Nonlinear Shell Theory With Finite Rotation and Stress Function Vectors[1]

ERIC REISSNER.[2] The results of this paper introduce a very interesting and, as far as the writer knows, new element into nonlinear shell theory, by succeeding in the formulation of strain-displacement relations in terms of finite translational *and* rotational displacement components, through the ingenious use of an "intermediate" triad of base vectors.

The writer would like to take issue with the authors in regard to a number of minor points. The first of these is the authors' statement "The introduction of an intermediate triad . . . provides a convenient mechanism for describing bending, transverse shear, and rotational strains. These latter two strains are initially included to indicate how one may generalize Reissner's theories of shallow shells and of symmetrically deforming shells of revolution suffering transverse shearing strains and supporting stress couples turning about the midsurface-normal. Subsequently, we restrict attention to conventional shell theory." The reason why the writer, in 1968–1969, formulated nonlinear shallow shell theory and symmetrical shell-of-revolution theory with the inclusion of transverse shear deformation and midsurface-normal stress-couple components was the observation that by so doing the theory became more harmonious and in certain ways actually simpler than "conventional" shell theory, with the possibility remaining to recover conventional theory by appropriate specialization of constitutive equations. The writer is persuaded that adoption of his point of view by the present authors played a more essential role in their developments than is indicated by the sentences just quoted.

It is interesting to observe that Simmonds and Danielson establish their results by independent considerations of kinematics and statics (dynamics), followed by the statement ". . . compatibility conditions (i.e., strain-displacement relations), and equations of motions must be supplemented by constitutive equations . . . ." "The *proper form of these missing ingredients* may be inferred from the equation of balance of mechanical power . . . ." In the light of the foregoing reasoning it is fortunate that the authors' kinematical and statical results are in fact *consistent* via an application of the equation of balance of mechanical power (or equivalently the principle of virtual work), so that it is indeed physically reasonable to relate the authors' strain measures to their measures of stress. However, when this is done it should be recognized that the authors are dealing with a strictly *two*-dimensional theory and that for such a theory the problem of formulating constitutive equations—by way of suitably designed experiments, or by way of a step from given three-dimensional constitutive equations to the desired two-dimensional system is not actually considered by them. It seems appropriate to emphasize this point in the light of the authors' statement "For elastically isotropic shells undergoing small extensional strains, the strain-energy densities may be taken (as in Koiter's 1960 Delft Symposium paper) . . . ."

Authors' Closure

Professor Reissner's close reading of our paper is appreciated. However, we feel that he has drawn certain inferences that are unwarranted.

First, nowhere do we introduce *translational* displacement components. As is evident from the first two paragraphs of our paper, an essential feature of our theory is that it avoids explicit introduction of displacement components. (Their existence is guaranteed by satisfaction of our compatibility condition (12).)

Second, we happily acknowledge the influence of Reissner's work on our own; references [7–10] in our paper indicate this. At the same time, we fail to see how inclusion of effects beyond the scope of classical first-approximation theory, such as transverse shearing strains and stress couples turning about the midsurface normal, simplifies the derivation of our theory. We introduced transverse shearing and rotational strains into our kinematic development precisely for the reason stated in the sentences quoted by Reissner and no other. In this connection, we note that the key kinematic relations used in the paper under discussion were developed in an antecedent paper without any reference to transverse shearing or rotational strains. Furthermore, if one introduces the assumptions $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}$, $\gamma_\alpha = M^\alpha = \mathbf{l} = \mathbf{m} = 0$ of conventional shell theory at the start, one may move easily and directly from (11), (19), and (20) of our paper via the principle of virtual work to (48)–(50).

Third, Reissner's remark that it is "fortunate" that our static and kinematic results are consistent with the principle of virtual work is misleading since the whole point of the section of our paper entitled "Modifications" was to replace the conventional stress measures by ones that *were* consistent with the principle of virtual work.

Finally, Reissner is quite right in pointing out that our theory is strictly two-dimensional. However, in the first paragraph on page 1087 of the December, 1972, issue of the JOURNAL OF APPLIED MECHANICS, we indicated that the portions of our equations which contained the (conventional) two-dimensional stress measures could be obtained by integrating the analogous terms in the three-dimensional equations of motion across the thickness. It was because of this connection that we felt justified in appealing to Koiter's derivation of two-dimensional stress-strain relations from three-dimensional ones. An up-to-date discussion of this point may be found in Section 3 of [1].[3]

Reference

1 Koiter, W. T., and Simmonds, J. G., "Foundations of Shell Theory," Report No. 473, Laboratorium voor Technische Mechanica, Delft, 1972; also to appear in the *Proceedings of the 13th International Congress of Theoretical and Applied Mechanics*, Moscow, Aug. 1972.

[3] Number in brackets designates Reference at end of Closure.

[1] By J. G. Simmonds and D. A. Danielson, published in the December, 1972, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 39, No. 4, TRANS. ASME, Vol. 94, Series E, pp. 1085–1090.

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The First Passage Time For Simple Structural Systems[1]

J.-N. YANG.[2] The authors are to be congratulated for presenting a very fine analysis for the first passage problem of a single-degree-of-freedom system excited by white noise. The integral equation presented in the paper is very interesting and is worthy of pursuing further research so that the method can be applied to cases involving not only nonwhite and nonstationary excitations, but also complex systems. Although the authors did suggest an extension to a more complex system, it becomes apparent that for a n-degree-of freedom system $(2n - 1)$ folds of numerical integrations (see equation (12) for $n = 2$) has to be carried out before solving the integral equation. This is insurmountable even with a high-speed digital computer. The restriction on white-noise input and the difficulty involved in extending the integral

[1] By J. W. Shipley and M. C. Bernard, published in the December, 1972, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 39, No. 4, TRANS. ASME, Vol. 94, Series E, pp. 911–917.

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[3] Numbers in brackets designate References at end of Discussion.

equation approach to complex systems stem from the fact that the Markov property of the response vector should be satisfied. The writer wishes to propose herein to make approximation directly on $k(t, \eta)$ so that nonwhite and nonstationary excitations as well as complex systems can be dealt with.

Note that $k(t, \eta)dt$ is the probability of excursion (outward crossing) in the time interval $(t, t + dt]$ given that the sample paths have first excursion in the time interval $(\eta, \eta + d\eta]$. Approximating the event of first excursion in $(\eta, \eta + d\eta]$ by the event of excursion in $(\eta, \eta + \eta]$, i.e., neglecting what happened about $X(t)$ before η, except the initial condition, one obtains,

$$k(t, \eta)dt \simeq P \text{ [excursion in } (t, t + dt] | \text{excursion in } (\eta, \eta + d\eta], x, \dot{x}, 0]$$

$$= \frac{N(t, \eta | x, \dot{x}, 0)}{N(\eta | x, \dot{x}, 0)} dt \qquad (A)$$

in which $N(t, \eta | x, \dot{x}, 0)dtd\eta$ is the probability of simultaneous excursions in the time intervals $(t, t + dt]$ and $(\eta, \eta + d\eta]$, and $N(\eta | x, \dot{x}, 0)d\eta$ is the probability of excursion in $(\eta, \eta + d\eta]$ (see equation (10)). The explicit expressions of $N(t, \eta | x, \dot{x}, 0)$ and $N(\eta | x, \dot{x}, 0)$ for structural responses have been given in [3, 6].

With the substitution of equation (A) into equation 7, the integral equation becomes

$$N(t | x, \dot{x}, 0) \simeq f(t | x, \dot{x}, 0) + \int_0^t \frac{N(t, \eta | x, \dot{x}, 0)}{N(\eta | x, \dot{x}, 0)} f(\eta | x, \dot{x}, 0) d\eta \qquad (B)$$

In conjunction with the point process approximation [1, 2],[3] similar approach as equation (B) has been used [3] and it is found that the results for both stationary and nonstationary excitations are very satisfactory in comparison with the simulation results. Therefore, it is the writer's belief that equation (B) will produce satisfactory results for $f(t | x, \dot{x}, 0)$.

References

1 Yang, J.-N., and Shinozuka, M., "On the First-Excursion Probability in Stationary Narrow-Band Random Vibration," JOURNAL OF APPLIED MECHANICS, Vol. 38, No. 4, TRANS. ASME, Vol. 93, Series E, Dec. 1971, pp. 1017–1022.

2 Yang, J.-N., "First-Excursion Probability in Nonstationary Random Vibration," *Journal of Sound and Vibration*, Vol. 27, No. 2, 1973.

3 Yang, J.-N., "First-Passage Analysis Under Nonstationary Random Excitation," to appear in the special issue of the *Journal of Acoustical Society of America*, 1973.

Authors' Closure

The possibility of direct kernel approximation which Dr. Yang has noted does appear to increase the range of application of the integral equation procedure which the authors proposed, and his comparison with some of his own recent work is interesting. Several comments regarding the paper and Dr. Yang's discussion may add perspective.

The method proposed by the authors is applicable to stationary excitation. This case is discussed in reference [4],[4] and the results agree quite favorably with numerical simulation studies.

Equation (11) in the paper appears to be equivalent to Dr. Yang's equation (B). The kernel for equation (11) was expressed as a $2n - 1$ fold integral to show the relationship between processes of higher dimension and the harmonic oscillator. The authors did not intend to compute a $2n - 1$ fold numerical integration but to use the crossing rate functions for the higher-order system to obtain the kernel. (These crossing rate functions are similar in form to those expressed in the paper for the damped harmonic oscillator.) In fact, for a Gaussian process this interpretation of the kernel will lead rapidly to the result that all the integrations but two in the $2n - 1$ fold integral can be represented in closed form. This leaves numerical integrations which are no more difficult than those indicated for the harmonic oscillator.

[4] Number in bracket designates Reference at end of Closure.

Reference

4 Bernard, M. C., and Shipley, J. W., "The First Passage Problem for Stationary Random Vibration," *Journal of Sound and Vibration*, Vol. 24 (1) 1972, pp. 121–132.

Flow Patterns of a Circular Vortex Ring With Density Difference Under Gravity[1]

C. F. CHEN.[2] Buoyant vortex rings are of current interest because they offer the possibility of carrying effluents from a stack to a much higher level thus relieving the area immediately surrounding the stack. To spread out the pollution over a much wider area at a lower intensity, however, may not be the ultimate solution. The height of rise (descent) of a buoyant (heavy) vortex ring has been thoroughly investigated by Turner.[3] (This reference seems to have escaped the authors' attention.) The present results show that the heavy vortex rings generated usually travel about 10 dia from the orifice, stop, then break up into one of the three patterns shown in the authors' Fig. 2. According to Turner's experience, buoyant vortex rings rise in a uniform surrounding to considerable heights (>45 dia). It is when the surrounding is stratified the buoyant vortex ring loses its buoyancy by continuous entrainment of the surrounding fluid and ultimately stops. The density stratification in Turner's experiment corresponds to a temperature stratification of about 1° F/ft in air. I wonder if the authors took special precautions to maintain an isothermal state in the test tank.

Again referring to the authors' Fig. 2, the breakdowns at stage 3 for the patterns A and C are quite similar to those observed by Maxworthy[4] for neutral density vortex rings. Gravitational effect comes into play later to generate the "mushroom" from the lumps in pattern A. In pattern C, the gravitational effect is minimized by the vigorous mixing in the early stages of the vortex motion thus reducing any initial differences in densities of the vortex ring and the surrounding.

Pattern B, with its disklike vortices, must be a genuine density-difference effect. In a neutral density ring, the velocity field of a vortex is neutrally stable with respect to centrifugal instabilities. However, if the core is heavier than the surrounding, this is no longer the case. This perhaps explains why such instabilities have not been observed either in buoyant or neutral density vortex rings.

G. E. MATTINGLY.[5] This paper contains several interesting experimental observations of vortex ring motion. In commenting

[1] By C.-J. Chen and L.-M. Chang, published in the December, 1972, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 39, No. 4, TRANS. ASME, Vol. 94, Series E, pp. 869–872.

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[3] Turner, J. S., "Buoyant Vortex Rings," *Proceedings, Royal Society*, London, Series A, Vol. 239, 1957, pp. 61–75.

[4] Maxworthy, T., "The Structure and Stability of Vortex Rings," *Journal of Fluid Mechanics*, Vol. 51, 1972, pp. 15–32.

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