equation approach to complex systems stem from the fact that the Markov property of the response vector should be satisfied. The writer wishes to propose herein to make approximation directly on $k(t, \eta)$ so that nonwhite and nonstationary excitations as well as complex systems can be dealt with.

Note that $k(t, \eta)dt$ is the probability of excursion (outward crossing) in the time interval $(t, t + dt]$ given that the sample paths have first excursion in the time interval $(\eta, \eta + d\eta]$. Approximating the event of first excursion in $(\eta, \eta + d\eta]$ by the event of excursion in $(\eta, \eta + \eta]$, i.e., neglecting what happened about $X(t)$ before η, except the initial condition, one obtains,

$$k(t, \eta)dt \simeq P\,[\text{excursion in } (t, t+dt] \,|\, \text{excursion in } (\eta, \eta + d\eta], x, \dot{x}, 0]$$

$$= \frac{N(t, \eta | x, \dot{x}, 0)}{N(\eta | x, \dot{x}, 0)}\, dt \qquad (A)$$

in which $N(t, \eta|x, \dot{x}, 0)dtd\eta$ is the probability of simultaneous excursions in the time intervals $(t, t + dt]$ and $(\eta, \eta + d\eta]$, and $N(\eta|x, \dot{x}, 0)d\eta$ is the probability of excursion in $(\eta, \eta + d\eta]$ (see equation (10)). The explicit expressions of $N(t, \eta|x, \dot{x}, 0)$ and $N(\eta|x, \dot{x}, 0)$ for structural responses have been given in [3, 6].

With the substitution of equation (A) into equation 7, the integral equation becomes

$$N(t|x, \dot{x}, 0) \simeq f(t|x, \dot{x}, 0) + \int_0^t \frac{N(t, \eta|x, \dot{x}, 0)}{N(\eta|x, \dot{x}, 0)} f(\eta|x, \dot{x}, 0) d\eta \qquad (B)$$

In conjunction with the point process approximation [1, 2],[3] similar approach as equation (B) has been used [3] and it is found that the results for both stationary and nonstationary excitations are very satisfactory in comparison with the simulation results. Therefore, it is the writer's belief that equation (B) will produce satisfactory results for $f(t|x, \dot{x}, 0)$.

References

1 Yang, J.-N., and Shinozuka, M., "On the First-Excursion Probability in Stationary Narrow-Band Random Vibration," Journal of Applied Mechanics, Vol. 38, No. 4, Trans. ASME, Vol. 93, Series E, Dec. 1971, pp. 1017–1022.

2 Yang, J.-N., "First-Excursion Probability in Nonstationary Random Vibration," *Journal of Sound and Vibration*, Vol. 27, No. 2, 1973.

3 Yang, J.-N., "First-Passage Analysis Under Nonstationary Random Excitation," to appear in the special issue of the *Journal of Acoustical Society of America*, 1973.

Authors' Closure

The possibility of direct kernel approximation which Dr. Yang has noted does appear to increase the range of application of the integral equation procedure which the authors proposed, and his comparison with some of his own recent work is interesting. Several comments regarding the paper and Dr. Yang's discussion may add perspective.

The method proposed by the authors is applicable to stationary excitation. This case is discussed in reference [4],[4] and the results agree quite favorably with numerical simulation studies.

Equation (11) in the paper appears to be equivalent to Dr. Yang's equation (B). The kernel for equation (11) was expressed as a $2n - 1$ fold integral to show the relationship between processes of higher dimension and the harmonic oscillator. The authors did not intend to compute a $2n - 1$ fold numerical integration but to use the crossing rate functions for the higher-order system to obtain the kernel. (These crossing rate functions are similar in form to those expressed in the paper for the damped harmonic oscillator.) In fact, for a Gaussian process this interpretation of the kernel will lead rapidly to the result that all the integrations but two in the $2n - 1$ fold integral can be represented in closed form. This leaves numerical integrations which are no more difficult than those indicated for the harmonic oscillator.

[4] Number in bracket designates Reference at end of Closure.

Reference

4 Bernard, M. C., and Shipley, J. W., "The First Passage Problem for Stationary Random Vibration," *Journal of Sound and Vibration*, Vol. 24 (1) 1972, pp. 121–132.

Flow Patterns of a Circular Vortex Ring With Density Difference Under Gravity[1]

C. F. CHEN.[2] Buoyant vortex rings are of current interest because they offer the possibility of carrying effluents from a stack to a much higher level thus relieving the area immediately surrounding the stack. To spread out the pollution over a much wider area at a lower intensity, however, may not be the ultimate solution. The height of rise (descent) of a buoyant (heavy) vortex ring has been thoroughly investigated by Turner.[3] (This reference seems to have escaped the authors' attention.) The present results show that the heavy vortex rings generated usually travel about 10 dia from the orifice, stop, then break up into one of the three patterns shown in the authors' Fig. 2. According to Turner's experience, buoyant vortex rings rise in a uniform surrounding to considerable heights (>45 dia). It is when the surrounding is stratified the buoyant vortex ring loses its buoyancy by continuous entrainment of the surrounding fluid and ultimately stops. The density stratification in Turner's experiment corresponds to a temperature stratification of about 1° F/ft in air. I wonder if the authors took special precautions to maintain an isothermal state in the test tank.

Again referring to the authors' Fig. 2, the breakdowns at stage 3 for the patterns A and C are quite similar to those observed by Maxworthy[4] for neutral density vortex rings. Gravitational effect comes into play later to generate the "mushroom" from the lumps in pattern A. In pattern C, the gravitational effect is minimized by the vigorous mixing in the early stages of the vortex motion thus reducing any initial differences in densities of the vortex ring and the surrounding.

Pattern B, with its disklike vortices, must be a genuine density-difference effect. In a neutral density ring, the velocity field of a vortex is neutrally stable with respect to centrifugal instabilities. However, if the core is heavier than the surrounding, this is no longer the case. This perhaps explains why such instabilities have not been observed either in buoyant or neutral density vortex rings.

G. E. MATTINGLY.[5] This paper contains several interesting experimental observations of vortex ring motion. In commenting

[1] By C.-J. Chen and L.-M. Chang, published in the December, 1972, issue of the Journal of Applied Mechanics, Vol. 39, No. 4, Trans. ASME, Vol. 94, Series E, pp. 869–872.

[2] Professor, Department of Mechanical and Aerospace Engineering, Rutgers University, The State University, New Brunswick, N. J.

[3] Turner, J. S., "Buoyant Vortex Rings," *Proceedings, Royal Society*, London, Series A, Vol. 239, 1957, pp. 61–75.

[4] Maxworthy, T., "The Structure and Stability of Vortex Rings," *Journal of Fluid Mechanics*, Vol. 51, 1972, pp. 15–32.

[5] Assistant Professor, Department of Civil and Geological Engineering, School of Engineering and Applied Science, Princeton University, Princeton, N. J.

further on this paper, I would suggest the authors replace the Froude number they mention by the more appropriate densometric Froude number. Results of this densometric Froude number plotted against the number of diameters of ring travel would be preferable for the application cited; namely, an alternative scheme for dispersing hot stack gases.

I also believe that ring stability is a strong function of the conditions under which the ring is generated. When the ring fluid is 50 percent heavier than the ambient fluid, I question the quiescence of the flow at the generating orifice when the ring motion is downward. For instance, is the orifice initially closed and the cavity about to be impulsed filled with heavy oil vapor and air? Is the orific quickly opened, the impulse suddenly applied and the ring formed outsde the orifice in the fluid disturbed by the opening orifice? Or is the heavy fluid allowed to leak downward from the open orifice when the impulse is applied? The authors need to examine these conditions and/or report them.

In the first paragraph of the "Experimental Results" portion of the paper, the inequality signs delineating the pertinent Strouhal limits are backwards.

I would question the sketches presented in Fig. 2 if these are to indicate the formation of vortex rings that differ according to the impulse time interval. For example, using Fig. 1, this discusser calculates that the range of impulse time varies between 0.04 sec to 2.6 sec. Surely, the ring formation phenomena for the 2.6 sec impulse time does not appear as shown in *B* or *C* in Fig. 2. A 2.6 sec impulse time for a 10-in-dia piston impulsing a 2-in-dia orifice produces a jet flow through the orifice which is not shown in Fig. 2.

I believe the authors' inference that ". . . . Although the experiment does not cover other combinations of fluids, it gives the complete flow patterns of a circular vortex for two fixed fluids under all ranges of Reynolds number (Froude number)," overstates their case. For instance, immiscible combinations of fluids, or combinations involving nonstationary ambient fluids, or combinations involving different Froude numbers are neither mentioned nor dealt with.

H. VIETS.[6] The purpose of this Discussion is to present some experimental results of the decay of a three-dimensional vortex ring which support the findings of the authors' paper [1].[7]

The experiment described here was performed in a water tunnel at the Polytechnic Institute of Brooklyn. A small tank of dyed water was placed in the water tunnel. The top of the tank contained a rectangular orifice of aspect ratio ten. The tank was connected to a pressurized source of dyed water so that a small amount of dyed water could be forced into the tank through a feed line, thereby expelling an equal amount of dyed water through the rectangular orifice.

The fluid thus expelled from the orifice formed a vortex ring which oscillated about a circular shape as it rose. These oscillations are described in reference [2]. The dyed water was slightly denser than the surroundings, thus the circulation of the ring decayed, gravity became dominant, and the ring's upward progress was halted and it began to fall. The portions of the ring containing the most dyed fluid fell the fastest and exhibited the "upside down mushrooms" described by the authors [1] and shown in Fig. 1.

The significance of this procedure, in view of your paper, is that dyed fluid which formed the mushrooms had accumulated in those portions of the vortex ring *before* the final decay began. Fig. 2 shows a top view of the vortex ring while it is undergoing the oscillations. The rectangular orifice may be seen in the background. It may be noted that the ring is thickest at those positions where it crosses the rectangle. These are the portions of the ring which form the upside down mushrooms after further oscillation.

Thus, in this case, it appears that the decay process is merely the falling of the denser portions of the ring (as described by you) rather than the flowing of the denser fluid into a downward distortion as suggested by Dabrowski [3].

[6] Visiting Research Associate, Energy Conversion Research Lab, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.

[7] Numbers in brackets designate References at end of Discussion.

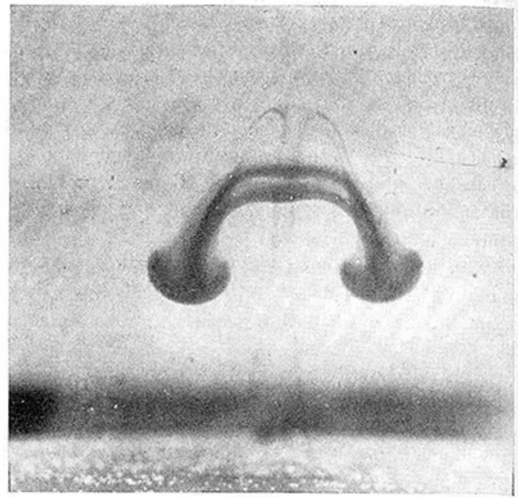

Fig. 1 3-D vortex ring with upside down mushrooms

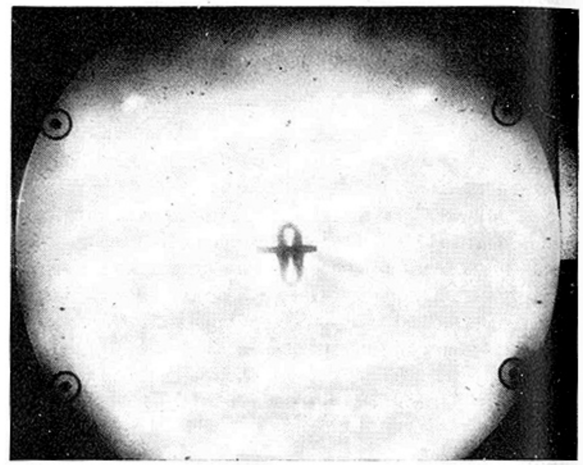

Fig. 2 3-D vortex with a thickness distribution

References

1 Chen, C. J., and Chang, L. M., "Flow Patterns of a Circular Vortex Ring With Density Difference Under Gravity," Journal of Applied Mechanics, Vol. 39, No. 4, Trans ASME, Vol. 94, Series E, Dec. 1972, pp. 869–872.

2 Viets, H., and Sforza, P. M., "Dynamics of Bilaterally Symmetric Vortex Rings," *Physics of Fluids*, Vol. 15, No. 2, 1972, pp. 230–240.

3 Dabrowski, N. J., "Physical Mechanism of Vortex-Ring Cascade," *Journal of the Aeronautical Sciences*, Vol. 24, 1957, pp. 708–709.

Authors' Closure

We appreciate the interest shown by Professors Chen, Mattingly, and Dr. Viets in our paper.

In regard to the distance that a vortex ring can travel we would like to point out that for a vortex ring with density difference under gravity, four distinguished cases, instead of two cases as pointed out by Professor Chen, must be considered. They are as follows:

1 A heavier vortex core moving downward.
2 A heavier vortex core moving upward.
3 A lighter vortex core moving downward.
4 A lighter vortex core moving upward.

In Cases 1 and 4, the vortex ring receives acceleration due to buoyant and gravitational forces. However, the vortex ring in Case 1, by having denser fluid in the core than the ambient fluid, is more unstable with respect to centrifugal instability mechanism than that of Case 4 which has lighter fluid in the core than the ambient fluid. Similarly, although in Cases 2 and 3 the vortex ring receives deceleration due to buoyant force, the vortex in Case 2, again by having denser fluid inside the vortex, is much more unstable with respect to centrifugal instability mechanism than that of Case 3. Therefore, the present investigation which is Case 1 should differ from Turner's results which is Case 4 not to mention the difference in different density ratio used. In our Fig. 4, the distance of laminar flow is the distance that a vortex with denser fluid inside can maintain the flow pattern at stage 2 as shown in Fig. 2 or the shape like photo 1. Therefore, a vortex ring, although unstable, may remain essentially a ring shape and travel a considerably longer distance than the laminar distance. It should be noted that in Turner's Fig. 12 the final height, which is the total distance, is not the distance of laminar flow as defined in our paper. Disregarding the effect of centrifugal instability, Turner's results show that in the case of a density difference of 20 percent (compared with our 50 percent) the final height may reach a distance only about 20 times that of the diameter. In addition, Turner experimented with a fixed input condition and varied the buoyance, while in our experiment we fixed the buoyance and varied the input condition. We do agree with Professor Chen that the breakdown at stage 2 of our Fig. 2 may also occur for neutral density vortex range. The centrifugal instability can be shown, for example, from Rayleigh theorem, to manifest outside the vortex core in the region where the azimuthal velocity, relative to the core center, decreases with the radius. This instability is known to occur in neutrally buoyant fluid such as investigated by Krutzsch in reference [3] of our paper. However, this instability should be stronger for the case of heavier vortex core than that of a neutral one and should be weaker for the case of buoyant vortex ring.

Concerning the initial exit condition, we agree with Professor Mattingly in that the ring stability should depend on the initial condition under which the ring is generated. We slowly fed the smoke, which is cooled to room temperature through a coil, into the vortex chamber radially while the orifice remains open. When the smoke is about to fill the whole chamber we observed that the smoke drifted out along the edge of orifice in a manner like a teapot effect. If the feeding is too fast a strong drift downward along the center of orifice is observed. However when the valve controlling the smoke flow is shut off there is a duration of about 5 sec in which the smoke is almost stopped from drifting downward even though the orifice is not closed. In this 5-sec duration the experiment began by turning off the d-c current to the magnet that controls the piston. We found that this process is quite satisfactory as it creates the least disturbance with our experiment. We know that when a vortex is generated in the observation chamber there is another vortex drawn into the smoke chamber. Therefore, several seconds will elapse before the smoke will finally drift out of the chamber again. However, the vortex in the observation chamber is too far away to be disturbed by the drift. Our data show that the range of impulse time varies from 0.04 to 0.4 sec. To obtain the actual time from our Fig. 1 we have $t(\text{sec}) = D^2 (\nu N_{st} N_{re})^{-1} = 170(N_{st} N_{re})^{-1}$.

Thus to get the longest impulse time is to choose the smallest Reynolds and Strouhal numbers. For $N_{re} = 1200$ and $N_{st} = 0.28$, we get $t = 0.5$ sec. We certainly agree that a 2.6-sec impulse time for a 10-in-dia piston impulsing a 2-in-dia orifice will produce a jet flow.

We would like to thank Dr. Viets for offering further observations on the formation of subring and the "upside down mushrooms" structures. This shows further that the subring formation is dominated by the gravitational force.

In conclusion, we feel that we covered the flow pattern for all ranges of Reynolds and Strouhal numbers that are capable of generating a circular vortex ring under the present arrangement of the experiment. Of course this does not cover the case of immiscible combination of fluids or the case of nonstationary ambient fluid.

Governing Equations for Vibrating Constrained-Layer Damping Sandwich Plates and Beams[1]

D. J. MEAD.[2] The authors are to be commended for the attempt to put a new slant on the theory of sandwich plates and to produce a simplified governing equation of motion. If equation (37) is valid, it will lead to considerable simplification in computing the plate dynamic properties.

I must confess to some uneasiness, however, about the simplification that has been achieved. In the first place, it leads to only *two* boundary conditions which need to be satisfied at each end of the beam (as shown in equations (42) (48)). That this is inadequate for a clamped boundary of a sandwich beam is easily shown. Such a boundary must prevent transverse displacement $\hat{W}$, and rotation $d\hat{W}/dx$ (as expressed by equations (43) and (44)) if there is no shear deformation in the face plates. In addition, *a condition must be imposed on the in-plane motion of the face-plates.* A fully clamped boundary must also have $U = 0$, but it is conceivable that the boundary might allow U displacements, at the same time as it maintains $\hat{W} = 0$ and $d\hat{W}/dx = 0$. In this case, say, we might have to impose a zero value on the mid-plane face-plate stresses, σ_x. Thus *three* conditions in all are required at the clamped end.

The governing differential equation, being biharmonic, only requires two boundary conditions for each end. It seems, then, that in deriving this simplified equation some important constraint has in effect been imposed, and this excludes the possibility of satisfying all three of the important boundary conditions.

In the second place, the results shown in Table 1 have me worried!

1 Because the results from the Mead/Markus equation should be the same as those from DiTaranto's (the two equations stem from identical assumptions, and were derived by similar analyses).

2 Because the Mead/Markus/DiTaranto results should be virtually exact for the wave numbers considered, which imply bending wavelengths of about 60 times the thickness of the thickest face plate. This means that face-plate shear deformation and rotatory inertia are negligible—which Mead/Markus and DiTaranto assumed.

This latter fact prompts the question "Why does the new theory of Yan and Dowell yield different results from Mead/Markus and DiTaranto?" I suggest that it is due to Yan and Dowell assuming zero transverse *stress* in the layers of the sandwich, whereas Mead/Markus and DiTaranto assumed zero transverse *strain* but finite (nonzero) transverse stress. The

[1] By M.-J. Yan and E. H. Dowell, published in December, 1972, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 39, No. 4, TRANS. ASME, Vol. 94, Series E, pp. 1041–1047.

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