Hydrodynamical Modes of the Superfluid He^4 with Its Vapor

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In our calculations, we restrict ourselves to the case where the volume of the vapor is very small and therefore the evaporation-condensation at the free surface cannot take place. Taking the effects of an external force into consideration, we have three kinds of modes for thick films or two kinds of modes for thin films. At the end, we comment on the fifth sound.

We study hydrodynamical modes of a superfluid He^4 with its vapor on it. Our system is confined in adiabatic rigid walls under the influence of a constant force (Fig. 1). Furthermore, a thickness of the vapor $H-h$ is assumed to be so small that the particle transfer parallel to the liquid surface hardly takes place by its own viscosity. Without the particle transfer, there is no evaporation-condensation mechanism at the liquid surface $z=\Sigma(x,t)$ and we therefore find that present fluctuations of the pressure and the temperature, $p'$ and $T'$, need not fulfill the Clausius-Clapeyron equation. In other words, the superfluid He^4 and its vapor suffer the superheating and the supercooling, respectively. Hereafter we assume $p=p(\rho)$ for our simplicity.

(i) Thick films of the superfluid He^4

For the density fluctuations of the liquid and the vapor, we have

$$\rho_i'<-c_i^2\nabla^2\rho_i'+gF\rho_i'=0, \quad (i=l, v)$$

where $i=l$ and $v$ represent the liquid and the vapor, respectively, and $g$ is the constant external force $g=(0,0,-g) \quad (g>0)$. $c_l$ is the sound velocity because $c_l^2=(\partial p_l/\partial \rho_l)_{\Sigma}=(\partial p_l/\partial \rho_l)_{\Sigma} \quad (C_v/C_p)=\partial p_l/\partial \rho_l)_{\Sigma}$. $C_v$ and $C_p$ are specific heats at constant pressure and at constant volume, respectively.

For the temperature fluctuations, we have

$$\nabla^2 T_i'=0 \quad (i=l, v)$$

and

$$C_{py} T_i'=(\kappa_{py}/\rho_{py}) \nabla^2 T_y' , \quad (i=l, v, 2)$$

where $c_i$ is the second sound velocity, $c_i^2=(\rho_{py}c_{\Sigma}T)/(\rho_{py}C_{\Sigma})$ and $\kappa_{py}$ is the thermal conductivity of the vapor. Deriving (1) ~ (3), we have replaced the equilibrium values $\rho_{\Sigma}(z)$, etc. and $c_l(z) \quad (i=l, v, 2)$ by their mean values, for example, $\rho_{\Sigma}=\int_0^H \rho_{\Sigma}(z) dz/h$. The suffix 0 represents the equilibrium value.

Next, we study the boundary conditions. First, we take up the law of mass conservation,

$$\frac{d}{dt} \int_0^L dx \int_0^H \rho_l(x,z,t) \rho_v(x,z,t) \rho_l(x,z,t) dz$$

Our assumption is valid for $H-h<\sqrt{\rho\omega}$ because, as is well known, the velocity field of the viscous fluid with a frequency $\omega$ strongly clings to the wall within the width of $\delta=\sqrt{\rho\omega}$ where $\rho$ and $\omega$ are the shear viscosity and the mass density of the vapor, respectively. Using the experimental results and $h/a=\Sigma T$, we have $\delta=10^{-4}$ cm at $T=1$ K. This length is about 10 times as large as the mean distance of gas particles at this temperature.
\[ + \int_{t(x,t)}^{\mu} \rho \psi(x, z, t) \, dz = 0. \tag{4} \]

\( L \) is assumed to be arbitrary but so large that we may ignore the disturbance by \( L \). Considering
\[ [v]_{z=0} = [v]_{z=H} = 0, \tag{5} \]
we obtain
\[ (\rho_{10} - \rho_{00}) z = 0 \times \left[ \rho_{00} v_{xz} + \rho_{00} \Phi_{xz} \right]_{z=H} \]
\[ + [\rho_{00} v_{y}]_{z=H} = 0 \tag{6} \]
from (4) up to first order in small fluctuating variables. In a similar way we have
\[ (\rho_{10} - \rho_{00}) z = 0 \times \left[ \rho_{00} v_{xz} + \rho_{00} \Phi_{yz} \right]_{z=H} \]
\[ + [\rho_{00} v_{y}]_{z=H} = 0 \tag{7} \]
from the law of entropy conservation. Deriving (7) we have used the second equality of
\[ \left( \frac{\partial T_{y}}{\partial z} \right)_{z=H} = \frac{\rho_{00} v_{x}}{\rho_{00}} \tag{8} \]
and
\[ \left( \frac{\partial T_{x}}{\partial z} \right)_{z=H} = \frac{\rho_{00} v_{y}}{\rho_{00}} \tag{9} \]
from the pressure balance at \( z = \xi(x, t) \).

For the sake of convenience, we rewrite (6), (7), and (9) as
\[ \left( \frac{\partial T_{y}}{\partial z} \right)_{z=H} = \frac{\rho_{00} v_{x}}{\rho_{00}} \tag{10} \]
and
\[ \left( \frac{\partial T_{x}}{\partial z} \right)_{z=H} = \frac{\rho_{00} v_{y}}{\rho_{00}} \tag{11} \]
respectively. Here we use \( (\partial T_{y}/\partial z) = 0 \) in (6) and (9). In (10) and (11), we have taken
\[ \xi(x, t) = v_{x}(x, y, t) = v_{x}(x, y, t) \tag{13} \]
into consideration. Here \( v_{x}(x, y, t) \) is defined by \( (\rho_{00} v_{x} + \rho_{00} \Phi_{x}) / \rho_{10} \) and we should note that (13) is not correct for the case where the evaporation-condensation takes place. Details of this problem are discussed in Refs. 2 and 3.

Inserting the type of solutions \( \rho_{1} \) of (1) into (5), (10) and (12), we have
\[ \left[ \frac{\rho_{10} - \rho_{00}}{\rho_{00}} \frac{H-h}{H} \right]_{\omega} \cdot \left[ \frac{\rho_{10} v_{x} + \rho_{10} \Phi_{x}}{\rho_{00}} \right]_{\omega} \left[ \frac{\rho_{10} v_{y} + \rho_{10} \Phi_{y}}{\rho_{00}} \right]_{\omega} = 0 \tag{14} \]
in the approximation up to first order in \( gh / \rho_{00} \). By (14), we have two types of the solution,
\[ \omega^{2} = \left( \frac{1 + \rho_{10} v_{x} h_{x} - h_{x}}{\rho_{00}} \right) \left( \frac{1 + \rho_{10} v_{y} h_{y} - h_{y}}{\rho_{00}} \right) \left( \frac{\rho_{00} v_{y} h_{y} - h_{y}}{\rho_{00}} \right) \cdot \frac{c_{z}^{2} k_{x} k_{y}}{c_{z}^{2}} \tag{15} \]
and
\[ \omega^{2} = \left( \frac{1 + \rho_{10} v_{x} h_{x} - h_{x}}{\rho_{00}} \right) \left( \frac{1 + \rho_{10} v_{y} h_{y} - h_{y}}{\rho_{00}} \right) \left( \frac{\rho_{00} v_{y} h_{y} - h_{y}}{\rho_{00}} \right) \cdot g h k_{x} \tag{16} \]
Strictly speaking, we have \( (d/dt) \int d v_{x} = d w_{x}(T_{y})^{2} / T_{y}^{2} \) in the vapor. However, we neglect the right-hand side because we use the linearized equations.
The first which agrees with the "classical" one obtained by Putterman et al. is the compressional mode and its wave function \( \rho'(z) \) spreads over the whole system, \( 0 < z < H \). The second is the so-called ripplon mode and \( \rho'(z) \) is localized near \( z = h \).

Combining the type of solutions \( T_i' \) of (2) and (3) with (8), (11) and \( T_i'(h) = T_i(h) \), we have

\[
\omega^2 = \epsilon_i^2 k^2 \left( 1 + \frac{\rho_{\text{so}}}{\rho_{\text{lo}}} \cdot \frac{\rho_{\text{fo}}}{\rho_{\text{lo}}} \cdot \frac{C_{\text{so}}}{C_{\text{lo}}} \cdot \frac{H - h}{h} \right),
\]

where \( C_{\text{so}} \) and \( C_{\text{fo}} \) are specific heats at constant pressure of the liquid and the vapor, respectively. \( c_2^2(h) \) is the second sound velocity determined by \( \{1 + \rho_{\text{so}}/\rho_{\text{fo}}\} \) and \( P_{\text{so}}(h) \).

The damping term relating to \( \rho' \) is neglected in (17) because it is the higher order correction in \( kh \ll 1 \).

Using the type of solution \( \rho_i' \) of (19) and the boundary conditions (5), (12), and (20), we obtain

\[
\left[ \frac{\rho_{\text{so}}}{\rho_{\text{lo}}} \cdot \frac{\rho_{\text{fo}}}{\rho_{\text{lo}}} \cdot \frac{C_{\text{so}}}{C_{\text{lo}}} \cdot \frac{H - h}{h} \right] \cdot \epsilon_i^2 k^2 \cdot \frac{1}{h} \cdot \rho_i'(h) + \frac{\rho_{\text{so}}}{\rho_{\text{fo}}}(h) \cdot \frac{\rho_{\text{fo}}}{\rho_{\text{lo}}} \cdot \frac{C_{\text{so}}}{C_{\text{lo}}} \cdot \frac{H - h}{h} \cdot \frac{1}{h} = 0.
\]

Seeking the dispersions \( \omega \sim k(hk \ll 1) \) for \( gh/c_i^2 < gH/c_i^2 < 1, \)

\[
\omega^2 = \left( \frac{\rho_{\text{so}}}{\rho_{\text{lo}}} \cdot \left( \frac{\rho_{\text{fo}}}{\rho_{\text{lo}}} \cdot \frac{C_{\text{so}}}{C_{\text{lo}}} \cdot \frac{H - h}{h} \right) \cdot \frac{1}{h} \cdot \rho_i'(h) + \frac{\rho_{\text{so}}}{\rho_{\text{fo}}}(h) \cdot \frac{\rho_{\text{fo}}}{\rho_{\text{lo}}} \cdot \frac{C_{\text{so}}}{C_{\text{lo}}} \cdot \frac{H - h}{h} \cdot \frac{1}{h} \right) = 0.
\]

This disagreement comes from the fact that they study the case where \( \rho' \) and \( T' \) can be assumed to fulfil the Clausius-Clapeyron equation.

(ii) Thin films of the superfluid He

We study the case of very thin films with \( v_n = 0 \). Instead of (1) and (2), we have

\[
\rho_i' - c_i^2 \epsilon_i^2 \rho_i' + \left( \rho_{\text{so}}/\rho_{\text{lo}} \right) \cdot c_i^2 A, \quad A = 1 + \frac{\rho_{\text{so}}}{\rho_{\text{fo}}}(h) \frac{c_i^2}{c_f^2}.
\]

The boundary condition (10) is modified by

\[
\left( \rho_{\text{so}} \cdot \frac{H - h}{h} \right) \cdot \rho_{\text{fo}} A + \rho_{\text{so}} A = 0.
\]

In this case, the external force is the van der Waals force, \(-\alpha/z^4\). But we have known that replacing \(-\alpha/z^4\) by \(-\alpha/h^4\) is a fairly good approximation (see Ref. 4).

the condition \( v_n = 0 \) as

\[
\left( \frac{\rho_{\text{so}}}{\rho_{\text{fo}}}(h) \cdot \frac{c_i^2}{c_f^2} \right) \rho_i'(h)
\]

In this case, the van der Waals force is the external force, \(-\alpha/z^4\). But we have known that replacing \(-\alpha/z^4\) by \(-\alpha/h^4\) is a fairly good approximation (see Ref. 4).
where we also use $|1 - A/A(h)| \ll 1$ for our simplicity. $A$ is the mean value of $A(z)$, $A = \int_0^H A(z) (dz/h)$. Equation (22) gives us two types of solutions. One is the compressional mode of $\rho'_{s'}$:

(22)

$$
\omega^2 = \left[ 1 + \frac{\rho_{s0} \rho_{s0}(h)}{\rho_{s0} \rho_{s0}(h) A(h)} \cdot \frac{H-h}{h} \right]^2 \left[ 1 + \frac{\rho_{s0} \rho_{s0}(h)}{\rho_{s0} \rho_{s0}(h) A(h)} \cdot \frac{H-h}{h} \right],
$$

and the other is the ripplon mode,

$$
\omega^2 = \frac{\rho_{s0} - \rho_{s0}(h)}{\rho_{s0} \rho_{s0}(h)} \cdot \frac{H-h}{h} \cdot g h k^2.
$$

Behaviors of $\rho'(z)$ of (23) and (24) are similar to those of (15) and (16), respectively.

We note that our results (23) and (24) are obtained in the separable form contrary to

$$
\omega^2 = \left[ \frac{\rho_{s0} \rho_{s0}(h)}{\rho_{s0} \rho_{s0}(h) A(h)} \right] k^2,
$$

which is obtained by Putterman for the thin superfluid helium film without the evaporation-condensation. In (25), the superfluid is assumed to be incompressible. Jelatis et al. proposed that the so-called fifth sound $c_{s5}^2 = (\rho_{s0}/\rho_{s0}) c_{s3}^2$ can be under-

stood by (25) with $g > 0$. However, we cannot obtain a similar result corresponding to (25).

Before ending this short note, we comment on one possibility for the fifth sound, $c_{s5}^2 = (\rho_{s0}/\rho_{s0}) c_{s3}^2$. When the system is kept at the end point of the metastable region of the isothermal curve given by the van der Waals state equation, we may expect $\omega^2 = c_{s5}^2 k^2$ by (23) because of $(\partial p_t/\partial T)_s = 0$ at this point and $H \sim h^k$.

The most interesting case is the case where $H-h$ is sufficiently large (the evaporation-condensation can take place) and the film thickness is very small. In this case, there seem to be two constraints on $\rho'$ and $T'$: one is the Clausius-Clapeyron equation and the other is $\rho_{s3} (\partial s_{s3}/\partial T)_s T_s = 0$ coming from $v_n = 0$. We do not study this case in this paper.


4) H. Kawamoto and T. Ohmi, Prog. Theor. Phys. 61 (1979), 983.

5) S. Putterman, Superfluid Hydrodynamics (North-Holland, Amsterdam, 1974), chap. V.


* Here, we use the relation $(\partial p/\partial T)_s = v T (\partial p/\partial T)_s C_T$ for $(\partial p/\partial T)_s = 0$ and the fact that the effect of $(\partial p/\partial T)_s$ is extremely small.