Two Models for Cavity Flow, a Theoretical Summary and Application[1]

A. J. ACOSTA.[2] I would like to thank Professors Street and Larock for presenting a very interesting and useful summary of their recent work in cavity flow theory. As mentioned by them, a number of different models for the nonlinear free streamline flow past hydrofoil shapes have been developed in the past; the present summary brings attention to newer models for the termination of the constant pressure cavity since this evidently leads to certain analytical simplifications. I wonder if it would be possible for the authors to describe somewhat more fully than is evident in Fig. 1 what the physical differences are between the two spiral models. Perhaps it would be of interest also to hear why these models are preferable to some of the other schemes proposed in the past.

The calculated lift coefficient as a function of spray sheet thickness shows an impressive agreement with the experimental results of reference [15]. It would be interesting to know for the various conditions of Fig. 3 the lift slope value, as this is a useful design parameter. In connection with the spray sheet thickness, there is a point upon which I am not quite clear; namely, the definition of submergence and spray sheet thickness. As the authors clearly point out, the two are not the same nor is it necessary that they be assumed to be the same even in the linear theory (see, for example, reference [18]). The calculations of the latter reference—though for zero cavitation number—show very clearly that the submergence for flows of finite depth can easily be negative. This is, in fact, suggested by the dotted lines of Fig. 2 of the present work. I wonder if the authors would care to comment further on the definition of submergence, especially in regard to the experimental situation that is faced in water tunnels and towing tanks. I would also like to ask if the particular model assumed in Fig. 1(*a*) for the calculation of submergence depends in any way upon the making of the potential at the points E and D the same, and if there might be some advantage in making the potential at these two points different.

C. S. SONG.[3] It may be interesting to point out that the author's "double-spiral model," initially named so by Tulin, is equivalent to one of the two models proposed by Song[4] and called "solution with constant wake pressure." The model was justified by the result of a pressure survey in the wake. It was used in a perturbation theory with the acceleration potential applied to an unsteady counterpart of the problems presented in this paper. Since, for a steady flow, the complex acceleration potential is equivalent to the linearized version of ω as defined by equation (6), the solution given by equation (8) is equivalent to the solution given by Song. It should also be noted that the solution (8) can be obtained by invoking the principle of minimum singularity without considering the "double-spiral model."

It can readily be observed that the "double-spiral model" leads to a logarithmically singular solution at the tail of the cavity, while the "single-spiral model" admits simple poles at the tail of the cavity. Therefore, it is not a coincidence that the "double-spiral model" is simpler and, perhaps, more accurate than the "single-spiral model."

Authors' Closure

The authors are most appreciative of the comments by Drs. Acosta and Song.

For a complete description of the physical aspects and mathematical bases of the cavity models that have been used, as well as justification for and useful mathematical properties of the models, we refer the reader to Tulin's original and most complete paper on supercavitating flows delivered to the *International Symposium on the Applications of Analytic Functions in Continuum Mechanics*, Tbilisi, Georgia, USSR, in September 1963 (available as *Hydronautics, Inc.* TR 121-3, September 5, 1963, or reference [13]). In this work, Tulin also establishes the relations between the nonlinear and linear versions of cavity models, the importance of which Dr. Song has emphasized in his discussion.

Unfortunately the authors have not had available the computer time required to calculate the lift slope as suggested by Dr. Acosta. Because the lift slope is a nonlinear function of cavitation number, angle of attack and spray sheet thickness, a large number of computations would be required to construct a representative plot. Indeed, it may be more profitable at this time to pursue the suggestion of Dr. Acosta regarding a redefinition of submergence. As the authors noted in reference [6], the present definition of submergence, which is arbitrary, was made to give some idea of the relationship between spray sheet thickness, submergence, and the other parameters of the problem. In the present formulation the vertical location y of the free surface of the fluid for negative x is asymptotically proportional to the product of the lift coefficient and $\ln |x|$ for large negative x. If, as Dr. Acosta suggests, the potentials at points D and E were allowed to be different, or perhaps additional singularities were introduced at D and E, the extra parameters thereby made available could be used in the solution to make $\lim_{x \to -\infty} y$ finite. The submergence could then be defined unambiguously. An investigation of this idea would be worthwhile.

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[1] By R. L. Street and B. E. Larock, published in the June issue of the JOURNAL OF BASIC ENGINEERING, pp. 269–274.

[2] Division of Engineering and Applied Science, California Institute of Technology, Pasadena, Calif. Mem. ASME.

[3] Associate Professor, Department of Civil Engineering, University of Minnesota, Minneapolis, Minn.

[4] Song, C. S., "Two-Dimensional Supercavitating Plate Oscillating Under a Free Surface," *Journal of Ship Research*, Vol. 9, No. 1, June 1965 (also, St. Anthony Falls Hydraulic Laboratory Technical Paper No. 47, Series B, Dec. 1963).

[5] Van Dyke, Milton, *Perturbation Method in Fluid Mechanics*, Academic Press, 1964, pp. 53.

Computerized Method of Characteristics Calculations for Unsteady Pneumatic Line Flows[1]

MICHAEL A. STONER.[2] The author has presented a method of analysis for unsteady gas flow which is certain to receive much attention in the future as a design and analysis tool for pneumatic lines. The method of characteristics' ability to handle the nonlinear terms of the gas dynamics equations and the ease with which the resulting equations can be programmed on a digital computer leads one to this conclusion.

The author has chosen to go to a rectangular grid or a method of specified time intervals rather than use a free grid or characteristic grid for the x,t solution scheme of the characteristics equations. His reasons for doing so are acknowledged; however, it is possible that a characteristics grid scheme would give better results. In problems such as the author's where the

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[1] By J. R. Manning, published in the June 1968 issue of the JOURNAL OF BASIC ENGINEERING, TRANS. ASME, pp. 231–241.

[2] NSF Trainee, Civil Engineering Department, University of Michigan, Ann Arbor, Mich.

slope of the C+ and C− characteristic lines change significantly during the course of the computations, the rectangular grid method is not recommended because of the large interpolations required. An interesting consequence of using a procedure employing large interpolations is an increase in the effective propagation velocity; something which might have a great effect on the author's resonant frequency problem. In reference to the author's Fig. 1, assume that XRR is 80 percent of DELX, and that a disturbance exists at point 1. Because of the interpolation, the effects of this disturbance will arrive at the point labeled U,A about 20 percent earlier than in the real case. This effect is especially significant for steep fronted disturbances. To minimize this error one should try to keep the interpolation distances XRR and XLR as close to DELX as possible. This could be done in the author's program by allowing RATINC to change from one time step to the next. This is done by searching the just computed time line for the largest absolute value of U + A or U − A and changing DELTI accordingly. DELX is not changed since it is fixed by the pipe length and the number of reaches used.

In determining the number of reaches MX necessary to get a nonchanging representation of a given transient, the author simply increased the number until the result ceased to change significantly. It should be pointed out that the number of reaches necessary is dependent upon the relationship between the period of the pipe and the highest frequencies present in the disturbances to be propagated in the pipe.

In regard to the approximate area computations the question is raised as to why the more exact approach was not used. If the cross-sectional area variation term is kept in the basic continuity equation, the author's equation (1) becomes

$$5A_{TI} + 5UA_X + AU_X + UA\frac{F_X}{F} = 0 \qquad (1)$$

where F is a dimensionless representation of the cross-sectional area. This equation and equation (2) can be carried through the method of characteristics with no added difficulty. This permits a more representative solution of area-change problems since $\frac{F_X}{F}$ can have any desired functional relationship with x instead of having to be approximated by a step function.

In discussing the effect of superacoustic shock velocity on oscillation frequency in resonance tubes the statement is made that. . . . "This observation casts some doubt on the validity of the method of characteristics analysis for resonance tubes." It should be pointed out that it is not the method of characteristics which should be faulted but the validity of the equations to which the method of characteristics is applied. The inclusion of the proper terms in the basic equations and the use of the correct set of basic equations is the investigators responsibility. The method of characteristics only reduces the partial differential equations to a set of ordinary differential equations which can then be integrated.

W. A. WOODS.[3] I was very interested in Professor Manning's paper. I had been associated with the research mentioned in the author's reference [6] and since that time, the work at Liverpool has been extended to cover a number of other topics, and I wished to add to the communication of Dr. Woollatt and give a brief review of what they had covered.

The main difference between Professor Manning's paper and the approach used at Liverpool was that the former considered homentropic flow, and the slope of the characteristics in the position diagram were evaluated using the mean values of A and U at the beginning and end of the time step; the latter on the other hand used basically an Euler method of integration (see, for example, reference [1, p. 163][4]), using the values of A and U at the beginning of the time step. This procedure was necessary because the Liverpool approach treated nonhomentropic flow by including effects of pipe friction, heat transfer, and entropy gradients.

Situations including a threeway tee junction have been studied in reference [2] and configurations with area changes in reference [3]. The cylinder boundary problem has been studied for the poppet valve case, reference [4], and the piston ported engine is currently being investigated at Liverpool, The work on flow at branches has been extended to an inclined threeway junction [5] and also to a fourway branch [6].

The method of characteristics is also being applied to discharge problems in chemical engineering [7] and the writer is associated with a rapid expansion experiment being carried out at the National Engineering Laboratory in Britain. Computer programs are being developed for the latter two projects which are of special interest because they deal with both subsonic and supersonic unsteady flows.

A comprehensive investigation on the application of the method of characteristics by means of a computer program to flow in a high-speed, supercharger four-stroke diesel engine has recently been completed [8]. Some of the aforementioned and other work in this field was recently reviewed by Benson [9].

In conclusion the findings of our work at Liverpool are that the computer programs just mentioned can predict accurately gas exchange processes in various applications provided care is taken on representing the boundary conditions adequately and that irreversible effects such as pipe friction and heat transfer are included in the treatment of the characteristics. A final word about the boundary conditions is that the characteristic and boundary condition equations should be solved simultaneously at the end of each time step; otherwise there is a possibility that the program will become unstable. This is discussed fully in reference [6].

Additional References

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3 Benson, R. S.. Garg, R. D., and Woods, W. A., "Unsteady Flow in Pipes With Gradual or Sudden Area Changes," *Proceedings of The Institution of Mechanical Engineers*, Vol. 178, Part 3 I, Paper 9, 1963–1964, pp. 1–23.

4 Woods, W. A., and Khan, S. R., "Discharge From a Cylinder Through a Poppet Valve to an Exhaust Pipe," *Proceedings of The Institution of Mechanical Engineers*, Vol. 182, Part 3 H, 1967–1968, pp. 1–8.

5 Few, P. C., "Compressible Flow in an Inclined Junction," ME thesis, Liverpool University, 1966.

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[3] University of Liverpool, England.

[4] Numbers in brackets designate Additional References at end of discussion.

6 Daneshyar, H., "Unsteady Gas Flow Through Branched Exhaust Systems of Multi-Cylinder Engines," PhD thesis, Liverpool University, 1967.

7 Woods, W. A., and Thornton, R. E., "Calculation of Transient Forces During Emergency Escape of Gases From an Autoclave, With Special Reference to Design Methods," *Proceedings of the Institute of Chemical Engineers*, Third Symposium on Chemical Process Hazards, 1967.

8 Douthwaite, W., "Gas Flow Through a Supercharged Four-Stroke Diesel Engine," PhD thesis to be submitted to Liverpool University, 1968.

9 Benson, R. S., "Some Recent Research on Nonsteady Flow Problems," ASME Fluid Meters and Flow Measurements Conference, Pittsburgh, 1966.

D. WOOLLATT.[5] The advent of the digital computer resulted in the development of several very useful methods for the solution of unsteady flow problems. Many of these were intended primarily for shock wave calculations and used Lagrangian coordinates. Other methods, employing Eulerian coordinates, are more useful for most engineering problems. Two methods of this kind are that given by Lax[6] and that used in this paper. This second method, that I have also used, is preferable to that of Lax as complex boundary conditions may be more easily accommodated.

It is true that we were inexcusably unaware of Hartree's work when writing reference [6] of the paper. We were, however, aware of the even earlier work of Courant, Isaacson, and Rees[7] whose "rectangular mesh" method is identical to the method we used except in the choice of the point at which the slopes of the characteristics are evaluated. The most recent version[8] of this method for homentropic flow represents a further improvement in the calculation of these slopes. The method described there of using characteristics through the mesh points at the beginning of the time steps has some advantages for homentropic flows.

A feature of the method described in reference [6] of this paper was the variable time step (i.e., variable RATING). Some test calculations made at that time showed that, if the value of RATING is slightly too large, the calculation will proceed successfully for a considerable time and then suddenly become violently unstable. If RATING is slightly lower than necessary, reasonably accurate results will be obtained, but if RATING is significantly lower than the maximum value, then any discontinuties will be rapidly smoothed out and all high frequency pulsations will be removed from the solution which will otherwise be unaffected. Using a value of RATING that varies from time step to time step adds little to the complexity of the program and gives the fastest, most accurate solution. An investigation of the effect of reducing the number of meshes used (increasing the mesh size) showed that high frequency pulsations are dampened out, but that the amplitude and form of slower pulsations are unaffected.

I have been associated with the application of this method to problems of internal combustion engines and reciprocating compressors. A large number of boundary conditions (e.g., Poppet valves, nonreturn disk valves, sudden area changes, three-way pipe junctions, orifices, and gauzes) have been developed and the method used for hundreds of design and research calculations. The basic method has proved extremely satisfactory and I feel that users in the field of pneumatic line analysis can have every confidence in Dr. Manning's method. It is accurate, easy to program, and can easily be used for complicated piping systems if the necessary boundary conditions are known.

A word of warning is perhaps appropriate in the use of the energy equation in a boundary condition for a homentropic flow calculation (equation 22). This leads to the assumption of isentropic flow across the boundary. A better approximation can be obtained by assuming that AL and AR represent pressure rather than temperature; i.e., $AL = \left(\frac{p_1}{p_e}\right)\frac{k-1}{2k}$, etc. The momentum equations or empirical pressure loss data can then be used for the boundaries. This will give greatly improved accuracy in the prediction of pressure pulsations if the stagnation pressure loss at any boundary is significant. A simple way of feeding in empirical boundary conditions for both single and multiple pipe boundaries is to use polynomials to relate the Riemann variable ratios.[9] However, in this paper, the area changes are used to represent a gradual area change in which the flow would be homentropic. In this special case, the assumption of no loss in the area change is probably better than the correct solution. It would, incidentally, be possible to include a term in the characteristic compatibility equations (equations (15) and (16)) to account directly for gradual area changes.

It is quite feasible (reference [6] of the paper) to extend the method to flow with entropy gradients and discontinuties in an axial direction and with friction and heat transfer. However, the characteristic calculation and, even more important, the boundary conditions[10] are much more complicated in that case. Care must be taken in the "plotting" of the third characteristic, the path line. The path line is steep (large $\Delta t/\Delta x$) compared with the wave characteristic and, if the same grid is used for both, any longitudinal discontinuity of entropy will be rapidly smoothed out. We have therefore chosen to abandon the mesh method for path lines and to "plot" the path line without interpolation. However, it has been shown that, even in i.c. engine exhaust systems where there are very large entropy discontinuities, the homentropic solution gives good results as far as the pressure diagrams are concerned. It obviously cannot give even an approximately accurate temperature.

The solution of problems with both subsonic and supersonic flow is not easy and I hope that the author will continue with his investigation of these problems as well as the others he has mentioned. I look forward to seeing the results of work from Stanford in the near future.

Author's Closure

The author sincerely appreciates the careful and extensive comments of Dr. Stoner, Prof. Woods, and Dr. Woollatt; they will be of help both to users of the technique in its present state and to research efforts aimed at further improving the method.

The author agrees that a variable ratio of time-step to space-step size (RATINC) based on scanning the previous Tl line would be an easily realized, yet substantial improvement to the method of his paper. The best way to accommodate gradual area change in homentropic flow will remain unknown until more numerical results are available; it is certainly true that the concept mentioned by Dr. Stoner is more direct from the standpoint of fluid mechanics.

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[5] Worthington Corporation, Buffalo, N. Y.

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