, an increase in the variance of \( y \) would shorten the expected length of time between changing the capital stock when \( y \) is at one boundary, and then subsequently changing the capital stock in the opposite direction when \( y \) is at the other boundary. When \( R > 1 \) it is costly to reverse a change in the capital stock, and by increasing the expected frequency of such reversals, an increase in the variance of \( y \) would increase the expected costs facing the firm for given \( y_L \) and \( y_U \). To mitigate this increase in expected costs, the firm increases the wedge between the upper and lower boundaries.

For a given value of \( \gamma \), the wedge \( G_\gamma \) between the upper and lower trigger values of the marginal revenue product of capital is a decreasing function of \( r + \delta \). An increase in

6. In models in different economic contexts, Dixit (1991), Delgado and Dumas (1994) and Shreve and Soner (1994) have found a similar locally cubic property.

7. In the neighbourhood of \( R = 1 \), the optimal value of \( G \) is very close to one, which means that the wedge between the upper and lower boundaries on \( y \) is very small. The dynamics of \( y \) from one boundary to the other boundary infinitesimally far away, and thus the expected costs associated with reversals of changes in the capital stock, are governed by \( \sigma, dz \) rather than by \( \mu, dt \) because \( dt \) is of second order compared to \( dz \). Thus, as in equation (20), \( G \) is independent of \( \mu \), in the neighbourhood of \( R = 1 \).
r means that the firm discounts the future more heavily and thus attaches less weight to the cost associated with future reversals of changes in the capital stock. Therefore, the firm is willing to incur a higher expected frequency of such reversals and narrows the wedge between the upper and lower boundaries of y. An increase in \( \delta \) means that a smaller fraction of a current change in the capital stock will remain at any future date. Therefore, the expected cost associated with any future reversal of a current change in the capital stock will be smaller. Thus, as in the case of an increase in \( r \), the firm will choose to incur a higher expected frequency of such reversals and will narrow the wedge between the upper and lower trigger values.

V. THE USER COST OF CAPITAL

In a deterministic neoclassical model of costlessly reversible investment, Jorgenson (1963) showed that the user cost of capital is given by

\[
c_f \equiv (r + \delta)p_K - \hat{p}_K,
\]

where \( p_K \) is the price at which the firm can purchase or sell capital. The user cost has three components: (1) interest cost, \( rp_K \); (2) physical depreciation, \( \delta p_K \); and (3) the capital loss associated with the decline in the price of a unit of capital, \(-\hat{p}_K\). Of course, if \( p_K \) is constant, the Jorgensonian user cost is simply \( (r + \delta)p_K \).

In this section we extend the Jorgensonian concept of the user cost of capital to the general case of costly reversibility under uncertainty by making two straightforward modifications. First, we replace the purchase/sale price of capital, \( p_K \), by the shadow price of capital, \( q(y) \). In the costlessly reversible case analysed by Jorgenson, the shadow price \( q(y) \) is always equal to the purchase/sale price \( p_K \). However, when the shadow price differs from the purchase/sale price of capital, it is the shadow price that is relevant for the user cost of capital. The second modification takes account of uncertainty by using the expected value of the capital loss. With these two modifications, the user cost of capital, comprised of the interest cost, physical depreciation, and the expected capital loss, is

\[
c(y) \equiv (r + \delta)q(y) - \frac{1}{dt} E_t\{dq(y)\}.
\]

Using Ito's Lemma to calculate \( E\{dq(y)\} \) in equation (22) yields

\[
c(y) = (r + \delta)q(y) - \mu_y q'(y) - \frac{1}{2} \sigma_y^2 q''(y).
\]

Substituting equation (6) into equation (23) yields

\[
h y' = c(y).
\]

According to equation (24), optimal behaviour implies that the marginal revenue product of capital \( h y' \) is always equal to the user cost of capital. As shown below, the user costs at the triggers \( y_L \) and \( y_U \) play a special role. Let \( c_U \equiv c(y_U) \) be the user cost at the upper trigger, and observe from equations (19a) and (24) that

\[
c_U = b_U/\phi(G^{-1}).
\]

Similarly, defining \( c_L \equiv c(y_L) \) as the user cost at the lower trigger, and using equations (19b) and (24) we obtain

\[
c_L = b_L/\phi(G).
\]

Equations (25) and (26) imply \( c_L \leq b_L/\phi(1) = (r + \delta)b_L \leq (r + \delta)b_U = b_U/\phi(1) \leq c_U. \)
The two user costs, \( c_L \) and \( c_U \), allow us to characterize optimal investment behaviour by a simple rule: *Keep the marginal revenue product of capital from leaving the closed interval \([c_L, c_U]\) if \([b_L/\phi(G), b_U/\phi(G^{-1})]\).* To implement this rule, purchase capital to prevent its marginal revenue product from rising above the user cost \( c_U \), and sell capital to prevent its marginal revenue product from falling below \( c_L \). If the marginal revenue product of capital is in the interior of the interval \([c_L, c_U]\), then it is optimal to neither purchase nor sell capital.

In the case of costlessly reversible investment examined by Jorgenson (1963), \( R = 1 \), so \( G = 1 \) and \( c_L = c_U = (r + \delta)b \) where \( b \) is the common value of \( b_U \) and \( b_L \). In this case, the interval \([c_L, c_U]\) collapses to a single point \((r + \delta)b\) which is the Jorgensonian user cost of capital. Maintaining the marginal revenue product of capital in this degenerate interval requires that the marginal revenue product of capital is continuously equated with the Jorgensonian user cost.

In the case of costly reversibility, \( R > 1 \), and Proposition 3 implies that \( c_U/c_L \), the ratio of the user costs, which equals \( G^* \), is greater than \( R \), the ratio of the purchase and sale prices of capital. In the case of complete irreversibility, which has been widely studied in the literature, \( b_L = 0 \) so \( R = \infty \) and \( G = \infty \) which implies that \( c_L = 0 \) and \( c_U = b_U/\phi(0) = b_U(r + \delta)(1 - \gamma/\alpha) > (r + \delta)b_U \). The optimal policy under complete irreversibility is to purchase capital to prevent the marginal revenue product of capital from rising above \( c_U \).  

### VI. CONCLUSION

For the sake of simplicity, Arrow (1968) and much of the subsequent literature chose to examine the special case of irreversible investment, spawning a literature that has produced a variety of insights about optimal investment under uncertainty. While several papers have explored the spectrum between irreversibility and costless reversibility, we show that the solution to this optimal investment problem may be characterized entirely in terms of the width of the range of inaction. Further, the width of the range of inaction is characterized by a single nonlinear equation for which there is a simple iterative solution algorithm and a useful local approximation.

Our analysis of costly reversibility offers a prescription for optimal investment that encompasses the entire spectrum from costless reversibility to complete irreversibility. This prescription is based on a natural extension of Jorgenson's definition of the user cost of capital to the case of costly reversibility under uncertainty. We define and calculate \( c_U \), the user cost relevant for purchasing capital, and \( c_L \), the user cost relevant for selling capital. Using these definitions, the prescription for optimal investment is simply stated: Purchase and sell capital as needed to prevent the marginal revenue product of capital from leaving the closed interval \([c_L, c_U] \).

### APPENDIX

Define \( b = a - a_N > a = a - \gamma > 0 \) and observe that

\[
\theta(x) = \frac{1 - x^a}{1 - x^b},
\]

(\text{A.1})

8. Dixit and Pindyck (1994, p. 145) thank Giuseppe Bertola for pointing out that the trigger value for the marginal revenue product of capital can be interpreted as the user cost of capital. However, neither they nor Bertola show that the value of the trigger is equal to the sum of the costs associated with interest, physical depreciation, and the expected capital loss on a unit of capital, as we have done here. This demonstration gives economic content to an otherwise formal definition of user cost, and illustrates how this definition of user cost is based on Jorgenson's definition.
It follows immediately from equation (A.1) that
\[ \theta(\infty) = 1. \]  

(A.2)

To calculate \( \theta(0) \), multiply the numerator and denominator of equation (A.1) by \( x^b \) to obtain
\[ \theta(x) = \frac{x^b - x^{b-a}}{x^b - 1}. \]  

(A.3)

It follows immediately from equation (A.3) that
\[ \theta(0) = 0. \]  

(A.4)

To evaluate \( \theta(x) \) at \( x = 1 \), apply L'Hôpital's Rule to equation (A.1) to obtain
\[ \theta(1) = \frac{a}{b} = \frac{a_p - \gamma}{a_p - a_N} > 0. \]  

(A.5)

Differentiating equation (A.1) with respect to \( x \) yields
\[ \theta'(x) = \frac{N(x)}{D(x)}, \]  

(A.6a)

where
\[ N(x) = abx^{-\frac{a+b+1}{b}} \left( \frac{x^b - 1}{b} - \frac{x^a - 1}{a} \right). \]  

(A.6b)

and
\[ D(x) = (1 - x^b)^2 \geq 0, \quad \text{with strict inequality if } x \neq 1. \]  

(A.6c)

Observe that \( N(1) = N'(1) = D(1) = D'(1) = 0, N''(1) = ab(b-a) \) and \( D''(1) = 2b^2 \) so
\[ \theta'(1) = \frac{a}{b} = \frac{\gamma - a_N}{2} \text{, } \theta(1) > 0. \]  

(A.7)

Lemma 1. Define \( \phi(z) = (x^z - 1)/z \) for \( x > 0 \) and \( z > 0 \). If \( x \neq 1 \), then \( \phi'(z) > 0. \)

Proof. Observe that \( \phi(z) = (e^{z \ln x} - 1)/z \) so that
\[ \phi'(z) = \frac{x^z \ln x - x^z - 1}{z^2} = \frac{(z \ln x - 1)x^z + 1}{z^2}, \]  

where \( v(z) = (z \ln x - 1)x^z + 1 \). Differentiating \( v(z) \) yields \( v'(z) = x^z \ln x + (z \ln x - 1)x^z \ln x = z(\ln x)^2 x^z > 0 \) for \( z > 0 \) and \( x \neq 1 \). Note that \( v(0) = 0 \). Therefore, \( v(z) > 0 \) for \( z > 0 \) which implies that \( \phi'(z) > 0 \) for \( z > 0 \).

Using the definition of \( \phi(z) \) in Lemma 1 and the definition of \( N(x) \) in equation (A.6b) yields
\[ N(x) = abx^{-\frac{a+b+1}{b}}[\phi(b) - \phi(a)] > 0 \quad \text{for } 0 < x \neq 1, \]  

(A.8)

where the inequality follows from \( b > a > 0 \) which implies \( \phi(b) > \phi(a) \). Therefore, (since we have already shown that \( \theta'(1) > 0 \)), we have
\[ \theta'(x) > 0 \quad \text{for } x > 0. \]  

(A.9)

Differentiating equation (15) and evaluating the derivative at \( x = 1 \) yields
\[ \phi'(1) = \frac{\gamma H(1/a_p - 1/a_N)}{h} \theta'(1). \]  

(A.10)

Using the definition of \( H \) and the expression for \( \theta'(1) \) in equation (A.7) yields
\[ \phi'(1) = \frac{\gamma}{\rho(\gamma)} \frac{a_N - a_p}{a_p a_N} \frac{\gamma - a_N}{2} \theta(1). \]  

(A.11)
Substituting the expression for $\theta(1)$ from equation (13) into equation (A.11) and simplifying using equation (11) yields

$$\phi'(1) = \frac{\gamma}{2} \frac{1}{r + \delta}. \quad (A.12)$$

Now use the expression for $\phi(1)$ in equation (16) to rewrite equation (A.12) as

$$\phi'(1) = \frac{\gamma}{2} \phi(1). \quad (A.13)$$

Second and third derivatives of $\theta(x)$ and $\phi(x)$:

Write $\theta(x) = n(x)/d(x)$, where $n(x) = 1 - x^{-a}$ and $d(x) = 1 - x^{-b}$. Let $m_n(x) = n(x)$ so that $\theta(x) = m_n(x)/d(x)$. In general, the $j$-th derivative of $\theta(x)$ is $\theta^{(j)}(x) = m_{n+j}(x)/d(x)$ where $m_{n+j}(x) = m_j(x) - \theta^{(j)}(x)d'(x)$. Now evaluate the derivatives at $x = 1$. Since $n(1) = d(1) = 0$, we have $\theta(1) = m_n(1)/d'(1)$ which implies that $m_n(1) = 0$. Indeed, $m_j(1) = 0$ for all $j$. Therefore,

$$\theta^{(j)}(1) = m_{n+j}(1)/d'(1). \quad (A.14)$$

Straightforward but tedious application of equation (A.14) using equations (A.5) and (A.7) yields

$$\theta''(1) = (1/3)\theta''(1)(b - 2a - 3) \quad (A.15)$$

$$\theta'''(1) = \theta'(1) \left( -\frac{ab}{2} + \frac{a^2}{2} - b + 2a + 2 \right) \quad (A.16)$$

Because $\phi(x)$ is a linear function of $\theta(x)$, equations (A.15) and (A.16) imply

$$\phi''(1) = (1/3)\phi''(1)(b - 2a - 3) \quad (A.17)$$

$$\phi'''(1) = \phi'(1) \left( -\frac{ab}{2} + \frac{a^2}{2} - b + 2a + 2 \right). \quad (A.18)$$

Approximation of $J(R, \theta)$ around $R = \theta = 1$

A third-order Taylor series approximation of equation (18) around $R = \theta = 1$ yields

$$J(R, \theta) \approx J(1, 1) + J_R(1, 1)(R - 1) + J_{\theta}(1, 1)(\theta - 1) + 3J_{RR}(1, 1)(R - 1)^2 + 2J_{R\theta}(1, 1)(R - 1)(\theta - 1) + J_{\theta\theta}(1, 1)(\theta - 1)^2 \quad (A.19)$$

Partially differentiate equation (18) with respect to $R$ to obtain

$$J_R(R, \theta) = \phi(\theta). \quad (A.20)$$

Using equation (16) in equation (A.20) yields

$$J_R(1, 1) = \frac{1}{r + \delta}. \quad (A.21)$$

Differentiating equation (A.20) with respect to $\theta$ yields $J_{\theta\theta}(R, \theta) = \phi'(R)$ which implies (using equation (A.12)) that

$$J_{\theta\theta}(1, 1) = \frac{\gamma}{2(r + \delta)}. \quad (A.22)$$

Differentiating equation (A.20) twice with respect to $\theta$ yields $J_{\theta\theta\theta}(R, \theta) = \phi''(R)$ which implies (using equation (A.12)) that

$$J_{\theta\theta\theta}(1, 1) = \phi''(1). \quad (A.23)$$

Next, observe from equation (18) that $J(R, \theta)$ is linear in $R$ so that

$$J_{RR}(R, \theta) = J_{R\theta}(R, \theta) = J_{\theta\theta}(R, \theta) = 0. \quad (A.24)$$
To compute $J_G(R, G)$, partially differentiate equation (18) with respect to $G$ to obtain

$$J_G(R, G) = R \phi' (G) - \gamma G^{-1} \phi' (G^{-1}) + G^{-2} \phi'' (G^{-1}).$$  \hfill (A.24)

Evaluate equation (A.24) at $R = G = 1$ and use equation (17) to obtain $J_G(R, G) = 0$. Next, partially differentiate equation (A.24) with respect to $G$ to obtain

$$J_{GG}(R, G) = R \phi'' (G) - \gamma (\gamma - 1) G^{-2} \phi' (G^{-1}) + 2(\gamma - 1) G^{-3} \phi'' (G^{-1}) - G^{-4} \phi''' (G^{-1}).$$  \hfill (A.25)

Evaluate equation (A.25) at $R = G = 1$ and use equation (17) to obtain $J_{GG}(1, 1) = 0$. Differentiating equation (A.25) with respect to $G$ and evaluating the derivative at $R = G = 1$ yields

$$J_{GGG}(1, 1) = 2 \phi'' (1) - 3(\gamma - 2) \phi' (1) + 3(\gamma - 1)(\gamma - 2) \phi' (1) - \gamma (\gamma - 1)(\gamma - 2) \phi (1).$$  \hfill (A.26)

Use equations (A.13), (A.17), and (A.18) to replace $\phi (1)$, $\phi'' (1)$, and $\phi''' (1)$ by expressions that are proportional to $\phi' (1)$. Then simplify the resulting expression to obtain

$$J_{GGG}(1, 1) = (a + \gamma)(a + \gamma - b) \phi' (1).$$  \hfill (A.27)

Use equation (A.12) and the facts that $a + \gamma = a_\nu$ and $a + \gamma - b = a_\nu$ to rewrite equation (A.27) as $J_{GGG}(1, 1) = a_\nu a_\gamma / (2(r + \delta))$. Then use the fact that $a_\nu a_\gamma = -2(r + \delta) / a_\gamma^2$ to obtain

$$J_{GGG}(1, 1) = -\frac{\gamma}{a_\gamma^2}.$$  \hfill (A.28)

Now substitute these results into the approximation in equation (A.19). From above, $J_{R}, J_{RR}, J_{RRC}, J_{RG}, J_{GG}, J_{G}, J_{RGG}, J_{GGG}, J_{GGG}$ are zero, so we obtain

$$J(R, G) \approx J_R(1, 1)(R - 1) + J_{RG}(1, 1)(R - 1)(G - 1) + \frac{1}{8}[J_{RGG}(1, 1)(R - 1)(G - 1)^2 + J_{GGG}(1, 1)(G - 1)^2].$$  \hfill (A.29)

Setting $J(R, G) = 0$ in equation (A.29) yields

$$-\frac{1}{8}J_{GGG}(1, 1)(G - 1)^2 \approx [J_R(1, 1) + J_{RG}(1, 1)(G - 1) + \frac{1}{2} J_{RGG}(1, 1)(G - 1)^2](R - 1).$$  \hfill (A.30)

The terms $J_{RG}(1, 1)(R - 1)(G - 1)$ and $J_{RGG}(1, 1)(R - 1)(G - 1)^2$ on the right-hand side of equation (A.30) are negligibly small (relative to $J_R(1, 1)(R - 1)$) in the neighbourhood of $R = G = 1$. Using equations (A.21) and (A.28) we have

$$(G - 1)^2 \approx -\frac{6}{J_{GGG}(1, 1)} J_R(1, 1)(R - 1) = \frac{6 a_\gamma^2}{\gamma (r + \delta)} (R - 1)$$  \hfill (A.31)

which implies equation (20) in the text.

Acknowledgements. The authors thank two anonymous referees for helpful comments, Francisco Delgado for discussions, and seminar participants at Brown University, Columbia University, the Federal Reserve Bank of Atlanta, Georgetown University, M.I.T., Northwestern University, the University of Chicago, the University of Maryland, the University of Montreal, Wayne State University, and the Penn Macro Lunch Group for comments. Financial support from the National Science Foundation and a Sloan Foundation Research Fellowship is gratefully acknowledged.

REFERENCES


