prove useful if the boundary has to be extended to a distance where the series convergence becomes a problem.

In a bar of finite length, both the authors' procedure and the simple graphical procedure apply only in the region of the x, t-plane below the first reflected elastic wave from the far end (slope $-c_0$). Above this line a numerical solution is required. But if an explicit representation in terms of x and t can be written for the solution below this line, then this line can be taken as the boundary for a computer solution above the line. This is particularly convenient, since then the numerical solution does not have to be extended down into the region near $x = 0$, $t = 0$ where the functions are varying rapidly; see, for example, [3].

The author's procedure may be especially suitable for such finite bar analyses, if the required unloading boundary near the impacted end is short enough that rapid convergence of the series occurs.

References

1 Ponomarev, S. D., et al., "Resistance Calculus in Construction of Machines" (Mashigiz, Moscow) Tom III, 1959, pp. 553–580, (in Russian).

2 Cristescu, N., *Dynamic Plasticity*, Amsterdam and New York, North-Holland, Wiley, 1967.

3 Cristescu, N., "The Unloading in Symmetric Longitudinal Impact of Two Elastic-Plastic Bars," *International Journal of Mechanical Sciences*, Vol. 12, 1970, pp. 723–738.

Authors' Closure

The authors wish to thank Professor Malvern for his thoughtful comments and kind remarks. They agree that an extension of the technique for the determination of the unloading boundary to noncentered loading waves would be valuable, and are grateful to the discusser for bringing to their attention the work cited in reference [1] of the Discussion.

Extending the unloading boundary to greater distances from the origin of the x-t plane is no problem. In the authors' paper the section entitled "Boundary Asymptote" describes one possible approach. Some others were cited by Professor Malvern, and yet another possibility appears in a paper by Clifton and Bodner.[4]

Professor Malvern mentions that in some problems the series expansion technique can be used to start the solution, and numerical procedures applied to continue it for larger times. Such a problem is solved in a forthcoming paper.[5] The unloading boundary is determined in an elastic-plastic string under transverse impact by the series expansion technique near the region $x = 0$, $t = 0$, and by numerical integration away from this region.

[4] Clifton, R. J., and Bodner, S. R., "An Analysis of Longitudinal Elastic-Plastic Pulse Propagation," Journal of Applied Mechanics, Vol. 33, Trans. ASME, Vol. 88, Series E, 1966, pp. 248–255.

[5] Tuschak, P. A., and Schultz, A. B., "Determination of the Unloading Boundary in Transverse Impact of an Elastic-Plastic String," ASME Paper No. 72-APM-12.

Bounds on the Maximum Contact Stress of an Indented Elastic Layer[1]

J. DUNDURS.[2] The reader should realize that the layer will separate from the substrate if there is no bond capable of transmitting tensile tractions between the two bodies.[3,4,5] Physically, frictionless contact implies in most cases that no tensile tractions can be developed. The only exceptions might be some phenomena on a very small scale when there is a lubricating film between the layer and the substrate, no penetration by a gas is possible, and the liquid does not cavitate. As the layer bends away from the substrate, it has the tendency to wrap itself onto the curved indentor, and the extent of contact between the indentor and the layer will increase beyond the values predicted in the paper for the frictionless support. The increase can be expected to be particularly significant when the extents of contact at the top and the bottom of the layer are of comparable magnitudes. This will happen when the indentor is relatively flat (large radius of curvature in comparison to the thickness of the layer), the level of loading is high, and the layer has a large compliance in comparison to those of the indentor and the substrate. It is difficult to anticipate whether the separation between the layer and the substrate strongly affects the maximum pressure between the indentor and the layer. However, the profile of the contact pressure can be expected to change a great deal.

It also might be noted that Figs. 4 and 5 distort the essential dependence of the results on the elastic constants. The natural elastic constant in plane problems involving smooth advancing contacts is $G/(\kappa + 1)$, where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for generalized plane stress.[6] Thus the Young's modulus E is a natural constant only for plane stress. The first-order effect of the elastic constants in the problem considered is that the extent of contact between the rigid indentor and the layer is inversely proportional to $G_l/(\kappa_l + 1)$. If the quantity $p_0R(\kappa_l + 1)/2G_la$ were plotted versus a/h in Fig. 5, all curves would originate from the point (0,1) and practically coincide for moderate values of a/h. Even for the fairly large extent of contact $a/h = 1$, displayed in Fig. 4, the differences would be reduced by about a half if $G_l/(\kappa_l + 1)$ were used instead of E_l in the normalization.

P. K. GUPTA[7] and J. A. WALOWIT.[7] The plane-strain problem of an elastic layer bonded to a rigid half space and indented by a rigid indenter has been asymptotically solved by Meijers [1].[8] It has been shown that the asymptotic solutions for small and large a/h match so well that the obtained solution is valid over any arbitrary range of a/h values.

The authors have presented numerical solutions to Meijers' problem. However, they have failed to compare their results with those already published by Meijers. For $\nu_l = 0.5$ and $a/h = 1$, the authors' results for contact pressure variations on the surface as shown by them in Fig. 4 can be replotted in the Discussers' Fig. 1 against the results determined by formulas given by Meijers. A substantial difference between the two solutions is clearly seen. The solutions for maximum contact pressure as a function of a/h ratios for $\nu_l = 0.5$ as shown by the authors in Fig. 5, may be compared with Meijers' results in the Discussers' Fig. 2. It is clearly seen that the differences be-

[1] By Y. C. Pao, T.-S. Wu, and Y. P. Chiu, and published in the September, 1971, Journal of Applied Mechanics, Vol. 38, Trans. ASME, Vol. 93, Series E, pp. 608–614.

[2] Professor of Civil Engineering, Northwestern University, Evanston, Ill.

[3] Weitsman, Y., "On the Unbonded Contact Between Plates and an Elastic Half Space," Journal of Applied Mechanics, Vol. 36, Trans. ASME, Vol. 91, Series E, 1969, pp. 198–202.

[4] Pu, S. L., and Hussain, M. A., "Note on the Unbonded Contact Between Plates and an Elastic Half Space," Journal of Applied Mechanics, Vol. 37, Trans. ASME, Vol. 92, Series E, 1970, pp. 859–861.

[5] Keer, L. M., Dundurs, J., and Tsai, K. C., "Problems Involving a Receding Contact Between a Layer and a Half Space," Journal of Applied Mechanics, in press.

[6] Dundurs, J., and Stippes, M., "Role of Elastic Constants in Certain Contact Problems," Journal of Applied Mechanics, Vol. 37, Trans. ASME, Vol. 92, Series E, 1970, pp. 965–970.

[7] Mechanical Technology Incorporated, Latham, N. Y.

[8] Numbers in brackets designate References at end of Discussion.

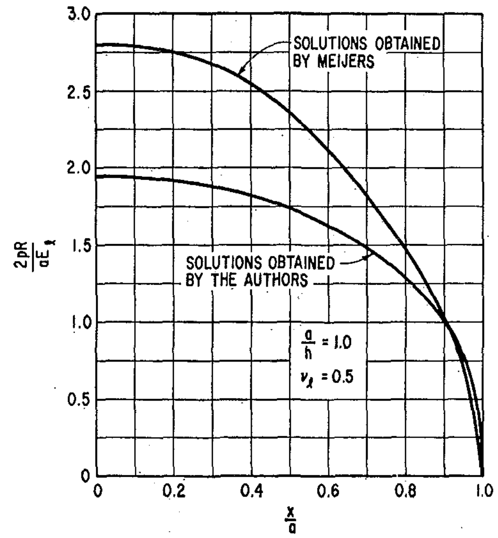

Fig. 1 Comparison of contact pressure variations as obtained by authors and solutions obtained by Meijers. $a/h = 1.0$, $\nu_l = 0.5$, rigid indenter and layer rigidly adhered to foundation.

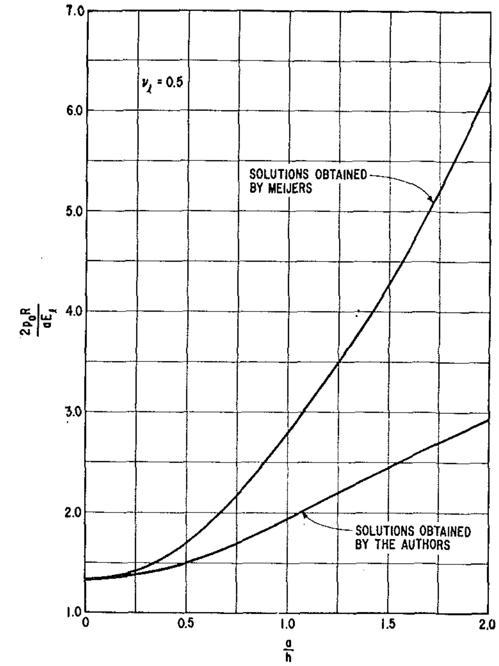

Fig. 2 Differences in solutions of maximum contact pressure as obtained by Meijers and authors; $\nu_l = 0.5$, rigid indenter and layer rigidly adhered to rigid foundation.

tween the two solutions increase rapidly with increasing a/h.

It seems that the authors have committed a rather serious mistake in writing equation (9) from Sneddon's work [2]. The correct form of this equation should read

$$\frac{\partial v}{\partial x} = -\frac{1+\nu_l}{\pi E_l}\int_0^\infty \left[(1-\nu_l)\frac{\partial^3 G^{(A)}}{\partial y^3} + (\nu_l - 2)\xi^2 \frac{\partial G^{(A)}}{\partial y}\right]\frac{\sin(\xi x)}{\xi}\,d\xi$$

A constant factor of 2 in equations (8) and (9) will not introduce any error in analysis of stresses. However, the differences between the quantities in parentheses in equation (9) and the foregoing equation may introduce substantial errors in the final results. A similar error may also be noted in the last of equations (3) in the authors' earlier work [3].

Aside from the previous discrepancies, the applicability of the results published by the authors to any practical problem, is very much restricted. For the case when the layer is bonded to rigid half space, they have given the values for contact pressures as a function of true half width of contact but no solutions of contact forces are given and thus the true half width for a given problem cannot be readily determined. From a practical standpoint it is necessary to have a plot, similar to the Discussers' Fig. 2, of the contact force as a function of a/h values for a layer rigidly adhered to the foundation.

References

1 Meijers, P., *Applied Scientific Research*, Vol. 18, 1968, p. 353.
2 Sneddon, I. N., *Fourier Transforms*, McGraw-Hill, New York, 1951.
3 Wu, T., and Chiu, Y. P., *Quarterly of Applied Mathematics*, Vol. XXV, 1967, p. 233.

Authors' Closure

The authors appreciate Professor Dundurs' comments regarding an alternative way of normalizing and plotting the maximum stress for various a/h ratios. In Fig. 5, if the curves for the frictionless case are plotted according to the way Professor Dundurs suggested, they would all coincide for the reason that the kernel involved is independent of the Poisson's ratio. Since our study is intended for the practical problem of a strip compressed by two rollers, one on each side of the strip, the wrapping phenomenon of the strip onto the roller will not happen and has therefore not been considered. However, we believe that the axisymmetric problems, mentioned by Professor Dundurs, of the layer wrapping onto the curved punch are certainly of great interest.

We are grateful to Drs. Gupta and Walowit for pointing out the typographical error of equation (9) in our paper. (Equation (9) should read as the one given by Gupta and Walowit in their Discussion of our paper.) What they have written is correct as we have done so ourself in our earlier work.[9] In fact, it is this correct form we have used in the derivation of the other equations in our paper, so this inadvertently copied incorrect equation has no consequence on the numerical results presented.

Concerning the discrepancies between our results and those given by Meijer,[10] we wish to point out that the analytical approach employed in the present paper is basically different from Meijer's approximate solution which is based on the truncation of a series expression for the kernel function. We doubt very much the accuracy of Meijer's results for the cases of $0.5 < a/h < 2$ for the reason that Meijer has taken only three terms in the $K(x)$ series for his asymptotic solution. Not only that for the $a/h > 0.5$ cases the smallness precondition is violated, the adequacy of taking merely three series terms is also highly questionable in view of the fact that the $K(x)$ series is alternating in signs as well as with coefficients increasing in magnitude.

Although Meijer claimed that the small-value and large-value asymptotic solutions match well for the immediate values of a/h, two wrongs do not make a right. We await to see the more convincing conclusion (such as taking more terms of the series to show the convergence) of Meijer's approach than that in its present form. Until then, our solution to the problem is apparently exact, except for the part of numerical calculations of data, and more versatile.

We agree with Drs. Gupta and Walowit that a graph of the contact force versus a/h is of practical value. Such a plot can be easily accomplished by numerical integrating the area under the solid curves given in Figs. 3 and 4 of the paper.

[9] Wu, T. S., Pao, Y. C., and Chiu, T. P., "Analysis of a Finite Elastic Layer Containing a Griffith Crack," *International Journal of Engineering Science*, Vol. 8, 1970, pp. 575–582.

[10] Meijer, P., *Applied Scientific Research*, Vol. 18, 1968, p. 353.