IMPRIMITIVE PARAMETRIZATION OF ANALYTIC CURVES
AND FACTORIZATIONS OF ENTIRE FUNCTIONS

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Abstract
Let \( f(z) \) and \( g(z) \) be transcendental entire functions. Fuchs and Song proved that if \((f(z), g(z))\) parametrizes some complex algebraic curve, then \( f \) and \( g \) must have a transcendental common right factor. The paper proves this result by a different method that also allows a similar result to be proved for some transcendental curves. This result is then used to solve some factorization problems of entire functions.

1. Introduction and main results

Let \( f(z) \) and \( g(z) \) be entire functions of one complex variable and \( \Phi(x, y) \) be a complex polynomial in both \( x \) and \( y \). The pair \((f(z), g(z))\) is called a parametrization of a complex algebraic curve defined by \( \Phi(x, y) = 0 \) if \( \Phi(f(z), g(z)) \equiv 0 \) on the complex plane. The parametrization \((f(z), g(z))\) is called imprimitive if there exists a non-linear entire function \( h(z) \) such that \( f(z) = f_1(h(z)) \) and \( g(z) = g_1(h(z)) \), where \( f_1, g_1 \) are analytic on the image of \( h \), which will be denoted by \( \text{Im}(h) \). By the little Picard theorem, \( \text{Im}(h) \) can either be \( \mathbb{C} \) or \( \mathbb{C} - \{a\} \) for some complex number \( a \). When both \( f_1, g_1 \) are entire, we call \( h \) a common right factor of \( f \) and \( g \). In [5], Fuchs and Song proved that if both \( f, g \) are transcendental and \((f(z), g(z))\) parametrizes some complex algebraic curve, then the parametrization must be imprimitive. In fact, they proved something more.

**Theorem A.** Let \( f(z), g(z) \) be two transcendental entire functions and \( \Phi(x, y) \) be a non-zero complex polynomial such that \( \Phi(f(z), g(z)) \equiv 0 \) on the complex plane. Then there exists a transcendental entire function \( h(z) \) such that \( f(z) = f_1(h(z)) \) and \( g(z) = g_1(h(z)) \), where \( f_1, g_1 \) are both rational functions with at most one pole.

**Example 1.** Let \( f(z) = \sin z \), \( g(z) = \cos z \) and \( \Phi(x, y) = x^2 + y^2 - 1 \). Then \( \Phi(f(z), g(z)) \equiv 0 \) on \( \mathbb{C} \). In this case, we can take \( h(z) = e^{iz} \), \( f_1(z) = (i/2)(z - z^{-1}) \) and \( g_1(z) = 1/2(z + z^{-1}) \) such that \( f(z) = f_1(h(z)) \) and \( g(z) = g_1(h(z)) \).

Theorem A has many applications to the factorization and sharing value problems of entire functions. Note that Fuchs and Song only considered the case \( \Phi(x, y) = p(x) - q(y) \), where \( p, q \) are polynomials. However, their proof works for general \( \Phi \). Their proof is based on the following result of Picard [11], which can also be proved by using the Nevanlinna theory (see [7, p. 232]).

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Then $\Phi(f(z), g(z)) \equiv 0$
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on the complex plane. Then the complex algebraic curve defined by $\Phi(x, y) = 0$ has genus less than or equal to 1.

In this paper, we give a different proof of Theorem A. Our method depends on a result of Grauert [6] about complex analytic equivalence relations. Using this method, we can prove a result similar to Theorem A for some transcendental curve. In fact, we have the following.

**Theorem 1.** Let $n \geq 1$ and $\Phi(x, y) = \sum_{i=0}^{n} a_i(x)y^i$ be a polynomial in $y$ with entire functions $a_i(x)$ as coefficients such that $a_n \not\equiv 0$. Suppose that $f(z)$ and $g(z)$ are transcendental entire functions such that

$$\Phi(f(z), g(z)) = \sum_{i=0}^{n} a_i(f(z))g(z)^i \equiv 0$$

on the complex plane. Then there exists a transcendental entire function $h$ such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$, where $f_1, g_1$ are analytic on $\text{Im}(h)$.

**Example 2.** Let $f(z) = \cos z$, $g(z) = \sin z e^{\cos z}$ and $\Phi(x, y) = (1 - x^2) e^{2x} - y^2$. Then $\Phi(f(z), g(z)) \equiv 0$ because $g(z)^2 = \sin^2 z e^{2\cos z} = ((1 - w^2) e^{2w}) \circ \cos z = ((1 - w^2) e^{2w}) \circ f(z)$. In this case, we can take $h(z) = e^z$, $f_1(z) = (1/2)(z + z^{-1})$, $f_2(z) = (i/2)(z - z^{-1}) e^{1/2i(z+z^{-1})}$ such that $f(z) = f_1(h(z))$, $g(z) = g_1(h(z))$ and $f_1, f_2$ are analytic on $\mathbb{C} - \{0\}$. Note that $f_2$ has an essential singularity at 0.

To prove Theorem 1, we first establish in Section 2 a main lemma that gives a sufficient condition for the existence of a non-linear generalized common right factor of two entire functions. We then further develop some criteria on the existence of a transcendental entire common right factor in Section 3. These results in turn allow us to prove Theorems 1 and A. To illustrate the usefulness of the main lemma, let us state the following result, which will be deduced from the main lemma in Section 2.

**Theorem 2.** Let $p(z)$ and $q(z)$ be two polynomials. If $p(x) - p(y)$ and $q(x) - q(y)$ have a common factor of degree $d \geq 2$ in $\mathbb{C}[x, y]$, then $p(z)$ and $q(z)$ have a common right factor of degree greater than or equal to $d$. Furthermore, $p(x) - p(y)$ and $q(x) - q(y)$ have a common factor of maximal degree $d$ in $\mathbb{C}[x, y]$ if and only if $p(z)$ and $q(z)$ have a greatest common right factor of degree $d$.

In Section 4, we shall mainly concern ourselves with the applications of those criteria to factorization problems of entire functions. We shall generalize some known results which were proved through the use of several different arguments before. In fact, the main purpose of this paper is to provide a more systematic way of solving factorization problems of entire functions. Our method depends very much on the ideas and results in Eremenko and Rubel’s paper [3]. In particular, most of the contents of Section 2 are taken from [3].
2. The main lemma

In this section, we state and prove the main lemma of this paper (Lemma 2), which gives a sufficient condition on the existence of a non-linear entire common right factor for any two non-constant entire functions. Its proof is based on the following result of Grauert [6] on complex analytic equivalence relations. Throughout this paper, $X$ denotes either $\mathbb{C}$ or $\mathbb{C} - \{a\}$, where $a$ is a complex number.

**Theorem C.** Let $R$ be any equivalence relation on $X$ with a graph $G = \{(x, y) \in X \times X \mid xRy\}$; that is a complex analytic subset of $X \times X$ containing no vertical or horizontal lines (that is, subsets of the form $\{x\} \times X$ or $X \times \{y\}$). Suppose that $G$ is of pure dimension 1 (that is, $G$ is everywhere of the same dimension 1). Then there exists a holomorphic map $h$ from $X$ onto a Riemann surface $S$ such that $xRy$ if and only if $h(x) = h(y)$.

In [3, Appendix A], Eremenko and Rubel gave a more elementary and direct proof when $X = \mathbb{C}$. The same proof also works for the case $X = \mathbb{C} - \{a\}$. The basic terminology and properties of complex analytic sets can be found in [1].

**Definition 1.** Let $f$ be an analytic function on $X$. We say that $h$ is a general right factor (denoted by $h \leq f$) of $f$ if $h$ is a holomorphic map from $X$ to a Riemann surface $S$ and there exists a holomorphic map $f_1$ from $S$ to $\mathbb{C}$ such that $f = f_1 \circ h$.

Note that the word ‘map’ here always means a mapping between two Riemann surfaces, while ‘function’ means a mapping with its range in the complex plane.

**Definition 2.** Let $f$ and $g$ be entire functions. An entire function $h$ is a greatest common right factor of $f$ and $g$ if

(i) $h$ is a right factor of both $f$ and $g$;

(ii) every right factor of $f$ and $g$ must be a right factor of $h$.

Note that if $h$ is a greatest common right factor, then so is $L \circ h$ for any linear function $L$. It is proved in [3] that a greatest common right factor of $f$ and $g$ always exists and is unique up to a composition of a linear function. A non-constant holomorphic map $k$ from $X$ onto a Riemann surface $S$ can induce an equivalence relation $R$ in $X$ defined by $xRy$ if and only if $k(x) = k(y)$. Let $G_k = \{(x, y) \in X \times X \mid k(x) = k(y)\}$; then $G_k$ is a complex analytic set of pure dimension 1 that does not contain any vertical or horizontal line because $k$ is non-constant. Such $G_k$ is called the graph of equivalence relation induced by $k$. Let $G_h$ and $G_k$ be the graphs of the equivalence relation induced by surjective holomorphic maps $h : X \rightarrow S_1$ and $k : X \rightarrow S_2$ respectively, where $S_1, S_2$ are two Riemann surfaces. The following lemma states the relation between $h$ and $g$ when $G_h$ is a subset of $G_k$ (see [3, p. 338]).

**Lemma 1.** $G_h$ is a subset of $G_k$ if and only if $h \leq k$.

**Proof.** It is clear that if $h \leq k$, then $G_h \subseteq G_k$. Now suppose that $G_h \subseteq G_k$. This actually means that for any $x, y \in X$, $h(x) = h(y)$ implies that $k(x) = k(y)$. Hence the function $f : S_1 \rightarrow S_2$ defined by $f(s) = k(h^{-1}(s))$ is single-valued. Clearly
\( k(x) = f \circ h(x) \) on \( X \). It remains to show that \( f \) is holomorphic on \( S_i \). Take any \( s \in \text{Im}(h) = S_i \), and let \( w \) be any point in \( h^{-1}(s) \). We can always choose suitable local charts \((z_i, U_i)\) of \( w \) and \((\beta_j, V_j)\) of \( s \) such that \( \beta_j \circ h \circ z_i^{-1}(z) = z^n \) in a neighborhood of zero for some positive integer \( n \). Let \((\gamma_k, W_k)\) be a local chart of \( k(w) \). Then \( \gamma_k \circ k \circ z_i^{-1}(z) \) is analytic near zero. Now \( \gamma_k \circ k \circ h^{-1} \circ \beta_j^{-1}(z) = \gamma_k \circ k \circ z_i^{-1} \circ z_i \circ h^{-1} \circ \beta_j^{-1}(z) = \gamma_k \circ k \circ z_i^{-1} \circ z^{1/n} \), which is single-valued, analytic in a deleted neighborhood of zero, and continuous at zero. Hence it is analytic at zero and \( f \) is holomorphic at \( s \). □

The following lemma is very crucial.

**Lemma 2.** Let \( f, g \) be two analytic functions on \( X \). For \( i = 1, \ldots, k, k \geq 2 \), let \( S_i = \{z_{in} \mid n \in \mathbb{N}\} \) be a sequence of distinct complex numbers with limit point \( z_i \). Suppose that all the limit points \( z_i \) are distinct and, for all \( n \in \mathbb{N} \),

\[
\begin{align*}
 f(z_{1n}) &= f(z_{2n}) = \ldots = f(z_{kn}) \\
g(z_{1n}) &= g(z_{2n}) = \ldots = g(z_{kn}).
\end{align*}
\]

Then there exists a holomorphic function \( h(z) : X \to \mathbb{C} \) (which depends only on \( f \) and \( g \)) that satisfies \( h \leq f, h \leq g \) and \( h(z_i) = h(z_{1}) \) for all \( 2 \leq i \leq k \).

The proof of Lemma 2 is very similar to that of [3, Theorem 1.1]. For completeness, we sketch the proof below.

**Proof of Lemma 2.** Let \( G_f \) and \( G_g \) be the graphs of the equivalence relation induced by \( f \) and \( g \), respectively. Then \( G_f \cap G_g \) is a complex analytic set (see [1, p. 62]), but it may not have pure dimension 1, so we consider its derived set \( H = (G_f \cap G_g)' \) (that is, the set of limit points). Then \( H \) is a pure dimension 1 complex analytic set, and it does not contain any vertical or horizontal line. The non-trivial fact that \( H \) is still a graph of some equivalence relation is proved in [3, p. 338]. By Theorem C, we conclude that \( H \) is a graph of the equivalence relation induced by some holomorphic map \( h \) from \( X \) to some Riemann surface \( S \). Clearly \( h \) depends only on \( f \) and \( g \). Now \( H \) is a subset of both \( G_f \) and \( G_g \), so, from Lemma 1, we have \( h \leq f, h \leq g \). From assumption \((*)\) of the lemma, we have \((z_{1n}, z_{jn}) \in G_f \cap G_g \) for all \( 2 \leq j \leq k \) and \( n \in \mathbb{N} \). Therefore, for all \( 2 \leq j \leq k \), \((z_{1n}, z_{jn}) \in H = (G_f \cap G_g)' \), and hence \( h(z_{1}) = h(z_{j}) \).

From the uniformization theorem, we know that \( S \) is conformally equivalent to \( S_0/G \), where \( G \) is a fixed-point free discrete group of isometries of \( S_0, S_0 \) is any one of \( \mathbb{C}_\infty, \mathbb{C} \) or the unit disk \( \Delta \). If \( X = \mathbb{C} \), we claim that \( S_0 \) cannot be \( \Delta \), for otherwise, \( S \) will have a holomorphic universal covering from \( S_0 = \Delta \). As \( X = \mathbb{C} \) is simply connected, \( h \) can be lifted to a holomorphic map from \( \mathbb{C} \) to \( \Delta \), which must be constant by Liouville’s theorem. Hence \( h \) must also be a constant, which is a contradiction. For \( X = \mathbb{C} - \{a\} \), we can get the same contradiction by considering \( h(z^2 + a) \) instead of \( h \). Therefore, \( S_0 \) is either \( \mathbb{C}_\infty \) or \( \mathbb{C} \). From this fact, it is not difficult to show that \( S \) can only be one of the following: a Riemann sphere, a complex plane, a punctured plane, or a torus (see [4, p. 193]). As \( h \leq f \) and \( h \leq g \), there exist holomorphic maps \( h_1 \) and \( h_2 \) from \( S \) to \( \mathbb{C} \) such that \( f = h_1 \circ h \) and \( g = h_2 \circ h \). If \( S \) is a sphere or a torus, then \( S \) is compact. As \( h_1 \) is holomorphic on \( S \), \( h_1 \) and hence \( h \) must be a constant, which is a contradiction. Therefore, \( S \) is the whole plane or punctured plane. This completes the proof of Lemma 2. □
Proof of Theorem 2. Let \( R(x, y) \in \mathbb{C}[x, y] \) (with \( \deg R(x, y) = n \geq 2 \)) be a common factor of \( p(x) - p(y) \) and \( q(x) - q(y) \). Assume without loss of generality that \( \deg R(x, y) = \deg_R R(x, y) = n \geq 2 \). We claim that there are only finitely many \((a, b) \in \mathbb{C}^2\) such that
\[
R(a, b) = 0, \quad R_y(a, b) = 0.
\]

Since \( p(x) - p(y) = R(x, y)S(x, y) \) for some \( S(x, y) \in \mathbb{C}[x, y] \), \( -p''(y) = R_y(x, y)S(x, y) + R(x, y)S_y(x, y) \). Therefore, if \((a, b)\) satisfies (1), then \( p(a) = p(b) \) and \( p''(b) = 0 \), which in turn can only be satisfied by finitely many \((a, b)\).

Now we can choose some \( a \) such that \( R(a, y) = 0 \) has \( n \) distinct solutions \( b_i \) with \( R_y(a, b_i) \neq 0 \). By the implicit function theorem, for each \( 1 \leq i \leq n \), there exists a unique analytic function \( w_i(x) \) on an open set \( A_i \) of \( a \) such that \( R(x, w_i(x)) \equiv 0 \) on \( A_i \). Hence we have on an open neighborhood \( A = \bigcap_{i=1}^{n} A_i \neq \emptyset \) of \( x = a \)
\[
p(x) = p(w_1(x)) = \ldots = p(w_n(x)),
q(x) = q(w_1(x)) = \ldots = q(w_n(x)).
\]

Take a sequence of distinct terms \( \{a_k\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} a_k = a \). Define \( z_k = w_i(a_k) \) and \( S_i = \{z_k \}_{k \in \mathbb{N}} \). According to the implicit function theorem, all \( z_k \) are distinct and \( \lim_{k \to \infty} z_k = \lim_{k \to \infty} w_i(a_k) = b_i \); hence the sequences \( S_i \) satisfy the requirements in Lemma 2. Therefore, there exists an entire function \( h(z) \) that is a generalized right factor of both \( p \) and \( q \). Note that \( h \) must be a polynomial, for otherwise \( p \) and \( q \) will be transcendental. Since all \( b_i \) are distinct and
\[
h(b_1) = \ldots = h(b_n),
\]
h is a polynomial of degree at least \( n \geq 2 \), and we have proved the first part of Theorem 2.

We first remark that for any non-constant polynomial \( k(z) \), we always have
\[
k(x) - k(y) = (x - y)K(x, y)
\]
for some \( K(x, y) \in \mathbb{C}[x, y] \). Thus if \( p(z) \) and \( q(z) \) have a common right factor \( h(z) \) that is of degree \( n \), then \( p(x) - p(y) = \{h(x) - h(y)\}P(x, y) \), and \( q(x) - q(y) = \{h(x) - h(y)\}Q(x, y) \) has a common factor \( h(x) - h(y) \) that is of degree \( n \). Now let \( R(x, y) \in \mathbb{C}[x, y] \) be a common factor of \( p(x) - p(y) \) and \( q(x) - q(y) \) with maximal degree. Then, for any common right factor \( h \) of \( p \) and \( q \), \( h(x) - h(y) \) divides \( R(x, y) \).

If \( n = \deg R(x, y) = 1 \), then it follows from the above argument that any common right factor of \( p(z) \) and \( q(z) \) must be of degree 1. Hence the greatest common right factor is of degree 1. If \( n = \deg R(x, y) \geq 2 \), from the first part of this theorem, there exists a common right factor \( h \) that is of degree at least \( n \). On the other hand, \( h \) must have degree less than or equal to \( n \) because \( h(x) - h(y) \) divides \( R(x, y) \). Hence \( \deg h = n \) and \( h(x) - h(y) = cR(x, y) \), where \( c \) is a non-zero complex number. It remains to show that \( h \) is a greatest common right factor of \( p \) and \( q \). Let \( k \) be any common right factor of \( p \) and \( q \). Then \( k(x) - k(y) \) divides \( R(x, y) = (1/c)(h(x) - h(y)) \).

This is equivalent to \( G_k \subseteq G_h \). By Lemma 1, we have \( k \leq h \). Since \( k \) is a polynomial, \( k \) is actually a right factor of \( h \).

3. Criteria for the existence of a non-linear common right factor

In many situations, Lemma 2 is not so easy to use because of the difficulties in finding the sequences required in the lemma. In this section, we deduce the following
two useful criteria for the existence of a non-linear common right factor of two entire functions.

**Theorem 3.** Let \( f \) and \( g \) be two holomorphic functions defined on \( X \). Suppose that there exists a non-constant entire function of two complex variables \( \Phi(x, y) \) such that \( \Phi(f(z), g(z)) \equiv 0 \) on \( X \). Suppose further that there exist \( n \geq 2 \) distinct points \( z_1, \ldots, z_n \) such that \( \Phi_{i}(f(z_i), g(z_i)) \neq 0 \), \( f'(z_i) \neq 0 \) for all \( i \), and

\[
\begin{align*}
  f(z_1) &= f(z_2) = \ldots = f(z_n) \\
  g(z_1) &= g(z_2) = \ldots = g(z_n).
\end{align*}
\]

Then there exists a holomorphic function \( h : X \to \mathbb{C} \) (which is independent of \( k \) and \( z_i \)) with \( h(x) = 0 \), \( h(z_i) = h_{2i-1}(z_i) \) for all \( 2 \leq i \leq n \).

**Proof.** For each \( 2 \leq i \leq n \), define \( a_i(s, t) = f(z_i + t) - f(z_i + s) \). Then \( a_i(0, 0) = 0 \) and

\[
\frac{\partial a_i}{\partial t}(0, 0) = f'(z_i) \neq 0.
\]

According to the implicit function theorem, there exists a unique analytic function \( \phi_i \) defined in a neighborhood \( A_i \) of \( s = 0 \) such that \( \phi_i(0) = 0 \) and \( a_i(s, \phi_i(s)) = 0 \) on \( A_i \), that is,

\[
f(z_1 + s) = f(z_1 + \phi_i(s)). \quad (2)
\]

Since \( \Phi(f(z_1), g(z_1)) = 0 \) and \( \Phi_{i}(f(z_1), g(z_1)) \neq 0 \), by the implicit function theorem again, there exists a unique analytic function \( k : W \to k(W) \) such that \( \Phi(w, k(w)) \equiv 0 \) on an open neighborhood \( W \) of \( f(z_1) \). Note that \( f(z_1) \in W \), \( g(z_1) = g(z_1) \in k(W) \). As \( 0 \in A_i \), \( \phi_i(0) = 0 \), \( f \) and \( \phi_i \) are continuous, we can choose \( A_i \) small enough such that

\[
f(z_1 + s) = f(z_1 + \phi_i(s)) \in W \quad g(z_1 + s), g(z_1 + \phi_i(s)) \in k(W).
\]

On the other hand, we always have

\[
\Phi(f(z_1 + s), g(z_1 + s)) = 0, \quad \Phi(f(z_1 + \phi_i(s)), g(z_1 + \phi_i(s))) = 0.
\]

Since for each \( 2 \leq i \leq n \), \( f(z_1 + s) = f(z_1 + \phi_i(s)) \) on \( E = \bigcap_{i=2}^{n} A_i \neq \phi \) of \( s \), it follows from the uniqueness part of the implicit function theorem that \( g(z_1 + s) = g(z_1 + \phi_i(s)) \) on \( E \) for all \( 2 \leq i \leq n \). Take a sequence \( \{s_i\}_{i \in \mathbb{N}} \) of distinct complex numbers such that \( \lim_{i \to \infty} s_i = 0 \). It is clear that we get the required sequences in Lemma 2, and the result follows.

We shall prove the following result, which is slightly more general than Theorem 1. It will be used to prove Theorem A in Section 4.

**Theorem 4.** Let \( n \geq 1 \) and \( \Phi(x, y) = \sum_{i=0}^{n} a_i(x)y^i \) be a polynomial in \( y \) with entire functions \( a_i(z) \) coefficients such that \( a_n \neq 0 \). Suppose that \( f, g : X \to \mathbb{C} \) are holomorphic functions defined on \( X \) such that

\[
\Phi(f(z), g(z)) = \sum_{i=0}^{n} a_i(f(z))g(z)^i \equiv 0
\]

on \( X \). If both \( f \) and \( g \) are transcendental (that is, they have an essential singularity
at infinity), then there exists a holomorphic function \( h \) defined on \( X \) such that \( h \leq f \), \( h \leq g \) and \( h^{-1}(s) \) is infinite for some complex number \( s \).

**Remark 1.** Let \( f(z) = z^2 \), \( g(z) = z e^{z^2} \) and \( \Phi(x, y) = x e^{2x} - y^2 \). Then \( \Phi(f(z), g(z)) \equiv 0 \) on \( \mathbb{C} \) because \( z e^{z^2} \circ f = z e^{z^2} \circ z^2 = z^2 e^{z^2} = z^2 \circ (z e^{z^2}) = z^2 \circ g \). Note that there exists no transcendental entire \( h \) with \( h \leq f \) and \( h \leq g \). Therefore, the condition that \( f \) and \( g \) are both transcendental is needed.

**Remark 2.** Let \( f(z) = e^z + z \), \( g(z) = e^z \) and \( \Phi(x, y) = e^z - ye^z \). Then \( \Phi(f(z), g(z)) \equiv 0 \) because \( e^z \circ f = e^z \circ (e^z + z) = e^z e^z = e^{2z} \circ (e^z) = e^{2z} \circ g \). Note that there exists no transcendental entire function \( h \) with \( h \leq f \) and \( h \leq g \), because \( f \) is prime (for the definition, see Section 4). Therefore, the condition that \( \Phi(x, y) \) is a polynomial in \( y \) is also needed.

**Proof of Theorem 4.** Define

\[
E = \{ f(z) \mid \Phi_3(f(z), g(z)) = 0 \text{ and } \Phi(f(z), g(z)) = 0 \} \cup \{ f(z) \mid f'(z) = 0 \}.
\]

Then \( E \) is a countable set. Since \( f \) is transcendental, it follows from the little Picard theorem that we can choose \( A \in \mathbb{C} - E \) so that the equation \( f(z) = A \) has infinitely many distinct roots \( \{ z_n \}_{n \in \mathbb{N}} \). Hence \( \Phi(A, g(z_n)) = \Phi(f(z_n), g(z_n)) = 0 \) for all \( n \). Thus the \( g(z_n) \) are roots of the equation \( \Phi(A, y) = 0 \), which has only finitely many roots. Hence there exists an infinite subsequence of \( \{ z_n \}_{n \in \mathbb{N}} \) (which we denote by the same \( \{ z_n \}_{n \in \mathbb{N}} \) such that \( g(z_1) = g(z_2) = \ldots = g(z_n) \). Note that \( f(z_1) = f(z_2) = A \) for all \( n \) and \( \Phi(f(z_n), g(z_n)) \neq 0 \), \( f'(z_n) \neq 0 \). By Theorem 3, there exists a holomorphic function \( h \) with \( h \leq f \), \( h \leq g \) and \( h(z_1) = h(z_2) = \ldots = h(z_n) \) for all \( n \in \mathbb{N} \). As all \( z_n \) are distinct, \( h^{-1}(h(z_1)) \) is infinite.

**4. Some applications**

In this section, we show how Theorems 3 and 4 can be used to solve some factorization problems of entire functions. Let us recall some basic definitions. As an analogue to prime numbers, we define a non-linear entire function \( F \) to be prime if \( F \) cannot be expressed as a composition of two non-linear entire functions. Examples of prime entire functions are polynomials of prime degrees, \( e^z + z \), \( ze^z \), \( \sin ze^z \), and so on (see [2] for more examples). In fact there are plenty of prime functions, as Noda proved in [10] that for any transcendental entire function \( f \), \( f(z) + az \) is prime for all \( a \in \mathbb{C} - E_f \), where \( E_f \) is some countable set. To prove \( F \) to be prime, we need to prove that it is pseudo-prime first, that is, \( F \) cannot be expressed as a composition of two transcendental entire functions. In [13], Steinmetz proved a very useful criterion for an entire function to be pseudo-prime. It says that any entire solution of a linear complex differential equation with polynomial coefficients is pseudo-prime. It follows that \( e^z \) is pseudo-prime. Given two prime entire functions \( f \) and \( g \), it is natural to ask whether \( f \circ g \) is uniquely factorizable, which means that, if we express \( f \circ g = f_1 \circ g_1 \) for non-linear entire functions \( f_1 \) and \( g_1 \), then \( f = f_1 \circ L \) and \( g = L^{-1} \circ g_1 \) for some linear \( L \). Note that \( f \circ g \) is not always uniquely factorizable. For example, \( ze^{z^2} \circ z^2 = z^2 \circ (ze^{z^2}) \), where all the factors involved are prime. In general, even when \( f \circ g \) is uniquely factorizable, it is usually difficult to prove it. In [8], Kobayashi showed that for the prime functions \( e^z + z \) and \( ze^z \), their composition \( F(z) = ze^z \circ (e^z + z) \) is uniquely factorizable. He proved this
result by using Nevanlinna theory. Using Theorem 3, we can give a much simpler proof.

Suppose that \( z e^z (e^z + z) = f_1 \circ g_1 \) for some non-linear entire functions \( f_1, g_1 \). Since \( e^z + z \) has infinitely many zeros which are all simple, so does \( F(z) = (e^z + z)e^{e^z + z} \). Therefore, \( f_1 \) must have at least one zero. If \( f_1 \) has only finitely many zeros \( a_1, \ldots, a_n \), then there exists some \( a_j \) such that \( g_1(z) = a_j \) has infinitely many roots, for otherwise \( f_1 \circ g_1 \) only has a finite number of zeros. If \( f_1 \) has an infinite number of zeros \( \{a_i\}_{i \in \mathbb{N}} \), then there exists some \( a_j \) such that \( g_1(z) = a_j \) has at least two distinct roots because \( g_1 \) is non-linear. In any case, we can get some \( a_j \) and two distinct \( z_1, z_2 \) such that \( f_1(a_j) = 0 \) and \( g_1(z_1) = g_1(z_2) = a_j \). Note that \( z_1, z_2 \) are zeros of \( F(z) = (e^z + z)e^{e^z + z} \). Hence \( e^z + z_1 = e^z + z_2 = 0 \). Let \( \Phi(x, y) = xe^x - f_1(y) \); then \( \Phi(e^z + z, g_1(z)) \equiv 0 \). Since \( e^z + z \) and \( F(z) = (e^z + z)e^{e^z + z} \) has simple zeros only, \( e^z + 1 \neq 0 \) and \( \Phi_i(e^z + z_1, g_1(z_1)) = f_1^i(g_1(z_1)) \neq 0 \) for \( i = 1, 2 \). By Theorem 3, there exists a non-linear entire function \( h \) with \( h \leq e^z + z \) and \( h \leq g_1 \). Hence, \( e^z + z = h_1 \circ h \) and \( g_1 = h_2 \circ h \), where \( h_1, h_2 \) are analytic on \( \text{Im}(h) \).

If the image of \( h \) is \( \mathbb{C} - \{a_i\} \), then \( h = a + e^q \) for some entire function \( q \). We can assume that \( a = 0 \), so that \( e^z + z = h_1(e^z) \circ q(z) \). The primeness of \( e^z + z \) forces \( q(z) \) to be linear. Hence \( e^z + z \) is periodic, which is impossible. Therefore, the image of \( h \) must be the whole plane. This implies that both \( h_1, h_2 \) are entire. \( e^z + z \) is prime and \( h \) is non-linear, so \( h_1 \) must be linear. It follows that \( g_1(z) = h_2 \circ h_1^{-1} \circ (e^z + z) \). From \( z e^z (e^z + z) = f_1 \circ g_1 \), we get \( z e^z = f_1 \circ h_2 \circ h_1^{-1}(z) \). The fact that \( z e^z \) is prime and \( f_1 \) is non-linear forces \( L = h_2 \circ h_1^{-1} \) to be linear, and we are done.

Clearly, by using very similar arguments, we can actually prove the following more general result.

**Theorem 5.** Let \( f(z), g(z) \) be two prime entire functions. Suppose that \( f \) is of the form \( z e^z \), where \( z(z) \) is a nonconstant entire function. If \( g(z) \) is non-periodic and has infinitely many zeros, where all but finitely many of them are simple, then \( f(g(z)) \) is uniquely factorizable.

**Remark 3.** When \( z \) is a polynomial or periodic entire function, it is known that \( z e^z \) is prime.

The following result was first proved in [9].

**Theorem 6.** Let \( f \) be a transcendental entire function not of the form \( h \circ q \), where \( h \) is a periodic entire function and \( q \) is a polynomial. If \( f \) is pseudo-prime (that is, \( f \) cannot be expressed as a composition of two transcendental entire functions), then so is \( p \circ f \), where \( p \) is a non-constant polynomial.

**Remark 4.** In [12], the periodic function \( f(z) = \sin z e^{\cos z} \) was showed to be pseudo-prime, while \( w^2 \circ f(z) = ((1 - w^2) e^{2w}) \circ \cos z \) is not. Therefore, the condition that \( f \) is not of the form \( h \circ q \) is needed. Whether a similar result holds for \( f \circ p \) remains open.

**Proof of Theorem 6.** Assume that \( p \circ f \) is not pseudo-prime, that is, \( p \circ f = k \circ g \) for some transcendental entire functions \( k \) and \( g \). Applying Theorem 4 to \( \Phi(x, y) = k(x) - p(y) \), we get a transcendental entire function \( h \) with \( h \leq f \) and \( h \leq g \). Hence
\[ f = h_1 \circ h \text{ and } g = h_2 \circ h, \text{ where } h_1, h_2 \text{ are analytic on the image of } h. \text{ If the image of } h \text{ is } \mathbb{C} - \{a\}, \text{ then } h = a + e^z \text{ for some entire function } q. \text{ We can assume that } a = 0, \text{ so that } f(z) = h_1(e^z) \circ q(z). \text{ The pseudo-prime} \text{ness of } f \text{ forces } q(z) \text{ to be a polynomial. This contradicts the assumption on the form of } f. \text{ Therefore, } \text{Im}(h) = \mathbb{C}, \text{ and both } h_1 \text{ and } h_2 \text{ are entire. Hence } p \circ h_1(z) \equiv k \circ h_2(z) \text{ on } \mathbb{C}. \text{ Since } h \text{ is transcendental, } h_1 \text{ must be a polynomial as } f \text{ is pseudo-prime. It follows that } p \circ h \text{ is a polynomial, which is impossible, as } k \text{ is assumed to be transcendental. Therefore } p \circ f \text{ must be pseudo-prime.} \]

Finally, we proved Theorem A using an alternative method.

**Proof of Theorem A.** By Theorem 4, there exists a transcendental entire function \( h \) such that \( f = f_1 \circ h \) and \( g = g_1 \circ h \), where \( f_1, g_1 \) are analytic on \( X = \text{Im}(h) \). By the little Picard theorem, \( X = \mathbb{C} \) or \( \mathbb{C} - \{a\} \) for some complex number \( a \). Now we have \( \Phi f_1(z), g_1(z) \equiv 0 \) on \( X \). Using the great Picard theorem and the fact that \( \Phi \) is a polynomial in both \( x \) and \( y \), it is not difficult to show that \( f_1 \) and \( g_1 \) must both be transcendental (that is, with an essential singularity at infinity) or both not. Suppose that both \( f_1 \) and \( g_1 \) are transcendental. It follows from Theorem 4 that there exists a transcendental holomorphic function \( h_0 : X \to \mathbb{C} \) such that \( h_0 \leq f_1 \) and \( h_0 \leq g_1 \). This implies that \( h_0 \circ h \leq f \circ g \), and hence \( \text{G}_{h_0 \circ h} = (\text{G}_{h_0} \cap \text{G}_{h}) \cap (\text{G}_{f} \cap \text{G}_{g}) = \text{G}_{h} \). By Lemma 1, there exists a holomorphic function \( h_1 \) such that \( h_1 \circ h_0 \circ h = h \), and hence \( h_1 \circ h_0 \equiv \text{id}_X \) on \( X \). This is impossible as \( h_0 \) is transcendental. Therefore, we must have that both \( f_1 \) and \( g_1 \) are rational and holomorphic on \( X \). \( f_1 \) and \( g_1 \) can have at most one pole as \( X = \mathbb{C} \) or \( \mathbb{C} - \{a\} \).

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