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A Semi-Classical Treatment of Non-Local Potentials

Hisashi HORIUCHI

Department of Physics, Kyoto University, Kyoto 606

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A semi-classical treatment of the three-dimensional Schrödinger equation containing a non-local potential is discussed by the use of the WKB method. An equivalent local potential is derived and the Perey effect is investigated. In deriving the equivalent local potential, it is shown that the concept of the parity-dependence should be introduced for the proper use of the WKB method. By using the equivalent local potential we discuss properties of the non-local potential, namely, its spatial, energy and angular-momentum dependence including the parity-dependence.

§ 1. Introduction

Recent development of the microscopic treatment of nucleus-nucleus scattering by the use of the resonating group method (RGM) and the generator coordinate method (GCM) has enabled us to study the interaction between complex nuclei even with fairly large mass number from the microscopic viewpoint. Various systems have been investigated by RGM or GCM and the knowledge of the inter-nucleus interaction has been accumulated for those individual systems.

Since the exchange kernels of RGM or GCM are now calculable in completely analytical forms, the analysis of these non-local kernels (without solving the RGM or GCM equation of motion) is expected to enable us to make general discussions (not restricted to individual cases) on the characteristic features of the inter-nucleus force.

An important contribution in this direction was done by Brussels group on the basis of the analysis of the parity-projected diagonal kernels of GCM, namely, the parity-projected energy curves. In their study they assume that the parity-projected energy curve is directly related with the inter-nucleus force as seen in regarding the parity-range of the energy curve as equivalent with that of the inter-nucleus force. This assumption is, however, somewhat doubtful, and we discuss this point in more detail in another paper. Another important contribution was given by LeMere and Tang who analysed the equivalent local potentials obtained from the RGM norm kernels by the prescription of Greenlees and Tang. Their argument is based on the assumption that the essential features of the exchange kernels of RGM are already contained in the exchange norm kernel of RGM. This assumption was justified by the analysis of the RGM interaction kernels done by LeMere, Stubeda, Tang and the present author. The prescription of Greenlees and Tang which transforms the non-local kernel to the equivalent
local one, is based on the use of the Born approximation. Therefore its applicability seems to be limited in a usual sense to high energy problems. Practical RGM calculations in Ref. 8), however, showed that the results obtained by the Greenlees-Tang prescription are valid also in fairly low energy region. We will partly justify these numerical experiences in a separate paper by using the results obtained in the present paper.

The purpose of this paper is to give a semi-classical treatment of non-local potentials of general type by using the WKB method. This gives us a way to analyze the RGM non-local kernels even in low energy region if the system can be regarded as semi-classical, for example, when the mass numbers of scattering nuclei are large. This semi-classical treatment enables us to derive equivalent local potentials and to investigate the Perey effect. By using the equivalent local potentials we can discuss the properties of non-local potentials, namely, their spatial range, energy range and angular-momentum-dependence including their parity-dependence. Our WKB treatment of non-local potentials is done for usual three-dimensional problems, which is developed by extending the treatment of our previous paper where the non-local potentials in one-dimensional problems are discussed by the WKB method.

In the next section (§ 2) we discuss the formulation of a semi-classical treatment of non-local potentials by the WKB method without using the partial wave expansion. From the WKB wave function we can define the equivalent local potential in the sense of Austern and Fiedeldey. The explanation of the Perey effect is done in a way similar to in the one-dimensional case of Ref. 11). Section 3 is devoted to the application of our discussion to a special type of non-local kernels, namely, Frahn-Lemmer type ones. We will see that the damping factor of the wave function derived by our WKB treatment coincides with that given by Austern. In § 4 it is shown that for the proper application of the WKB method to non-local potentials the concept of the parity-dependence needs to be introduced. Here in this section, the general type of term appearing in RGM non-local kernels is analysed by our WKB method in view of its spatial range, energy range and parity-dependence. Application of our WKB treatment to the detailed analyses of RGM non-local kernels will be given in a separate paper on the basis of the results of this section. In the final section (§ 5) the non-local potential in each partial wave equation is treated by the WKB method using the Langer transformation. The results are compared with those obtained in § 2.

§ 2. WKB treatment of non-local potentials without partial wave expansion

2.1. WKB approximation and Wigner transform of non-local potential

The Schrödinger equation we treat in this paper is written as follows:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V(r) \psi(x) + \int G(x, x') \psi(x') dx' = E \psi(x),$$

(1)
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where \( V(r) \) is a local central potential depending only on \( r = |x| \) and \( G(x, x') \) is a symmetric and rotationally invariant non-local potential. The symmetric property of \( G(x, x') \) means

\[
G(x, x') = G(x', x),
\]

and the rotational invariance of \( G(x, x') \) means

\[
G(Rx, Rx') = G(x, x'),
\]

where \( R \) is an arbitrary rotation matrix. The rotational invariance can be equivalently stated in different ways: \( G(x, x') \) is a function of three scalar products \( x^2, x'^2, x \cdot x' \) or alternatively \( (x + x')^2, (x - x')^2, (x + x') \cdot (x - x') \). As in the RGM case, \( G(x, x') \) can depend on the energy \( E \), which is irrelevant to our formulation of the WKB treatment.

As in Ref. 11), we first rewrite the integral operator \( G(x, x') \) into a differential operator as follows:

\[
\int G(x, x') \psi(x') dx' = \int G(x, x + s) \psi(x + s) ds = \int \left[ G(x, x + s) \exp \left( \frac{i}{\hbar} s \cdot p_{\text{op}} \right) ds \right] \psi(x) = \hat{G}(x, p_{\text{op}}) \psi(x),
\]

\[
p_{\text{op}} = \frac{\hbar}{i} p,
\]

where we have used the relation \( \exp \left((i/\hbar)s \cdot p_{\text{op}}\right)f(x) = f(x + s) \). Equation (1) now takes the form

\[
H(x, p_{\text{op}}) \psi(x) = \hat{G}(x, p_{\text{op}}) \psi(x) = E \psi(x),
\]

\[
H(x, p_{\text{op}}) = \frac{p_{\text{op}}^2}{2m} + V(r) + \hat{G}(x, p_{\text{op}}).
\]

By expressing \( \psi(x) \) as \( \psi(x) = \exp[(i/\hbar)S(x)] \) and by multiplying \( \exp[-(i/\hbar)S(x)] \) from the left to Eq. (5), we obtain

\[
H(x, p_{\text{op}} + \hbar S(x)) \cdot 1 = E.
\]

Following the philosophy of the WKB approximation: we expand \( S(x) \) in a power series of \( \hbar \) as \( S(x) = S_0(x) + \hbar S_1(x) + \cdots \), which gives us

\[
H(x, \hbar S_0(x) + \frac{1}{i} \hbar p + \cdots) \cdot 1 = E.
\]

When we regard as if \( \hat{G}(x, p_{\text{op}}) \) were independent of \( \hbar \) as a function of \( x \) and \( p_{\text{op}} \),
we obtain from the zeroth and first power terms of $\hbar$ of Eq. (7) the following equations:

$$H(x, pS_0(x)) = E,$$

$$\sum_j \left\{ 2i \left( \frac{\partial S_0(x)}{\partial x_j} \right) \frac{\partial}{\partial p_j} \right\} \frac{\partial H}{\partial p_j} = 0. \quad (8)$$

Here the first equation is the Hamilton-Jacobi equation for an action function $S_0(x)$. The solved action function $S_0(x)$ contains three integral constants $a_1, a_2, a_3 = E$.

The second equation of Eq. (8) is known to have the following solution:

$$S_1(x) = \frac{1}{2i} \log \left[ \det \left( \frac{\partial^2 S_0}{\partial \alpha \partial \alpha} \right) \right] + \text{const}, \quad (9)$$

which is easily checked by substituting into Eq. (8).\(\partial^2 S_0/\partial \alpha \partial \alpha\) means a $3 \times 3$ matrix defined by $(\partial^2 S_0/\partial \alpha \partial \alpha)_{ij} = \partial^2 S_0/\partial \alpha_i \partial \alpha_j$ where $\alpha_i$ are the integral constants mentioned above. $\det(\partial^2 S_0/\partial \alpha \partial \alpha)$ is known as the Van Vleck determinant. From Eq. (9), the WKB wave function for Eq. (1) is expressed as

$$\psi(x) = \text{const} \sqrt{\det \left( \frac{\partial^2 S_0}{\partial \alpha \partial \alpha} \right)} \exp \frac{i}{\hbar} S_0. \quad (10)$$

It is here to be noticed that, when a $c$-number vector $p$ is inserted in place of the $q$-number vector $p_{np}$ in $H(x, p_{np})$, the resulting Hamiltonian becomes

$$H(x, p) = \frac{p^2}{2m} + V(r) + G^w(x, p),$$

$$G^w(x, p) = \int ds \exp \left( \frac{i}{\hbar} s \cdot p \right) G \left( x - \frac{s}{2}, x + \frac{s}{2} \right). \quad (11)$$

$G^w(x, p)$ is the so-called Wigner transform of the non-local operator $G(x, x')$. $G^w(x, p)$ has the following two properties. Firstly from Eq. (2) we can easily prove

$$G^w(x, -p) = G^w(x, p). \quad (12)$$

Secondly from Eq. (3) we obtain

$$G^w(Rx, Rp) = G^w(x, p). \quad (13)$$

$R$ being any rotation matrix. This can be equivalently stated that $G^w(x, p)$ is a function of three scalar products $x', p', (x \cdot p)$. From Eq. (12) we know that $(x \cdot p)$ should appear in the form of $(x \cdot p)^3$. Thus Eqs. (12) and (13) are unified to

$$G^w(x, p) = g(x^3, p^3, (x \cdot p)^3). \quad (14)$$
It is worth mentioning that for a Frahn-Lemmer type non-local potential \( G(x, x') = U(\frac{1}{2}|x+x'|) \cdot H(|x-x'|) \) there does not appear any \((x \cdot p)^2\) dependence in \( G^w(x, p) \).

In our WKB treatment, we have regarded as if \( G^w(x, p) \) were independent of \( \hbar \) as a function of \( x \) and \( p \),\(^{11,20}\) If we let \( \hbar \to 0 \), \( G^w(x, p) \) vanishes and the classical Hamiltonian contains no effect of the non-local potential present in the quantal Hamiltonian. From Eq. (11) we know that to retain \( \hbar \) in \( G^w(x, p) \) is allowed when the following relation is satisfied:

\[
\beta p/\hbar \leq \pi ,
\]

where \( \beta \) is the non-locality range of \( G(x, x') \) and \( p \) is the local momentum of \( \psi(x) \). Therefore our WKB treatment is subject to the condition \((\beta p/\pi) \ll \hbar \) (characteristic action of the system), where the right-hand-side inequality is the usual short-wave-length condition necessary for the application of the WKB method.

2.2. Solution of Hamilton-Jacobi equation

The Hamilton-Jacobi equation in Eq. (8) is written in polar coordinates as follows:

\[
\frac{1}{2m} (\mathbf{p}_S)^2 + V(r) + g(r^2, (\mathbf{p}S)^2, (x \cdot \mathbf{p}S)^2) = E ,
\]

\[
(x \cdot \mathbf{p}S)^2 = r^2 \left( \frac{\partial S}{\partial r} \right)^2 ,
\]

\[
\frac{1}{2m} \left( \mathbf{p}_S \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial S}{\partial \phi} \right)^2 = E ,
\]

(16)

The solution of Eq. (16) is obtained as

\[
S_\phi = \pm \int^r r' p_r(r', E, \alpha_\theta) dr' + \int^\theta \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2 \theta}} d\theta + \alpha_\phi \cdot \phi ,
\]

(17)

where \( \alpha_\theta \) \((= \alpha_\theta') \) and \( \alpha_\phi \) \((= \alpha_\phi') \) are integral constants which correspond to the magnitude and the \( z \)-component of the angular momentum, respectively. The quantity \( p_r(r, E, \alpha_\theta) \) in Eq. (17) is a solution of the following equation:

\[
\frac{1}{2m} \left[ p_r + \frac{\alpha_\phi}{r^2} \right]^2 + \frac{\partial^2 S}{\partial r \partial \theta} \left( r^2 \frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial \phi^2} \right) = E ,
\]

(18)

It should be noted that if \( p_r = p_r(r, E, \alpha_\theta) \) is a solution of Eq. (18), \( p_r = -p_r(r, E, \alpha_\theta) \) is also a solution of Eq. (18), which is the reason of the appearance of \( \pm \) sign in front of \( p_r(r, E, \alpha_\theta) \) in Eq. (17). From Eq. (17), the Van Vleck determinant becomes

\[
\det \left( \frac{\partial^2 S}{\partial \alpha \partial \theta} \right) = \frac{1}{r^2 \sin \theta} \frac{\partial p_r(r, E, \alpha_\theta)}{\partial E} \frac{\alpha_\theta}{\sin^2 \theta} - \frac{\alpha_\phi}{\sin^2 \theta} .
\]
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\[ \frac{\partial p_r}{\partial E} = \frac{1}{m} \left( 1 + 2m \frac{\partial g}{\partial p} + 2mr^2 \right) \frac{\partial g}{\partial (x \cdot p)^2} \left( \frac{p^2 + q^2 + (x \cdot p)}{x \cdot p - r} \right) \] (19)

By putting \( \alpha_e = (l+1/2) \hbar, \beta_e = m \hbar \) in a usual way, the WKB wave function of Eq. (10) becomes

\[ \phi(x) \sim \text{const} \frac{1}{r} \frac{F(r)}{\sqrt{p_r}} \exp \left[ \pm \frac{i}{\hbar} \int p_r dr \right] \]

\[ \frac{1}{\sqrt{\sin \theta}} \int_{(l+1/2)}^{1/2} \frac{1}{\sin \theta} \exp \left[ \pm \frac{i}{\hbar} \int \left( \sqrt{l + 1/2} - \frac{\bar{m}^2}{\sin^2 \theta} d\theta \right) \right] \cdot e^{i\epsilon x} \]

\[ F(r) = \sqrt{1 + 2m \frac{\partial g}{\partial p} + 2mr^2 \frac{\partial g}{\partial (x \cdot p)^2}} \cdot (20) \]

2.3. Connection at turning points and equivalent local potential

What we need to do next is to linearly combine the solutions given in Eq. (20) so that the resulting WKB wave function satisfies correct connection conditions at turning points: If we denote \( \phi(x) \) as \( \phi(x) = (\mu_N(r)/r) \cdot \Theta(\theta) \cdot \exp(i\bar{m} \phi) \) the WKB approximation for \( \mu_N(r) \) and that for \( \Theta(\theta) \) should satisfy connection conditions at radial and angular turning points, respectively. Connection of the radial wave function at a turning point can be done as follows by the same argument as Ref. 11. According to the theory of Fleideldey18 treated by Coz, Arnold and McKellar,19 the Jost functions \( u_{N \pm}(r) \) of the radial Schrödinger equation with a non-local potential can be expressed by a common function times the Jost functions \( u_{L \pm}(r) \) of the equivalent local potential:

\[ u_{N \pm}(r) = \Lambda(r) u_{L \pm}(r). \] (21)

Jost functions are defined by the boundary conditions \( 1_{\text{mtr} \rightarrow \infty} \) \( \mu_N(r) = 1 \). Here \( k \) is the asymptotic wave number \( k = \sqrt{2mE}/\hbar \). \( \Lambda(r) \) in Eq. (21) is given by

\[ \Lambda^2(r) = \frac{-1}{2ik} \left[ u_{N \pm}(r) \frac{du_{N \mp}(r)}{dr} - u_{N \mp}(r) \frac{du_{N \pm}(r)}{dr} \right]. \] (22)

According to Eq. (20), in our WKB approximation, \( u_{N \pm}(r) \) are expressed as

\[ u_{N \pm}(r) \sim F(r) \sqrt{\frac{\hbar k r}{p_r}} \exp \left[ \pm \frac{i}{\hbar} \int p_r dr + \delta \right], \]

\[ \delta = kr_0 - \int_{r_0}^{r} \left( p_r - \hbar k \right) dr, \] (23)

where \( r_0 \) is an arbitrary fixed point. By substituting Eq. (23) into Eq. (22) we
From Eqs. (21), (23) and (24) we know
\[ u_{L}(r) \sim \sqrt{\frac{\hbar k}{p_r}} \exp \left[ \pm i \int_{r_1}^{r} \frac{p_r}{\hbar} dr + \frac{\pi}{4} \right]. \] (25)

The local potential \( V^{eq}(r) \) which gives the WKB Jost functions of Eq. (25) must clearly satisfy
\[ p_r(r, E, (l + \frac{1}{2}) \hbar) = \sqrt{2m \left( E - \frac{\hbar^2 (l + 1/2)^2}{2mr^2} - V^{eq}(r) \right)}. \] (26)

Equation (26) which the equivalent local potential \( V^{eq}(r) \) must satisfy can be rewritten as follows by substituting Eq. (26) into Eq. (18),
\[ V^{eq}(r) = V(r) + g(r^2, 2m(E - V^{eq}(r)), 2mr^2 \left( E - V^{eq}(r) - \frac{\hbar^2 (l + 1/2)^2}{2mr^2} \right)). \] (27)

Now, for the equivalent local potential, the ordinary WKB theory gives the WKB wave function \( u_L(r) \) which satisfies the connection condition at the turning point \( r_1 \) defined by \( p_r(r_1, E, (l + \frac{1}{2}) \hbar) = 0 \) as below:
\[ u_L(r) \sim \text{const} \sin \left[ \int_{r_1}^{r} \frac{p_r}{\hbar} dr + \frac{\pi}{4} \right]. \] (28)

From Eqs. (21), (24) and (28) we know the radial part of the WKB wave function of our problem which satisfies the connection condition at the turning point \( r_1 \) is given by
\[ u_{L}(r) \sim \text{const} \frac{F(r)}{\sqrt{p_r}} \sin \left[ \int_{r_1}^{r} \frac{p_r}{\hbar} dr + \frac{\pi}{4} \right]. \] (29)

As for the angular part of the WKB wave function of Eq. (20), the connection condition at the turning points \( \theta_1, \theta_2 \) defined by \( \sin \theta_{1,2} = \frac{m}{(l + \frac{1}{2})} \) gives us the following answer for the correctly-connected angular part (see Ref. 21),
\[ \theta(\theta) \sim \text{const} \frac{\sqrt{\sin \theta}}{\sqrt{(l + \frac{1}{2})^{\frac{3}{2}} - \frac{m^2}{\sin^2 \theta}}} \sin \left[ \int_{\theta_1}^{\theta} \sqrt{(l + \frac{1}{2})^{\frac{3}{2}} - \frac{m^2}{\sin^2 \theta}} d\theta + \frac{\pi}{4} \right]. \] (30)

* Strictly speaking, the relation of Eq. (26) may not be valid near the turning point \( r_1 \) of \( p_r \), since the WKB wave function of Eq. (23) cannot be used near \( r_c \). In such a case we need to regard \( p_r \) in Eqs. (28) and (29) as defined by Eq. (26) using \( V^{eq}(r) \) obtained by Fiedeldey’s theory.
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The relation
\[ \int_{-\pi}^{\pi} \sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{\lambda^2}{\sin^2 \theta}} \, d\theta = (l + |m| + \frac{1}{2}) \pi \]  
ensures an alternative expression for \( \Theta(\theta) \)
\[ \Theta(\theta) \sim \frac{\text{const}}{\sqrt{\sin \theta \sqrt{l + \frac{1}{2}}}} \sin \left[ \int_{-\pi}^{\pi} \sqrt{\left(l + \frac{1}{2}\right)^2 - \frac{\lambda^2}{\sin^2 \theta}} \, d\theta + \frac{\pi}{4} \right]. \]  
Equations (30) and (32) are known as the semi-classical expression of the associated Legendre function \( P_r^m(\cos \theta) \) if the constant factor is chosen to be \( \sqrt{(2l+1)/4\pi} \).

2.4. Perey effect and flux conservation

\( F(r) \) defined in Eq. (20) expresses (as is seen in Eq. (24)) the change of the amplitude of the wave function for a non-local potential from that for the equivalent local potential in the interaction region. As will be seen in \( \S \) 3, for attractive non-local potentials \( F(r) \) expresses the damping effect \( (F(r) < 1) \). This factor \( F(r) \) therefore gives an explanation of the Perey effect \( ^{(19,20,21)} \) for the non-local potentials.

The occurrence of the Perey effect is directly related to the conservation of the probability flux as is shown below. First we express \( S_r(x) \) as
\[ S_r(x) = \frac{1}{2i} \log \rho(x), \]  
which gives us \( \psi(x) \sim \sqrt{\rho(x)} \exp[(i/\hbar) S_r(x)] \), namely, \( |\psi(x)|^2 \sim \rho(x) \). Substitution of Eq. (33) into the second equation of Eq. (8) results in the following equation:
\[ \sum_i \frac{\partial}{\partial x_i} \left( \rho(x) \left[ \frac{\partial H}{\partial p_i} \right]_{p = p_s(x)} \right) = 0. \]  
From the Hamiltonian equation, \( (\partial H/\partial \rho) \) expresses the velocity \( v_i = x_i \). Therefore, if \( \rho(x) \) is regarded as a probability density, Eq. (34) implies the conservation of the probability flux: \( \nabla (\rho v) = 0 \). A characteristic point of the non-local potential problem is that the velocity \( v \) cannot be simply expressed as \( p/m = p S_s/m \). Instead we have
\[ v = \left[ \frac{\partial H}{\partial p} \right]_{p = p_s} = \frac{p S_s}{m} \left( 1 + 2m \frac{\partial g}{\partial p} \right) + 2x (x \cdot p S_s) - \frac{\partial g}{\partial (x \cdot p)}, \]  
(We here note that the quantity \( (\hbar/2im) [\hat{\psi} \hat{\phi} - \hat{\phi} \hat{\psi}] \) which gives \( \rho (p S_s/m) \) in the WKB approximation cannot be regarded as the probability flux density vector in the non-local potential problem.) From Eq. (35) the velocity components in \( r, \theta \) and \( \phi \)-directions are given by
By substituting these expressions of $v_r$, $v_\theta$ and $v_\phi$ into the flux conservation equation
\[ 0 = \mathbf{\nabla} \cdot (r \rho \mathbf{v}) \]
we can easily check that
\[ \rho = \text{const} \, F(r) \left[ r^2 p_r \sin \theta \sqrt{\left( l + \frac{1}{2} \right)^2 - \frac{m^2}{\sin^2 \theta}} \right] \]
is surely a solution of Eq. (37). In this derivation of $\rho$ we see that the appearance of the factor $F(r)$ is due to the change of the $r$-direction velocity $v_r$ from the usual value $(\partial S_0/\partial r)/m$ for local potential problems. Thus we can say that the Perey effect is a direct reflection of the probability-flux-conservation which involves the characteristic expression for the velocity for non-local potential problems different from that for local potential problems.

§ 3. Application to Frahn-Lemmer type non-local potentials

We apply in this section our treatment in § 2 to the non-local potentials of Frahn-Lemmer type:
\[ G(x, x') = \mathcal{U} \left( \frac{1}{2} |x - x'| \right) \mathcal{H} \left( |x - x'| \right), \]
\[ \mathcal{H}(|s|) \geq 0, \quad \int \mathcal{H}(|s|) \, ds = 1. \]  
(39)

The Wigner transform of this $G(x, x')$ is
\[ G^W(x, p) = g(r^2, p^2) = U(r) \int ds \, \exp \left( \frac{i}{\hbar} s \cdot p \right) \mathcal{H}(|s|) \]
\[ = U(r) \cdot 4\pi \int_0^m ds \cdot s^2 f_\hbar \left( \frac{s}{\hbar} |p| \right) \mathcal{H}(s). \]  
(40)

As we have noted before $G^W(x, p)$ has no $(x \cdot p)^2$-dependence and so in the above we have denoted $G^W(x, p)$ simply as $g(r^2, p^2)$ instead of $g(r^2, p^2, (x \cdot p)^2)$. The
function $g(r^2, p^2)$ has a property

$$|g(r^2, p^2)| < |U(r)| \left| \int ds H(|s|) = |U(r)|. \right.$$  \hspace{1cm} (41)

When the interaction consists only of the non-local potential of Eq. (39) without a local potential term, Eq. (27) shows us that the equivalent local potential $V_{eq}(r)$ is obtained from

$$V_{eq}(r) = g(r^2, 2m[E - V_{eq}(r)]),$$  \hspace{1cm} (42)

where $g(r^2, p^2)$ is given by Eq. (40). From Eq. (41), we know

$$|V_{eq}(r)| < |U(r)|.$$  \hspace{1cm} (43)

This relation gives an interpretation of the effect found by Percy and Buck by numerical experience; namely, the equivalent local potential to the non-local potential of Eq. (39) has always smaller depth than $U(r)$. The factor $F(r)$ of Eq. (20) is calculated to be

$$F(r) = 1 \sqrt{1 + \frac{2m}{\beta^2} \frac{\partial g}{\partial p^2}}$$

$$= 1 \sqrt{1 - \frac{4\pi m}{\hbar^2} U(r) \int_0^\infty ds s^3 s_j (s_p) H(s),}$$

$$\rho = \sqrt{2m(E - V_{eq}(r))},$$  \hspace{1cm} (44)

This result exactly coincides with the damping factor given by Austern. When $H(|x - x'|)$ has a form

$$H(|x - x'|) = (1/\pi \beta^2)^{k+1} \exp[-(x - x')^2/\beta^2],$$  \hspace{1cm} (45)

$g(r^2, p^2)$ is given by

$$G^w(x, p) = g(r^2, p^2) = U(r) \exp[-\beta^2 p^2/4\hbar^2],$$  \hspace{1cm} (46)

from which Eq. (42) for the equivalent local potential takes the form

$$V_{eq}(r) = U(r) \exp\left[-\frac{m\beta^2}{2\hbar^2} (E - V_{eq}(r))\right].$$  \hspace{1cm} (47)

This is the same result given by Percy and Buck. In this case the factor $F(r)$ of Eq. (44) becomes

$$F(r) = 1 \sqrt{1 - \frac{m\beta^2}{2\hbar^2} V_{eq}(r)},$$  \hspace{1cm} (48)

which indicates that $F(r)$ gives a damping effect for attractive non-local potentials. Equation (48) is of the same form as the damping factor found by Percy empiri-
Equation (47) can be interpreted as in the following way. When the range parameter $\beta$ is very small we have $H(|x-x'|) \approx \delta(x-x')$ and $G(x,x') \approx U(r) \times \delta(x-x')$. Measure of the validity of this approximation is obtained by comparing the range parameter $\beta$ with the local wave length $\lambda$ of the wave function $\phi(x)$. If $\lambda$ is far shorter than the non-locality length $2\beta$, the integration $\int G(x,x') \phi(x') dx'$ will vanish. In order to survive we need $2\beta < \lambda$. Since $\lambda$ is given by $\lambda = 2\pi/(p/\hbar)$ with $p = \sqrt{2m(E-V_{eq}(r))}$, the above criteria becomes $(\beta p/\pi \hbar) < 1$. This gives an interpretation of the appearance of the factor $\exp[-(\beta p/2\hbar)] = \exp[-(m\beta^2/2\hbar^2)(E-V_{eq}(r))]$ in Eq. (47). We should note the criterion $(\beta p/\pi \hbar) < 1$ is nothing but the relation of Eq. (15). In this argument we have assumed as usual that the range parameter of $U(r)$ is far longer than that of $H(s, \beta)$.

§ 4. Parity-dependence

4.1. General formulation

First we consider for the sake of interpretation a Frahn-Lemmer type non-local potential which we denote here as

$$G(x,x') = G_0 U_0 \left( \frac{1}{2} |x + x'| \right) H_0 \left( \frac{1}{2} |x - x'| \right),$$

$$\int dx' U_0 \left( \frac{1}{2} |x + x'| \right) = \int dx' H_0 \left( \frac{1}{2} |x - x'| \right) = 1. \quad (49)$$

If the range $\beta_H$ of $H_0(\frac{1}{2} |x - x'|)$ is far shorter than the range $\beta_U$ of $U_0(\frac{1}{2} |x + x'|)$, we have an approximation $G(x,x') \approx G_0 U_0(r) \delta(x-x')$. But on the contrary if the range parameter $\beta_U$ is far shorter than $\beta_H$ we should approximate $G(x,x')$ as $G(x,x') \approx G_0 H_0(r) \delta(x+x')$ instead of $G(x,x') \approx G_0 U_0(r) \delta(x-x')$. If we define the Majorana operator $P_H$ by $P_H \psi(x) = \psi(-x)$, we know that $\int G_0 H_0(r) \times \delta(x+x') \psi(x') dx' = G_0 H_0(r) P_H \psi(x)$. Thus the equivalent local potential to the non-local potential $G(x,x')$ with $\beta_U \ll \beta_H$ should be a parity-dependent potential.

The above conclusion on the appearance of the parity-dependent equivalent local potential results also quite naturally as we explain below through the process we derive an equivalent local potential by the WKB method. We denote the local wave length of $\psi(x)$ by $\lambda$ as in § 3 where $\lambda = 2\pi/(p/\hbar)$ with $p = \sqrt{2m(E-V_{eq}(r))}$ denoting the local momentum. In order to prevent the integral $\int G(x,x') \phi(x') dx'$ from vanishing the condition $\lambda > \min(2\beta_H, 2\beta_U)$ must be satisfied. When $\beta_H < \beta_U$ the application of the WKB method in §§ 2 and 3 is possible under the following short-wave-length condition. Namely, since in this case the range of the equivalent local potential is roughly given by $\beta_U$ from $V_{eq}(r) = G_0 U_0(r) \cdot \exp[-(m\beta_U^2/2\hbar^2) \times (E-V_{eq}(r))]$, the short-wave-length condition is $\lambda < 2\beta_U$. Thus our WKB treatment of the non-local potential in §§ 2 and 3 is found to be meaningful when
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To violate the condition $2\beta_H < \lambda$ simply means that the non-local potential $G(x, x')$ has no effect on the highly oscillating wave function $\psi(x)$. On the other hand, to violate the condition $\lambda < 2\beta_H$, means that we cannot apply the WKB approximation to the non-local potential $G(x, x')$. When $\beta_H > \beta_C$, the WKB approximation procedure in §§ 2 and 3 is not applicable directly to $G(x, x')$. This is because the short-wave-length condition $\lambda < 2\beta_H$ is not compatible with the survival condition $\lambda > \min(2\beta_C, 2\beta_H)$. In this case we need to treat the wave function $P_{\psi\psi}(x)$ instead of $\psi(x)$. We note the relation

$$JG(0, u_{\psi\psi}(x) - x') dx' = JG(x, -x') P_{\psi\psi}(x') dx'. $$

To the non-local potential $G(0, x, x')$ we can safely apply the WKB method in §§ 2 and 3 under the condition $2\beta_H < \lambda < 2\beta_H$. The condition $2\beta_H < \lambda$ is simply to avoid the vanishment of the integration $JG(x, x') \psi(x') dx' = JG(x, -x') P_{\psi\psi}(x') dx'$, and the condition $\lambda < 2\beta_H$ is a short-wave-length condition for the non-local potential $G(x, -x')$ to be treated by the WKB method.

Now we formulate the above discussions on the parity-dependence. We treat the general type of non-local potentials not restricted to the Frahn-Lemmer type. Non-local potentials are classified into two groups A and B; one group A being composed of usual non-local potentials whose non-locality range in $|x - x'|$ is shorter than that in $|x + x'|$, and the other group B being composed of the opposite type of non-local potentials whose non-locality range in $|x - x'|$ is longer than that in $|x + x'|$. The Schrödinger equation containing non-local potentials is written in general in the following way:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V(r) \psi(x) + \int [G_A(x, x') + G_B(x, x')] \psi(x') dx' = E \psi(x),$$

where $G_A(x, x')$ belongs to the group A and $G_B(x, x')$ to the group B. Both $G_A(x, x')$ and $G_B(x, x')$ satisfy Eqs. (2) and (3). In order to apply the WKB approximation, we need to transform Eq. (50) so that the resulting equation contains $G_B(x, -x')$ instead of $G_B(x, x')$. This can be done by rewriting Eq. (50) as follows,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi_\pm(x) + V(r) \psi_\pm(x) + \int [G_A(x, x') \pm G_B(x, -x')] \psi_\pm(x') dx' = E \psi_\pm(x),$$

$$\psi_\pm(x) = (1 \pm P_{\psi\psi}) \psi(x) = \psi(x) \pm \psi(-x).$$

A non-local potential $\tilde{G}_B(x, x')$ defined by $\tilde{G}_B(x, x') = G_B(x, -x')$ satisfies Eqs. (2) and (3) and can be treated by the WKB method. In deriving Eq. (51) from Eq. (50) we have used the relations $G_A(-x, -x') = G_A(x, x')$ and $\tilde{G}_B(-x, -x') = \tilde{G}_B(x, x')$ which are due to the fact that $G_A$ and $\tilde{G}_B$ are functions of $x^2$, $x'^2$ and $(x \cdot x')$. All the results in §§ 2 and 3 are available now if we replace
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\[ g(r^2, p^2, (x \cdot p)^2) \]

by \[ g_A(r^2, p^2, (x \cdot p)^2) + g_B(r', p', (x \cdot p')^2) \]

for even partial waves and by

\[ g_A(r^2, p^2, (x \cdot p)^2) - g_B(r^2, p^2, (x \cdot p)^2) \]

for odd partial waves, where \( g_A \) and \( g_B \) are the Wigner transforms of \( e_A \) and \( e_B \), respectively. Especially the equivalent local potentials \( V_{+eq}(r) \) for even partial waves and \( V_{-eq}(r) \) for odd ones are obtained from

\[
V_{\pm eq}(r) = V(r) + g_A(r^2, 2m(E - V_{\pm eq}(r)), 2mr^2(E - V_{\pm eq}(r)) - h^2(l + 1/2)^2)
\]

\[
2mr^2(E - V_{\pm eq}(r) - h^2(l + 1/2)^2/g_B(r^2, 2m(E - V_{\pm eq}(r)), 2mr^2(E - V_{\pm eq}(r)) - h^2(l + 1/2)^2).
\]

We can say that non-local potentials of the group B give the Majorana-type equivalent local potentials while those of the group A give the ordinary Wigner type ones.

4.2. Analyses of RGM kernels of two-spinless-cluster systems

RGM kernels of the system of two spinless clusters (or nuclei) are generally written as

\[
G(x, x') = \sum_n G_n(x, x'),
\]

\[
G_n(x, x') = P_n(x, x') \exp\left[ -a_n x^2 - b_n x'^2 - c_n x \cdot x' \right]
\]

\[
+ P_n(x', x) \exp\left[ -b_n x^2 - a_n x'^2 - c_n x \cdot x' \right],
\]

where \( P_n(x, x') \) are polynomials of \( x^2, x'^2 \), and \( (x \cdot x') \). We notice that \( G_n(x, x') \) for all \( n \) satisfy Eqs. (2) and (3). By rewriting the exponents in \( G_n(x, x') \) we have

\[
G_n(x, x') = \exp\left[ -\frac{a_n + b_n + c_n}{4} (x + x')^2 - \frac{a_n + b_n - c_n}{4} (x - x')^2 \right]
\]

\[
\times \left\{ P_n(x, x') \exp\left[ -\frac{a_n - b_n}{2} (x + x') \cdot (x - x') \right]
\right\}
\]

\[
+ P_n(x', x) \exp\left[ -\frac{b_n - a_n}{2} (x + x') \cdot (x - x') \right].
\]

Non-locality ranges of \( G_n(x, x') \) are mainly determined by the Gaussian part of \( G_n(x, x') \) on \( (x + x')^2 \) and \( (x - x')^2 \); namely, the non-locality range in \( |x + x'| \) is measured by \( \sqrt{4/(a_n + b_n + c_n)} \) while that in \( |x - x'| \) by \( \sqrt{4/(a_n + b_n - c_n)} \). We know thus if \( c_n < 0, G_n(x, x') \) belongs to the group A and if \( c_n > 0, \) to the group B. This classification can be justified also from the following partial wave expansion of the Gaussian part:
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\[ \exp[-a_x^2 - b_{x'}^2 - c_{x,x'}] \]

\[ = \exp[-a_x^2 - b_{x'}^2] \cdot 4\pi \sum_{l=0}^{\infty} i_l (-c_{x,x'}) \times \sum \mathcal{Y}_{lm}(x) \mathcal{Y}_{lm}(x') \]

where \( i_l(x) \) is a modified spherical Bessel function. When \( c_n < 0 \), \( \exp[-a_x^2 - b_{x'}^2] i_l (-c_{x,x'}) \) has the positive sign for all \( l \), but when \( c_n > 0 \), it has a regular alternative sign (-) \(^l\). This clearly indicates that \( G_n(x, x') \) with \( c_n > 0 \) has a Majorana-type character, namely, it belongs to the group B, while \( G_n(x, x') \) with \( c_n < 0 \) belongs to the group A. Our classification of \( G_n(x, x') \) according to the sign of \( c_n \) just coincides with the classification of Refs. 6) and 8) where use is made of the Born approximation technique in deriving equivalent local potentials.

The Wigner transform we need for \( c_n < 0 \) is that of \( G_n(x, x') \) and is as follows:

\[ G_n^{w}(x,p) = \int ds \exp(\frac{i}{\hbar} s \cdot p) G_n\left(x - \frac{s}{2}, x + \frac{s}{2}\right) \]

\[ = \exp[-4a_x b_n - c_n^2] \cdot \exp \left[-\frac{p^2}{(a_n + b_n - c_n)\hbar^2}\right] \times \left\{ \exp \left[i \frac{2(b_n - a_n)}{(a_n + b_n - c_n)\hbar} x \cdot p\right] Q_n(x, ip) \right. \]

\[ + \left. \exp \left[i \frac{2(a_n - b_n)}{(a_n + b_n - c_n)\hbar} x \cdot p\right] Q_n(x, -ip) \right\}, \quad (c_n < 0) \]

where \( Q_n(x, ip) \) is a polynomial of \( x^2, p^2 \) and \( i(x \cdot p) \) with real number coefficients. For \( c_n > 0 \), we need to use the Wigner transform of \( G_n(x, -x') \) which is given by

\[ \tilde{G}_n^{w}(x,p) = \int ds \exp(\frac{i}{\hbar} s \cdot p) G_n\left(x - \frac{s}{2}, -x - \frac{s}{2}\right) \]

\[ = \exp[-4a_x b_n - c_n^2] \cdot \exp \left[-\frac{p^2}{(a_n + b_n + c_n)\hbar^2}\right] \times \left\{ \exp \left[i \frac{2(a_n - b_n)}{(a_n + b_n + c_n)\hbar} x \cdot p\right] \tilde{Q}_n(x, ip) \right. \]

\[ + \left. \exp \left[i \frac{2(b_n - a_n)}{(a_n + b_n + c_n)\hbar} x \cdot p\right] \tilde{Q}_n(x, -ip) \right\}, \quad (c_n > 0) \]

where \( \tilde{Q}_n(x, ip) \) is also a polynomial of \( x^2, p^2 \) and \( i(x \cdot p) \) with real number coefficients. Equivalent local potentials are obtained by making substitutions \( p^2 \rightarrow 2m(E - V_{eq}(r)) \) and \( (x \cdot p)^2 \rightarrow 2mr^2 \cdot (E - V_{eq}(r) - \hbar^2(l+\frac{1}{2})/2mr^2) \) in these \( G_n^{w} \) and \( \tilde{G}_n^{w} \) as is shown in Eq. (52). From Eqs. (56) and (57) we know that a
characteristic parameter $R_n$ to determine the spatial range of the equivalent local potential is given for both $c_n \geq 0$ by

$$R_n = \sqrt{\frac{a_n + b_n + |c_n|}{4a_n b_n - c_n^2}}. \quad (58)$$

As for the energy dependence of the equivalent local potential, we first consider a case that a substitution $p^2 \rightarrow 2mE = \hbar^2 k^2$ is a good approximation. This approximation is good in high energy region where the effect of $V_{\pm \text{eq}}(r)$ can be neglected in comparison with $E$. Also at the surface region of $V_{\pm \text{eq}}(r)$ this approximation is valid. In this case we see from Eqs. (56) and (57) that the energy dependence is determined for both $c_n \geq 0$ by the following wave number parameter $k_n$:

$$k_n = \sqrt{a_n + b_n + |c_n|}. \quad (59)$$

which gives a measure of the energy range in the outside of which the potential is negligible. When the effect of $V_{\pm \text{eq}}(r)$ cannot be neglected, we need to use the correct substitution $p^2 \rightarrow 2m(2m-V_{\pm \text{eq}}(r))$. In this case we know clearly that the energy range is smaller than the energy range $\hbar^2 k_n^2 / 2m$ with $k_n$ of Eq. (59) as far as $V_{\pm \text{eq}}(r)$ is an attractive potential. Therefore in general the energy range of the non-local potential becomes smaller as we go inside the interaction region from the surface region.

Range parameters $R_n$ of Eq. (58) and $k_n$ of Eq. (59) are quantities to be compared with the corresponding range parameters $\tilde{R}_n$ and $\tilde{k}_n$ derived by LeMere and Tang:

$$\tilde{R}_n = \sqrt{\frac{2|c_n|}{4a_n b_n - c_n^2}}, \quad \tilde{k}_n = \sqrt{\frac{4a_n b_n - c_n^2}{a_n + b_n + |c_n|}}. \quad (60)$$

Although they look like different from each other at first sight, the substitution of the explicit formulas of $a_n$, $b_n$, and $c_n$ of RGM kernels into $(R_n, k_n)$ and $(\tilde{R}_n, \tilde{k}_n)$ shows that they are almost equivalent for important exchange kernels such as one-nucleon-exchange term and core-exchange term. Detailed discussions of this point will be given elsewhere.

§ 5. WKB treatment of non-local potentials with partial wave expansion

As is discussed in § 2.1, the Planck constant $\hbar$ contained in the Wigner transforms of non-local potentials is kept untouched in the WKB treatment where we expand all the other quantities in power series of $\hbar$. This incompleteness of the expansion in power series of $\hbar$ employed in the WKB treatment of non-local potentials can cause non-uniqueness of the obtained results. Namely if we apply the WKB approximation to a different solution-process of the Schrödinger equation than that adopted in § 2, we may obtain a result which is not exactly the same as that in § 2. However, from the arguments on Eq. (15) in § 2.1, we can say that the difference of the results which is due to the difference of the solution-
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processes of the Schrödinger equation cannot be large so long as the quantity \((\beta \hbar / \hbar)\) is regarded as sufficiently small \((\beta \hbar \ll \hbar)\).

Considering the above discussion we investigate in this section what is the result of the WKB treatment when it is applied to the radial Schrödinger equation of each partial wave. The radial Schrödinger equation for \(u_{\ell}(r)\) of \(\psi(x) = (u_{\ell}(r)/r) Y_{\ell m}(\hat{x})\) is obtained from Eq. (1) as follows:

\[
\begin{cases}
\frac{-\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2}{2m} \frac{(l+1/2)^2}{2} \end{cases} u_{\ell}(r) + \int_0^\infty G_{\ell}(r, r') u_{\ell}(r') dr' = Eu_{\ell}(r), \\
G_{\ell}(r, r') = r r' \int d\hat{x} d\hat{x}' Y_{\ell m}(\hat{x}) Y_{\ell m}(\hat{x}') G(\hat{x}, \hat{x}').
\]

In order to apply the WKB approximation, we introduce as usual the so-called Langer transformation:

\[
r = e^\tau, \quad (r' = e^{\tau'}), \\
u_{\ell}(r) = e^{i\varphi_0} \omega_{\ell}(x).
\]

Inserting Eq. (62) into Eq. (61) we get

\[
\begin{cases}
\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + e^{2\tau} \left[ \frac{\hbar^2 (l+1/2)^2}{2m} + V(e^\tau) - E \right] \omega_{\ell}(x) \\
\quad + \int_0^\infty f_{\ell}(x, x') \omega_{\ell}(x') dx' = 0, \\
f_{\ell}(x, x') = e^{i\varphi_0 x + x'} G_{\ell}(e^\tau, e^{\tau'}).\n\end{cases}
\]

We rewrite the integral operator \(f_{\ell}(x, x')\) into the differential operator like Eq. (4) as

\[
\int_0^\infty f_{\ell}(x, x') \omega_{\ell}(x') dx' = \tilde{f}_{\ell}(x, \tilde{p}) \omega_{\ell}(x),
\]

\[
\tilde{f}_{\ell}(x, \tilde{p}) = \int_0^\infty ds \exp \left( i s \tilde{p} \right) f_{\ell} \left( x - \frac{s}{2}, x + \frac{s}{2} \right) \exp \left( i s \tilde{p} \right),
\]

\[
\tilde{p} = \frac{\hbar}{i} \frac{d}{dx}.
\]

By expressing \(\omega_{\ell}(x)\) as \(\exp \left[ (i/\hbar) s(x) \right]\), we obtain from Eqs. (63) and (64),

\[
\hbar \left( x, \frac{ds(x)}{dx} + \tilde{p} \right) \cdot 1 = 0,
\]

\[
\hbar \left( x, \frac{ds(x)}{dx} + \tilde{p} \right) = \frac{\tilde{p}^2}{2m} + e^{2\tau} \left[ \frac{\hbar^2 (l+1/2)^2}{2m} + V(e^\tau) - E \right] + \tilde{f}_{\ell}(x, \tilde{p}).
\]
By expanding $s(x)$ in a power series of $h$ as $s(x) = s_0(x) + hs_1(x) + \cdots$, we obtain from the zeroth and first power terms of $h$ of Eq. (65) the following equations:

$$h \left( x, \frac{ds(x)}{dx} \right) = 0,$$

$$s_1(x) = \frac{i}{2} \log \left[ \frac{\partial h(x, p_1)}{\partial p_1} \right]_{p_1 = \text{const}} + \text{const.} \quad (66)$$

In deriving Eq. (66) we have regarded, as in deriving Eq. (8), as if $f_i(x, \vec{p})$ were independent of $h$ as a function of $x$ and $\vec{p}$. We notice here that when a $c$-number $p_1$ is inserted in place of the $q$-number $\vec{p}$ in $h(x, \vec{p})$, we have

$$h(x, p_1) = \frac{\hbar^2}{2m} + e^{x} \left[ \frac{h^2 (l+1/2)^2}{2me^{x}} + V(e^{x}) - E \right] + f_1^W(x, p_1),$$

$$f_1^W(x, p_1) = \int_{-\infty}^{\infty} ds \left( \exp \frac{i}{\hbar} s p_1 \right) f_i \left( x - \frac{s}{2}, x + \frac{s}{2} \right)$$

$$= e^{x} \int_{-\infty}^{\infty} ds \exp \left( \frac{i}{\hbar} s p_1 \right) G_1(\exp \left( x - \frac{s}{2} \right), \exp \left( x + \frac{s}{2} \right)). \quad (67)$$

The function $f_1^W(x, p_1)$ which is the Wigner transform of $f_i(x, x')$ has a property $f_1^W(x, -p_1) = f_1^W(x, p_1)$ and is therefore a function of $p_1^2$.

From Eq. (66), the WKB wave function of $w_N(x)$ is expressed as

$$\omega_N(x) \sim \frac{\text{const}}{\sqrt{|p_1 + m(\frac{\partial}{\partial p_1})f_1^W(x, p_1)|}} \exp \left[ \pm \frac{i}{\hbar} \int_{x_0}^{x} p_1(x) dx \right], \quad (68)$$

where $x_0$ is an arbitrary but fixed point and $p_1 = p_1(x)$ is obtained by solving

$$h(x, p_1(x)) = 0. \quad (69)$$

It is to be noticed that if $p_1 = p_1(x)$ is a solution of Eq. (69), $p_1 = -p_1(x)$ is also a solution of Eq. (69) because $h(x, p_1)$ is a function of $p_1^2$. Now we define $Q_i(r)$ by

$$Q_i(r) = e^{-i p_1(x)}. \quad (p_i(x) = rQ_i(r)) \quad (70)$$

From Eqs. (67) and (69), $Q_i(r)$ is found to satisfy

$$Q_i^2(r) + \frac{\hbar^2 (l+1/2)^2}{2m}\frac{\partial}{\partial r} + V(r) + \bar{G}_i(r, Q_i(r)) = E,$$

$$\bar{G}_i(r, Q_i) = \int_{-\infty}^{\infty} dt \exp \left( \frac{i}{\hbar} tQ_i \right) G_i(r e^{-t\alpha}, r e^{t\alpha}). \quad (71)$$

We also notice
where of course $r_0 = e^{i\phi}$. Furthermore we can prove the following relation easily:

$$e^{-i\theta} \left( \frac{\partial}{\partial \rho_1} \right)f_1^w(x, p_t(x)) = \left( \frac{\partial}{\partial \rho_3} \right) \tilde{G}_1^w(r, Q_t(r)).$$  \hspace{1cm} (73)

From Eqs. (70), (72) and (73) we obtain the WKB wave function of $u_N(x)$ as

$$u_N(x) = e^{i\phi_0} \omega_N(x)$$

$$\int_{r_i}^{x} \rho_i(x) dx = \int_{r_i}^{x} Q_i(r) dr,$$  \hspace{1cm} (72)

$$\sqrt{e^{-i\theta} \left( \frac{\partial}{\partial \rho_1} \right)f_1^w(x, p_t(x))} = \left( \frac{\partial}{\partial \rho_3} \right) \tilde{G}_1^w(r, Q_t(r)).$$

$$\tilde{G}_1^w(r, Q_t)$$ defined by Eq. (71) satisfies $\tilde{G}_1^w(r, -Q_t) = \tilde{G}_1^w(r, Q_t)$ and so it is a function of $Q_t$. This enables us to rewrite the amplitude of $u_N(x)$ as follows:

$$1 \sqrt{Q_t + m \left( \frac{\partial}{\partial Q_t} \right) \tilde{G}_1^w} = F_1(r) / \sqrt{Q_t},$$

$$F_1(r) = \sqrt{1 + 2m \left( \frac{\partial}{\partial Q_t} \right) \tilde{G}_1^w}.$$  \hspace{1cm} (75)

Application of Fiedeldey's theory, the derivation of an equivalent local potential and the connection of the WKB wave function at a turning point can be done entirely in parallel with § 2.3 and with Ref. 11). An equivalent local potential $V_1^{eq}(r)$ is given by

$$\sqrt{2m \left[ E - \frac{\hbar^2 (l+1/2)^2}{2mr^2} - V_1^{eq}(r) \right]} = Q_t(r),$$  \hspace{1cm} (76)

which can be rewritten by Eq. (71) as

$$V_1^{eq}(r) = V(r) + \tilde{G}_1^w(r, \sqrt{2m \left[ E - \frac{\hbar^2 (l+1/2)^2}{2mr^2} - V_1^{eq}(r) \right]}).$$  \hspace{1cm} (77)

The Perey effect of the wave function is expressed by the multiplicative factor $F_1(r)$ of Eq. (75). The WKB wave function which satisfies a connection condition at a turning point $r_i$ defined as a zero point of $Q_t(r)$ ($Q_t(r_i) = 0$) is given by

$$u_s(x) \sim \text{const} \frac{F_1(x)}{\sqrt{Q_t(x)}} \sin \left[ \int_{r_i}^{r} \frac{Q_t(r)}{\hbar} dr + \frac{\pi}{4} \right].$$  \hspace{1cm} (78)
Now we compare the above results of this section with those of § 2. We see clearly that, if $g(r', P') + \hbar^2((l + \frac{1}{2})^2/r^2, r^2p^2_\parallel)$ is equal to $G_{i, W}(r, P)$ for two independent variables $r$ and $P$, the results of § 2 are identical with those of § 5. Thus the similarity between the results of § 2 and those of § 5 is governed by the similarity between $g(r^2, p^2 + \hbar^2((l + \frac{1}{2})^2/r^2, r^2p^2_\parallel)$ and $G_{i, W}(r, P)$.

As we have discussed at the beginning of this section, if the non-locality range $\beta$ is so small as to satisfy $\beta \hbar \ll r$, $g$ and $G_{i, W}$ is expected to be approximately equal. To assure this point we below give an example. We consider for simplicity a non-local potential

$$G(x, x') = U \left( \frac{1}{2} |x + x'| \right) \left( \frac{1}{\pi \hbar^2} \right)^{\frac{3}{2}} \exp \left( - \frac{(x - x')^2}{\beta^2} \right) \quad (79)$$

with very small $\beta > 0$. We have

$$G_{i, W}(x, P) = g(r^2, P^2) = U(r) \exp \left( - \frac{\beta^2 P^2}{4 \hbar^2} \right), \quad (80)$$

which means

$$g(r^2, P^2 + \hbar^2((l + \frac{1}{2})^2/r^2)) = U(r) \exp \left( - \frac{\beta^2 P^2}{4 \hbar^2} \right) \exp \left( - \frac{r^2 + r'^2}{\beta^2} \right) \quad (81)$$

On the other hand $G_{i, W}(r, r')$ is given by

$$G_{i, W}(r, r') = 4\pi \left( \frac{1}{\pi \hbar^2} \right)^{\frac{3}{2}} r^2 U(r) i t \left( \frac{2r^2}{\beta^2} \right) \int_{-\infty}^{\infty} dt \exp \left( \frac{i}{\hbar} tp \right) \exp \left( - \frac{2r^2}{\beta^2} \cosh \frac{t}{r} \right) \quad (82)$$

where we have used an approximation $U\left( \frac{1}{2} |x + x'| \right) = U(\sqrt{rr'})$ because $x = x'$ in $G(x, x')$ due to the very small $\beta$. From Eq. (82) we get

$$\tilde{G}_{i, W}(r, P) = 4\pi \left( \frac{1}{\pi \hbar^2} \right)^{\frac{3}{2}} r^2 U(r) i t \left( \frac{2r^2}{\beta^2} \right) \int_{-\infty}^{\infty} dt \exp \left( \frac{i}{\hbar} t P \right) \exp \left( - \frac{2r^2}{\beta^2} \cosh \frac{t}{r} \right) \quad (83)$$

Due to the small $\beta$, we can use the following approximations in Eq. (83):

$$i t \left( 2r^2 \right) = i t \left( \frac{2r^2}{\beta^2} \right) \exp \left( 2r^2 - \frac{\beta^2 (l + 1)}{4r^2} \right) \quad (84)$$

We then obtain

$$\tilde{G}_{i, W}(r, P) \approx U(r) \exp \left( - \frac{\beta^2 P^2}{4 \hbar^2} \right) \exp \left( - \frac{r^2 + \hbar^2((l + \frac{1}{2})^2/r^2)}{\beta^2} \right) \quad (85)$$

which can be identified with Eq. (81) for large $l$. 

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Note: The document appears to be a scientific or technical text, possibly related to quantum mechanics or related fields, discussing various mathematical expressions and approximations. The text is presented as a single continuous paragraph with no explicit sub-sections or headings, typical of a research paper or advanced textbook entry.
A Semi-Classical Treatment of Non-Local Potentials

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References