Stresses and Deflections in an Elastically Restrainted Circular Plate Under Hydrostatic Normal Pressure Over a Segment

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The authors, "after a considerable amount of algebraic manipulation and reduction," tried to put the results in the simplest possible form. However, a simplification can be effected if we note that the terms involving the negative powers of \( t \) in \( \Phi(z, \gamma) \) and \( \Psi(z, \gamma) \) [equations (27) and (30) of the paper] are the singular parts of the terms involving

\[
\log \frac{1 - te^{-iy}}{1 - te^{iy}}
\]

Thus the combination of these terms yields the regular parts of the terms involving

\[
\log \frac{1 - te^{-iy}}{1 - te^{iy}}
\]

With this consideration \( \Phi(z, \gamma) \) of equation (27a), for example, can be written

\[
\Phi(z, \gamma) = -Ke^{i\gamma} \left[ tU(t, \gamma) + \gamma \left( \frac{t}{48} - \cos^2 \gamma \frac{2}{3} + \cos \gamma t \right) + \frac{1}{12} \left( \frac{7}{2} \cos \gamma + \frac{9}{2} \sin \gamma \right) t^2 \right]
\]

where \( \text{Reg} \) means the regular part of. This notation can be adopted whenever it is of any use.

An interesting point suggests itself in connection with the paper. This is the problem of a plane vertical base of a cylindrical tank of horizontal axis and which is partially filled with two immiscible liquids. Referring to Fig. 2 of the paper, the plate in this case is divided into three regions, where region 1 bounded by the chord \( AA' \) is free, region 2 bounded by the chords \( AA', EE' \), and region 3 bounded by the chord \( EE' \) are laterally loaded with respective load intensity \( px \) and \( p'x \), \( P < P' \).

Let the complex potential functions be

\[
\Phi(z) = \Phi(z)_{[i = 1, 2, 3]},
\]

and put

\[
K = \frac{p}{16Dr}, \quad K' = \frac{p'}{16Dr'} \quad \gamma = \angle OAX, \quad \gamma' = \angle OAE,
\]

\[
f = OB = c \cos \gamma, \quad f' = c \cos \gamma'.
\]

The particular integrals may be taken as

\[
W_1 = 0, \quad W_2 = \frac{\pi}{24} K_{x_{2y}}^2 (z + \delta),
\]

\[
W_3 = \frac{\pi}{24} K_{x_{2y}}^2 (z + \delta)
\]

The continuity conditions across \( AA' \) are given by

\[
[\Phi(z)]^2 = \pi K \left( \frac{1}{48} t^4 - \frac{1}{2} f_{t^2} + \frac{7}{3} \right)
\]

\[
[\Psi(z)]^2 = \pi K \left( \frac{1}{240} t^4 - \frac{1}{6} f_{t^2} + \frac{2}{3} f_{t^2} - \frac{1}{2} f_{t^2}^2 + \frac{4}{15} f^4 \right)
\]

as given in equations (23). The same conditions across \( EE' \) will be

\[
[\Phi(z)]^2 = \pi (K' - K) \left( \frac{1}{48} t^4 - \frac{1}{2} f_{t^2}^2 \right)
\]

\[
[\Psi(z)]^2 = \pi (K' - K) \left( \frac{1}{240} t^4 - \frac{1}{6} f_{t^2}^2 + \frac{2}{3} f_{t^2}^2 - \frac{1}{2} f_{t^2}^2 + \frac{4}{15} f^4 \right)
\]

The functions \( \Phi(z) \), \( \Psi(z) \) will be obtained by the principle of superposition. Thus if \( \Phi(z, p, \gamma), \Psi(z, p, \gamma) \) be the complex potential functions corresponding to hydrostatic pressure of intensity \( px \) over the segment of angle \( 2\gamma \) and \( \Phi(z, p' - p, \gamma), \Psi(z, p' - p, \gamma) \) be those corresponding to hydrostatic pressure of intensity \( (p' - p)x \) over the segment of angle \( 2\gamma' \), then

\[
\Phi(z) = \Phi(z, p, \gamma) + \Phi(z, p' - p, \gamma)
\]

\[
\Psi(z) = \Psi(z, p, \gamma) + \Psi(z, p' - p, \gamma)
\]

Applying equations (27) of the paper we obtain

\[
\Phi(z) = -Ke^{i\gamma} \left[ tU(t, \gamma) + \gamma \left( \frac{t}{48} - \cos^2 \gamma \frac{2}{3} + \cos \gamma t \right) + \frac{1}{12} \left( \frac{7}{2} \cos \gamma + \frac{9}{2} \sin \gamma \right) t^2 \right]
\]

\[
+ \sin \gamma \left( \frac{t}{48} \right)^2 + \frac{1}{12} \left( \frac{7}{2} \cos \gamma + \frac{9}{2} \sin \gamma \right) t^2 \left( \sin \gamma - \frac{5}{2} \sin 3\gamma \right) t
\]

\[
- \frac{1}{16} \left( \frac{7}{2} \cos \gamma + \frac{9}{2} \sin \gamma \right) t^2
\]

\[
+ \text{Reg} \left\{ -i \left( t - 2 \cos \gamma + t^{-1} \right)^2 \right\}
\]

\[
(t + 6 \cos \gamma + t^{-1}) \log \frac{1 - te^{-iy}}{1 - te^{iy}}
\]

\[
\frac{1}{6 \gamma} \left( t - 2 \cos \gamma + t^{-1} \right) \log \frac{1 - te^{-iy}}{1 - te^{iy}}
\]

\[
- (K' - K) e^{i\gamma} \left[ tU(t, \gamma) + \gamma \left( \frac{t}{48} - \cos^2 \gamma' \frac{2}{3} t \right) + \sin \gamma' \left( \frac{t}{48} \right)^2 + \frac{1}{12} \left( \frac{7}{2} \cos \gamma' + \frac{9}{2} \sin \gamma' \right) t^2 \right]
\]

\[
+ \frac{2 \cos \gamma' t}{2} \left( t - \cos \gamma' \right) t + \sin \gamma' \left( \frac{t}{48} \right)^2 + \frac{1}{12} \left( \frac{7}{2} \cos \gamma' + \frac{9}{2} \sin \gamma' \right) t^2
\]
- \frac{1}{24} \left( \sin \gamma - \frac{5}{2} \sin 3 \gamma \right) t - \frac{1}{16} \left( \gamma + 2 \sin 2 \gamma \right) + \frac{1}{2} \sin 4 \gamma t + \text{Reg} \left\{ \frac{2}{96} (t - 2 \cos \gamma + t^{-1})^2 \right. \\
\left. \quad \times (t + 6 \cos \gamma + t^{-1}) \log \frac{1 - te^{-2x}}{1 - te^{-x}} \right\}

\text{and}
\Psi(t) = K \phi \left[ U(t, \gamma) - \gamma \left( \frac{t}{240} - \frac{\cos^2 \gamma}{6} t^3 \right) \right.
\left. + 2 \cos^2 \gamma \frac{t}{3} - \frac{\cos \gamma}{2} t + \frac{4}{15} \cos(\gamma) \right]
\left. - \sin \gamma \frac{t}{240} - 2 \sin 2 \gamma \frac{t}{480} t^2 + \frac{1}{24} \left( \sin \gamma + \frac{29}{30} \sin 3 \gamma \right) t \right]
\left. - \frac{1}{48} \left( \gamma - \frac{10}{24} \sin 4 \gamma \right) t - \sin \gamma \frac{t}{24} + 5 \sin 3 \gamma \frac{t}{48} + \text{Reg} \left\{ \frac{t}{480} (t - 2 \cos \gamma + t^{-1})^3 \right. \\
\left. \times (t - 4t^{-1}) (t + 6 \cos \gamma + t^{-1}) - 16 \cos^3 \gamma \right\} \log \frac{1 - te^{-\gamma}}{1 - te^{-2\gamma}} \right\}

\text{Applying equations (2) we get } \Phi(x) \text{ and } \Psi(x) \text{ to which, if equations (3) are applied, we obtain } \Phi(x), \Psi(x). \text{ The problem can thus be solved completely.}


B. Auld. In one of the references given in the paper, R. M. Rosenberg shows that the equations of motion of the system are analogous to the equations of motion of a mass particle sliding around in a potential well. This picture appears to be quite useful in visualizing some of the phenomena discussed in this paper. In the case of the linear problem it may be shown that the contour lines of the potential surface are ellipses in the Xi - xr plane. It is quite apparent then that, if the particle is placed at a certain point in the potential well and released, the projection of the particle path on the Xi - xr plane will be a straight line only if the particle has been placed initially on the major or minor axes of the elliptical contours, Fig. 1 of this discussion. In the case of the nonlinear problem it may be shown that for small displacements from the potential minimum the contour lines are again elliptical. However, the higher potential contour lines will exhibit kinks resulting from the spring nonlinearities. Because of these kinks the walls of the potential will exhibit topographical features in the form of ridges and canyons.

In connection with this paper we are chiefly concerned with what might be called "linear" ridges and canyons; that is, cases where the x1 - x2 projection of the top of a ridge or the bottom of a canyon is a straight line, Fig. 2. In the figure there is a linear canyon along line M1 and it is clear that a particle placed on the base of the canyon will oscillate back and forth along the canyon. This will be a stable oscillation regardless of the initial potential level of the particle, because the walls of the canyon exert restoring forces on the particle. In the case of line M2, however, it is seen that the contours change from concave inward at low potentials to concave outward at high potentials. Thus topographical feature is a canyon at low elevations but turns into a ridge at high elevations. If a particle is initially placed on M2, it will tend to oscillate back and forth along this line. If the initial elevation is below the critical contour at which the curvature of the contours changes sign, the oscillation will be stable because it lies in a canyon. However, if the initial elevation is above the critical contour line, the particle spends part of its period on a ridge and the motion will be unstable.

Depending on the degree of nonlinearity of the problem, there may be more than two linear topographical features in the potential walls. However, it is apparent from Fig. 2 that a necessary condition for the existence of more than two linearly related modes is that the contour lines in the vicinity of the minimum must be circles. This is because a particle can move linearly only