Dynamical Properties of the Three-Dimensional \( XY \) Model at Low Temperatures

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On the basis of the generalized Langevin equations for the spin operators, the dynamical properties of the \( XY \) model at low temperatures are investigated in the spin wave approximation. The transverse as well as longitudinal dampings are calculated, with the use of which the hydrodynamic equations are derived. The frequency spectrum of spin wave thence obtained agrees with the result obtained hydrodynamically by Halperin and Hohenberg. The microscopic expressions for the various parameters appearing in their hydrodynamic equations are also determined.

§ I. Introduction

The \( XY \) model can be utilized as a reasonable model of a certain type of ferromagnets or antiferromagnets and also that of a quantum fluid. It is defined as a system with the Hamiltonian

\[
H = -2J \sum_{(i,j)} (S_i^x S_j^x + S_i^y S_j^y),
\]

where \( S_i^x \) and \( S_i^y \) denote the orthogonal \( x \)- and \( y \)-component of the spin operator \( S_i \) with magnitude one half of the \( i \)-th lattice site \((i=1, \cdots, N)\) and \( J \) does a coupling constant. The summation is taken over all nearest neighbour pairs of the lattice sites, \( N \) in total, and \( J \) will be assumed positive hereafter.

This model has been investigated by Matsubara and Matsuda as a special case of the quantum lattice model which was first introduced by Koide and further studied with the use of spin operators by them. It is believed to exhibit a phase transition at a sufficiently low temperature \( T_c \) in the three-dimensional case, where a spin component in the \( xy \)-plane, e.g.,

\[
S^x = \frac{1}{N} \sum_{i=1}^{N} S_i^x,
\]

is taken as an order parameter.

The expectation value \( \langle S^x \rangle \) of the operator (1.2) has a non-vanishing value below the transition temperature \( T_c \) mentioned above and vanishes above this temperature. The rotational symmetry of the Hamiltonian with respect to the \( z \)-axis is broken by the ordering, where it is to be noted that the order parameter
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operator given by (1·2) is incommutable with the Hamiltonian (1·1) of the system. This is contrastively different from the case of the Heisenberg model, where the order parameter commutes with the Hamiltonian.

Halperin and Hohenberg developed a hydrodynamic theory of the XY model which is quite analogous to the two-fluid hydrodynamics of liquid helium. They found a hydrodynamic low-frequency spin wave mode and calculated its damping rate below $T_c$. It is noticeable that the XY model as a simplified model of superfluid is amenable to a microscopic analysis at least at sufficiently low temperatures.

The aim of this paper is to study the dynamical behaviour of the XY model at low temperatures on the basis of Mori's theory of generalized Langevin equation, where the spin wave approximation is applied. Then, we calculate the transport coefficients which enter the hydrodynamic equations and examine the validity of hydrodynamic description.

In § 2, we give the Langevin equations of motion for macrovariables. In § 3, we apply the spin wave theory by making use of Holstein and Primakoff's transformation. Thus in §§ 4 and 5, the susceptibilities and damping constants are calculated. In § 6, the transport coefficient and thermodynamic parameters appearing in the hydrodynamic equations derived by Halperin and Hohenberg are microscopically determined by comparing our results with those equations. In conclusion in § 7, notes on the relaxations of the respective spin components in the hydrodynamical limit are given and the possibility of obtaining the first sound mode which survives even at the transition temperature is pointed out.

§ 2. Generalized Langevin equations of motion

According to Mori, we can write down for a set of any dynamical variables $\{A^a_q\}$ ($\alpha$ and $q$ denote the species of dynamical variable and the wave number vector, respectively) the generalized Langevin equation of motion,

$$\frac{d}{dt} A^a_q(t) = \sum_{(k,r)} \sum_{(p,0)} \left[ A^p_k(t) \left( A^a_q, A^a_{k'} \right)^{-1} (A^a_{k'}, \dot{A}^a_q) \ight]$$

$$- \int_0^t d\tau A^p_k(t-\tau) \left( A^a_q, A^a_{k'} \right)^{-1} (F^a_q(t), F^a_q(\tau)) \right] + F^a_q(t), \quad (2·1)$$

where $F^a_q$ represents the random force given by

$$F^a_q(t) = \exp[(1-L) i \mathcal{L} t] (1-L) \dot{A}^a_q. \quad (2·2)$$

The projection operator $\mathcal{P}$ in Eq. (2·2) is defined for any operator $G$ by

$$\mathcal{P} G = \sum_{(k,r)} \sum_{(q,\alpha)} A^a_k (A^a_{k'}, A^a_q)^{-1} (A^a_q, G). \quad (2·3)$$

The canonical product $(A, B)$ in Eq. (2·1) is defined by
\[ (A, B) = \frac{1}{\beta} \int d\xi \langle [A e^{-iH Be^{iH}}] - \langle A \rangle \langle B \rangle \rangle, \quad (2.4) \]

where \( \langle \cdots \rangle \) denotes the equilibrium ensemble average, \( \beta \) the inverse temperature and \( A^\dagger \) the Hermitian conjugate of an operator \( A \).

Henceforth, we concentrate on the dynamics in the ordered phase in which the spontaneous magnetization appears along the \( x \)-axis. We take the Fourier components of the spin and energy densities, \( \{S_q^x, S_q^y, S_q^z, H_q\} \) with small wave vector as the macrovariables \( \{A_q^a\} \), which describe the slow process in the system. Of these extensive variables, \( S_q^z \) and \( H_q \) are conserved quantities and \( S_q^x \) is a symmetry-restoring variable, which plays a vital role no less important than the conserved quantities in dynamics of the system with broken symmetry as was emphasized by Forster.\(^b\) We assume that \( S_q^z \) varies slowly in time as a result of broken symmetry. Provided that the set of macrovariables are taken enough to describe the slow process in the system sufficiently, Eq. (2.1) reduces to the one of Markovian character and can be rewritten as

\[
\frac{d}{dt} A_q^a(t) = \sum_{(k,n)} \sum_{(p,r)} \left[ A_p^a(t) (A_p^{a'}, A_k^{a'})^{-1} (A_k^{a'}, A_q^{a'}) \right. \\
\left. - A_p^{a'}(t) (A_p^{a'}, A_k^{a'})^{-1} \int_{\tau}^{\infty} d\tau (F_k^{a'}, F_q^{a'}(\tau)) \right].
\]

(2.5)

where the random force is averaged out in a time scale during which the fluctuation of random force exhibits so many periods of recursion and the macrovariables, on the other hand, cannot change appreciably.

In the ordered state, there are two symmetry operations that keep the interaction Hamiltonian (1.1) as well as the state invariant, as was pointed out by Kawasaki\(^\circ\) viz., the spin rotation by \( \pi \) around the \( x \)-axis and the combination of time reversal and spin rotation by \( \pi \) around the \( y \)-axis. As a consequence of these symmetries, \( S_q^y, H_q \) and the pair \( \{S_q^y, S_q^z\} \) do not couple dynamically with one another. Therefore we can deduce that each of these three groups forms a closed set of macrovariables. Consequently, for the longitudinal spin component, we obtain from Eq. (2.5)

\[
\frac{d}{dt} S_q^x(t) = -\sum_p S_p^x(t) \Gamma_{p,q}^x,
\]

(2.6)

and for the transverse one

\[
\frac{d}{dt} S_q^y(t) = \langle S^y_p \rangle S_q^y(t) - \sum_p S_p^y(t) \Gamma_{p,q}^y,
\]

(2.7)

\[
\frac{d}{dt} S_q^z(t) = -\langle S^y_p \rangle S_q^z(t) - \sum_p S_p^z(t) \Gamma_{p,q}^z,
\]

(2.8)

where the susceptibility \( \chi_{q}^a \) and the damping constant \( \Gamma_{p,q}^a \) are defined respec-
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3. Spin wave theory

In the presence of an external magnetic field $H_\text{ext}$ which is directed along the $x$-axis, the total Hamiltonian is given by

$$H = -2J \sum_{\langle \alpha, \beta \rangle} (S_\alpha^x S_\beta^x + S_\alpha^y S_\beta^y) - g \mu_\text{B} H_\text{ext} \sum_j S_j^z,$$  \hfill (3.1)

where $g$ denotes the Landé factor and $\mu_\text{B}$ the Bohr magneton.

Our problem is to find the eigenvalues of the Hamiltonian (3.1) for the case where the magnetic moment is nearly saturated, which is realized in the limit of low temperature. In the vicinity of the quasi-ground state of the XY model in which all the magnetic moments line up along the $x$-axis, we make use of the Holstein-Primakoff\textsuperscript{10} theory of spin waves. According to that theory, the spin operators are expressed as

$$S_j^+ = S_j^x + i S_j^y = \sqrt{2S} f_j(S) a_j,$$

$$S_j^- = S_j^x - i S_j^y = \sqrt{2S} a_j^* f_j(S),$$

$$S_j^z = S_j^z = S - n_j,$$  \hfill (3.2)

where the coordinate frame $(x, y, z)$ has been transformed into a new one $(\xi, \eta, \zeta)$, in such a way as shown in Fig. 1. The function $f_j(S)$ in Eq. (3.2), which is defined by

$$f_j(S) = \sqrt{1 - \frac{1}{2S} n_j} = 1 - \frac{1}{4S} n_j - \frac{1}{32S} n_j^2 - \cdots,$$  \hfill (3.3)

![Fig. 1. The coordinate transformation from the $(x, y, z)$ to $(\xi, \eta, \zeta)$ frame.](https://academic.oup.com/ptp/article-abstract/64/2/448/1869527)
is expanded into inverse power of $S$, where $n_j = a_j^* a_j$ is the spin deviation operator, $a_j^*$ and $a_j$ are the creation and annihilation operators of the spin deviation and they satisfy the commutation relation

$$[a_i, a_j^*] = \delta_{ij}.$$  

By making use of the operators $a_i^*$ and $a_j$, we obtain the following truncated form of the Hamiltonian (3.1):

$$H_{tr} = -\frac{J}{2} \sum_{\alpha j} [4(S^2 - 2S a_j^* a_j) - 2S(a_i^* - a_j^*) (a_j - a_j^*)]$$

$$- N g \mu_0 S H_m + g \mu_0 H_m \sum_j a_j^* a_j,$$  

where we have only retained the zeroth order term in the expansion (3.3) and discarded the fourth and higher order terms with operators $a_j^*$ or $a_j$.

By introducing the Fourier transforms of $a_j$ and $a_j^*$ by

$$a_k = \frac{1}{\sqrt{N}} \sum_j e^{-ik \cdot r_j} a_j, \quad a_k^* = \frac{1}{\sqrt{N}} \sum_j e^{ik \cdot r_j} a_j^*,$$

the Hamiltonian (3.5) for the case $S=1/2$ can be rewritten down as

$$H_{tr} = -\frac{1}{4} NJz - \frac{1}{2} N g \mu_0 H_m$$

$$+ \frac{1}{2} \sum_k \left[ \frac{1}{2} Jz (2 - r_k) + g \mu_0 H_m \right] (a_k^* a_k + a_k a_k^*)$$

$$+ \frac{1}{2} \sum_k \frac{1}{2} J z r_k (a_k^* a_k + a_k a_k^*),$$

where we have defined

$$r_k = \frac{1}{z} \sum_{\rho} e^{ik \cdot \rho}$$

with the vector $\rho$ to the nearest neighbour sites, and $z$ the total number of those sites or the coordination number of the lattice. Henceforth we assume the inversion symmetry of the lattice and therefore $r_k = -r_{-k}$.

The truncated Hamiltonian (3.7) is diagonalized by making use of a canonical transformation

$$a_k = U_k B_k - V_k B_k^*,$$

$$a_k^* = U_k B_k^* - V_k B_k,$$  

where $U_k$, $V_k$ are real numbers satisfying the $U_k^z - V_k^z = 1$. The insertion of (3.9) into (3.7) leads to
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\( H_{tr} = C + H_s, \)  \hspace{1cm} (3.10)

\[
C = -\frac{1}{4} NJ_z - \frac{1}{2} N g \mu_B H_m - \frac{1}{4} \sum_k \left[ J_z (2 - \gamma_k) + 2 g \mu_B H_m - 2 E_k \right], \hspace{1cm} (3.11)
\]

\( H_s = \sum_k E_k B_k^* B_k, \)  \hspace{1cm} (3.12)

\[
E_k = \frac{1}{2} \sqrt{J_z (2 - \gamma_k) + 2 g \mu_B H_m} \left[ 1 - (J_z \gamma_k) \right], \hspace{1cm} (3.13)
\]

where \( U_k \) and \( V_k \) have been chosen as

\[
U_k = \frac{1}{2} \left[ 1 + \frac{J_z (2 - \gamma_k) + 2 g \mu_B H_m}{2 E_k} \right],
\]

\[
V_k = \frac{1}{2} \left[ -1 + \frac{1}{2 E_k} \left( J_z (2 - \gamma_k) + 2 g \mu_B H_m \right) \right]. \hspace{1cm} (3.14)
\]

By assuming \( \alpha \ll 1 \) for the lattice constant \( a \), we can approximate

\[
1 - \gamma_k \approx \alpha^2 k^2,
\]

where \( \alpha \) denotes a constant length a little smaller than \( a \). In the case \( H_m = 0 \), we thus obtain from (3.13)

\[
E_k = J_z \alpha k. \hspace{1cm} (3.16)
\]

In the case of a finite field in which \( \alpha \ll g \mu_B H_m / J_z \), the energy spectrum (3.13) is approximated as

\[
E_k \approx \sqrt{1 + \frac{J_z}{g \mu_B H_m} \left( g \mu_B H_m + \frac{1}{2} J_z \alpha^2 k^2 \right)}. \hspace{1cm} (3.17)
\]

Therefore it turns out that the uniform mode in the XY model has no energy gap in the absence of external field. This fact is quite understandable as no preferable direction exists in the \( xy \)-plane in this system.

The ground state energy in the absence of field is obtained from (3.10) with (3.11) and (3.12) as

\[
E_0 = -\frac{1}{4} NJ_z - \frac{1}{2} J_z \sum_k (1 - \sqrt{1 - \gamma_k}). \hspace{1cm} (3.18)
\]

The ground state spin deviation can be calculated as

\[
\Delta \langle S^z \rangle = \langle a_i^* a_i \rangle = \frac{1}{N} \sum_k V_k^2. \hspace{1cm} (3.19)
\]

By carrying out the integrations in Eqs. (3.18) and (3.19) numerically over the Brillouin zone of the simple cubic lattice, we obtain the values \( E_0 / J_z = 1.58 \),
Recently Suzuki and Miyashita\(^8\) and Oitmaa and Betts\(^9\) have investigated the ground state of the \(xy\)-system by making use of a variational method, and of a numerical approach, respectively. The value of ground state energy given above is in good agreement with the estimates by these authors.

§ 4. Calculation of the susceptibilities

The susceptibilities can be calculated in the spin wave approximation. In the first place, by making use of Eqs. (3·2) and (3·3), the spin operators are expressed in terms of the operators \(a_j^*\) and \(a_j\) as

\[
S_q^z = \frac{\sqrt{2}S}{2} \left[ (a_{-q}^* + a_{q}^*) - \frac{1}{4NS} \sum_k \sum_p (a_{q+k-p}^* a_{q+p} a_{q+k}^* a_{q+p}^*) \right],
\]

\[
S_q^x = \frac{\sqrt{2}S}{2i} \left[ (a_{-q}^* - a_{q}^*) - \frac{1}{4NS} \sum_k \sum_p (a_{q+k-p}^* a_{q+p} a_{q+k}^* a_{q+p}^*) \right],
\]

\[
S_q^y = \sqrt{NS} \delta_{q,0} - \frac{1}{\sqrt{N}} \sum_k a_k^* a_{-k},
\]

where only up to first order terms have been retained in the inverse power expansion of \(S\) in (3·3). We restrict ourselves hereafter to the case, \(S=1/2\).

By making use of the definition (2·4), the expression (2·9) gives the explicit formula

\[
z_q^z = \int_0^\beta d\lambda [\langle S_q^z e^{-\beta H} S_q^z e^{i\lambda H} \rangle - \langle S_q^z \rangle \langle S_q^z \rangle].
\]

Substituting (4·3) into (4·4), we get the approximate expression for \(q \neq 0\):

\[
z_q^z = \frac{1}{N} \sum_k \sum_p \int_0^\beta d\lambda \left[ \langle a_{p}^* a_{p+q} e^{-\beta H} a_{k}^* a_{k-q} e^{i\lambda H} \rangle - \langle a_{p}^* a_{p+q} \rangle \langle a_{k}^* a_{k-q} \rangle \right],
\]

where \(\langle \cdot \rangle_0\) denotes an average taken over the canonical ensemble of systems with the Hamiltonian (3·12). By applying the canonical transformation (3·9) to Eq. (4·5) and noting that

\[
n_q = \langle B_{q}^* B_q \rangle_0 = -\frac{1}{\exp(\beta E_q) - 1},
\]

we obtain

\[
z_q^z = \frac{1}{N} \sum_p \langle U_p U_{p+q} + V_p V_{p+q} \rangle \exp(\beta E_{p+q}) - \exp(\beta E_p) E_{p+q} - E_p n_{p+q} - n_p^2 p+q
\]

\[
+ \frac{1}{N} \sum_p \langle U_p V_{p+q} + U_{p+q} V_p \rangle \exp(\beta (E_{p+q} + E_p) - E_{p+q} E_p - n_{p+q}^2) - n_{p+q}^2.\]

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In the case $H_0 = 0$, Eq. (4·7) can be approximated for $aq \ll k_B T / J z \ll 1$ as

$$
\chi_q \sim \frac{1}{(4\pi)^2 \rho J z \alpha^4} \left( \frac{k_B T}{J z} \right) \frac{1}{q} \int_0^\infty dx \frac{1}{x} \ln \left( \frac{1 + x}{1 - x} \right)^2 = \frac{k_B T}{16 \rho (J z)^2 \alpha^4} \frac{1}{q}, \tag{4·8}
$$

which is singular at the vanishing wave number.

In the case of finite field, Eq. (4·7) can be approximated for $aq \ll g \mu_B H_0 / J z \ll k_B T / J z \ll 1$ as

$$
\chi_q = \frac{1}{\pi \rho J z \alpha^4} \left( \frac{k_B T}{J z} \right) \sqrt{J z} \frac{1}{2 g \mu_B H_0} \int_0^\infty dx \frac{x^3}{(1 + x)} \left( \frac{1}{x} \right)^{1/4} \sqrt{H_0}, \tag{4·9}
$$

which is also singular at the limit of vanishing field.

It is noted that our results are very similar to the expressions for longitudinal susceptibility of the isotropic Heisenberg ferromagnet derived by Mori and Kawasaki. On the basis of phenomenological investigation, Patashinskii and Pokrovskii showed that such an unusually singular behaviour of the longitudinal susceptibility is found generally in the system with broken symmetry. Therefore the similarity of our results to Mori and Kawasaki’s is not accidental and both of the results are consistent with the conclusion of Patashinskii and Pokrovskii’s investigation. This also shows that it is adequate to take $S_q^x$ as a macrovariable.

In a similar way, $\chi_q^y$ is explicitly given by

$$
\chi_q^y = \int_0^\infty dx \left[ \langle S_{-q}^y e^{-iH^*} S_q^y e^{iH} \rangle - \langle S_{-q}^y \rangle \langle S_{q}^y \rangle \right]. \tag{4·10}
$$

Substituting (4·2) into (4·10), and using the canonical transformation (3·9), we obtain

$$
\chi_q^y = \frac{(U_q + V_q)^2}{2E_q} \left[ 1 - \frac{1}{N} \sum_p (2V_p + U_p V_p) - \frac{2}{N} \sum_p (U_p + V_p^2 + U_p V_p) n_p \right], \tag{4·11}
$$

where we have retained up to the fourth order correlation among the operators $a_{q^*}^y$ and $a_q^y$.

In the absence of the field, Eq. (4·11) can be approximated for $k_B T \ll J z$ in the limit $q \to 0$ as

$$
\lim_{q \to 0} \chi_q^y = \frac{1}{2 J z \alpha^4 q^4}. \tag{4·12}
$$

where we have neglected the second term in the square-brackets on the right-hand side of (4·11), which represents the effect of zero-point motion and is very small compared to the first term; that is, the ratio of the second term to the first is...
Our result is consistent with Forster's on the basis of the Bogoliubov inequality. He showed that the static correlation function of the symmetry-restoring variables in the system with broken symmetry has such a singularity as $1/q^2$ in the vanishing limit of $q$. Such a dependence of $\chi_q$ is related to the existence of a hydrodynamic spin wave and the $q$ dependence of its damping as will be shown in §§5 and 6.

In the presence of a finite field and in the vanishing limit of wave number, we can obtain for the case $k_B T \ll J_z$ and $k_B T \ll g \mu_B H_m$

$$\lim_{q \to 0} \chi_q = \frac{1}{2g \mu_B H_m}.$$  \hfill (4·13)

In the same way as in $\chi_q^s$ and $\chi_q^l$, we can similarly calculate $\chi_{q,l}^l$ in the limit of vanishing wave number and get the following results. In the case of no magnetic field in which $k_B T \ll J_z$

$$\lim_{q \to 0} \chi_q^l = \frac{1}{2J_z}.$$  \hfill (4·14)

and in the case of finite field in which $k_B T \ll J_z$ and $k_B T \ll g \mu_B H_m$

$$\lim_{q \to 0} \chi_q^l = \frac{1}{2(J_z + g \mu_B H_m)}.$$  \hfill (4·15)

§ 5. Calculation of the damping constants

By applying the spin wave approximation to the general expression (2·10) for the damping constant, we can estimate the damping due to the exchange interaction. As it is quite difficult to take into account the effect of the projection operator $1-P$ in the calculation of $\Gamma_{p,q}$, we neglect this projection operator and thus get

$$\Gamma_{p,q} \simeq \frac{1}{(S_p^s, S_p^s)} \lim_{\epsilon \to 0} \int_t^\infty dt \langle S_p^s, S_p^s(t) \rangle e^{-\epsilon t},$$  \hfill (5·1)

where $\epsilon$ is an infinitesimal and it is made to tend to zero after the limiting of $q$. It should be noted that the time correlation in the integrand of Eq. (5·1) includes a slow process with long time tail. In fact, the damping constant is only contributed from the rapid process causing a short time correlation, owing to the projection operator $1-P$ involved in the original form. So far as the macrovariables is properly chosen, it is considered that two kinds of correlation times which are quite distinct from each other appear in time correlation in Eq. (5·1) in the limit of $q$ vanishing: One is due to the rapid process with a sharp peak at $t=0$, and the other to the slow process with an infinitely long time tail. However, the convergence factor $\exp(-\epsilon t)$ in (5·1) prevents the slow process from
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... giving a misleading contribution to the time integral. In consequence, it is expected that the expression (5·1) reproduces essentially the same correct result as the full expression (2·10).

In the absence of external field, Heisenberg’s equations of motion for \( S_q^x \), \( S_q^y \) and \( S_q^z \) are respectively expressed as

\[
\dot{S}_q^x(t) = \frac{1}{i\hbar} [S_q^x(t), H] = -\frac{J_z}{\hbar \sqrt{N}} \sum_p \{ S_{q+p}^x(t) S_{q-p}^y(t) + S_{q-p}^x(t) S_{q+p}^y(t) \},
\]

(5·2)

\[
\dot{S}_q^y(t) = \frac{1}{i\hbar} [S_q^y(t), H] = \frac{J_z}{\hbar \sqrt{N}} \sum_p \{ S_{q+p}^y(t) S_{q-p}^x(t) + S_{q-p}^y(t) S_{q+p}^x(t) \},
\]

(5·3)

\[
\dot{S}_q^z(t) = \frac{1}{i\hbar} [S_q^z(t), H] = -\frac{J_z}{\hbar \sqrt{N}} \sum_p \{ S_{q+p}^z(t) S_{q-p}^x(t) - S_{q-p}^z(t) S_{q+p}^x(t) \}. \tag{5·4}
\]

To calculate \( I_{p,q}^z \), we make use of (5·2) to get

\[
\int_0^\infty dt \langle \dot{S}_p^x, \dot{S}_q^y \rangle e^{-it} = \frac{(J_z)^2}{\hbar \sqrt{N}} \sum_k \sum_{k'} \sum_{\tau \tau'} \tau \tau' \int_0^\infty dt e^{-it} \times \left[ (S_{q+k}^x S_{-k}^x S_{q+k'}^y(t) S_{-k}^y(t)) + (S_{q+k}^x S_{-k}^x S_{q+k'}^y(t) S_{-k}^x(t)) + (S_{q+k}^x S_{-k}^x S_{q+k'}^y(t) S_{-k}^x(t)) + (S_{q+k}^x S_{-k}^x S_{q+k'}^y(t) S_{-k}^x(t)) \right]
\]

\[
= I_x^x + I_x^y + I_y^x + I_y^y, \tag{5·5}
\]

where \( I_x^x \) is given by taking \( \hbar \) as unity by

\[
I_x^x = \frac{(J_z)^2}{\hbar \sqrt{N}} \sum_k \sum_{k'} \sum_{\tau \tau'} \tau \tau' \int_0^\infty dt \int_0^\beta d\lambda e^{-it} \left[ \langle S_{q-k}^x S_{-k}^x e^{iH(t+i\lambda)} \right. \times \left. S_{q+k-k'}^x S_{-k-k'}^x e^{-iH(t+i\lambda)} \rangle - \langle S_{q+k-k'}^x S_{-k}^y e^{iH(t+i\lambda)} \rangle \langle S_{q+k-k'}^x S_{-k}^y e^{-iH(t+i\lambda)} \rangle \right]. \tag{5·6}
\]

Substituting (4·1) and (4·2) into (5·6), and retaining up to the fourth order correlations with the operators \( a_q \) or \( a_q^* \), \( I_x^x \) is rewritten as

\[
I_x^x = \frac{(J_z)^2}{16N^2} \sum_k \sum_{k'} \sum_{\tau \tau'} \tau \tau' \int_0^\beta dt \int_0^\beta d\lambda e^{-it} \left[ \langle (a_{-k} - a_{-k}^*) (a_{k+p} + a_{-k-p}^*) \right. \times \left. e^{iH(t+i\lambda)} (a_{-q-k} + a_{q+k}^*) (a_{k'} - a_{-k'}^*) e^{-iH(t+i\lambda)} \rangle \right.
\]

\[
- \langle (a_{-k} - a_{-k}^*) (a_{k+p} + a_{-k-p}^*) \rangle \langle (a_{-q-k} + a_{q+k}^*) (a_{k'} - a_{-k'}^*) \rangle \rangle. \tag{5·7}
\]

Furthermore, by making use of the canonical transformation (3·9), \( I_x^x \) is reduced to
\[ I^x_i = \frac{\pi (Jz)^2}{8N} \sum_k [\gamma_k^2 (U_k + V_k)^2 (U_{q+k} - V_{q+k})^2 + \gamma_{q+k} \gamma_k] n_{q+k} n_k \]
\[ \times \exp \left( \frac{\beta (E_{q+k} + E_k)}{\beta (E_{q+k} + E_k)} - 1 \right) \frac{1}{\beta (E_{q+k} + E_k)^2 + \epsilon^2 \delta_{p,q}} \]
\[ + \frac{\pi (Jz)^2}{8N} \sum_k [\gamma_k^2 (U_k + V_k)^2 (U_{q+k} - V_{q+k})^2 - \gamma_{q+k} \gamma_k] n_{q+k} n_k \]
\[ \times \exp \left( \frac{\beta E_{q+k}}{\beta (E_{q+k} - E_k)^2} \right) \frac{1}{\beta (E_{q+k} - E_k)^2 + \epsilon^2 \delta_{p,q}}. \] (5.8)

We retain only the most dominant part in Eq. (5.8) in the limit of \( q \) vanishing, and then make \( \epsilon \) tend to zero. In consequence, the effect of the slow process can be eliminated. By making use of the relations for small \( q \) and \( k \),
\[ \gamma_k^2 (U_k + V_k)^2 (U_{q+k} - V_{q+k})^2 + \gamma_{q+k} \gamma_k \sim 2, \]
\[ \gamma_k^2 (U_k + V_k)^2 (U_{q+k} - V_{q+k})^2 - \gamma_{q+k} \gamma_k \sim \frac{1}{k^2} (q \cdot k), \] (5.9)
\[ E_{q+k} - E_k \sim \frac{Jz}{2k} (q^2 + 2q \cdot k), \]
we can further rewrite \( I^x_i \) as
\[ I^x_i = \frac{\pi (Jz)^2}{4N} \sum_k n_k \delta (2Jz \alpha k) \delta_{p,q} + \frac{\pi (Jz)^2}{8N} \sum_k \frac{(q \cdot k)}{k^2} n_k (1 + n_k) \]
\[ \times \delta \left[ \frac{Jz \alpha q}{2k} (q^2 + 2q \cdot k) \right] \delta_{p,q}, \] (5.10)
which gives
\[ I^x_i = \left( \frac{k_B T}{2\pi \rho \alpha^2} \right)^\frac{1}{2} \int_0^\infty dk \delta (2Jz \alpha k) \delta_{p,q} + \left( \frac{Jz}{2} \right)^2 \int_0^\infty dq \int_0^\pi dk \int_{-1}^1 d \xi k x n_k \left( 1 + n_k \right) \]
\[ \times \delta \left[ Jz \alpha q \left( x + \frac{q}{2k} \right) \right] \delta_{p,q} = \left[ \frac{k_B T}{2\pi \rho \alpha^2} \right]^\frac{1}{2} \frac{k_B T}{2\pi \rho \alpha^2} [q^n + q^n] \delta_{p,q}. \] (5.11)

In the same way, we can calculate the remaining three terms in Eq. (5.5) and obtain for each the result identical with that of \( I^x_i \). In consequence, we can get for \( \Gamma^x_{p,q} \)
\[ \Gamma^x_{p,q} = \frac{1}{2\pi q} \left[ \frac{k_B T}{2\pi \rho \alpha^2} Jz - \frac{1}{16\pi \rho \alpha^2} \right] \delta_{p,q}. \] (5.12)
Combining the previous result for \( \chi^x_q \), we get the following in the case that \( aq \ll k_B T / Jz \ll 1 \):
\[ \Gamma^x_{p,q} = D^x q^2 \delta_{p,q}, \] (5.13)
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\[ D^x = \frac{(Jz\alpha)^2}{2\pi\hbar k_B T}. \]  

(5.14)

By substituting (5.13) into (2.6), the motion of \( S^x_q \) is found to obey the diffusion equation,

\[ \frac{d}{dt} S^x_q (t) = -D^x q^2 S^x_q (t), \]  

(5.15)

so far as the following condition is satisfied:

\[ a q \ll \frac{\hbar k_B T}{Jz}. \]  

(5.16)

This result agrees with the one for the longitudinal spin component in the case of isotropic Heisenberg system. The temperature dependence of our diffusion constant is naturally different from the case of isotropic Heisenberg system in which the diffusion constant is independent of temperature.

We can perform the calculation of \( I_{p,q}^x \) in the same way. Use of Eq. (5.3) leads to

\[ \int_0^\infty dt \langle \hat{S}_p^y, \hat{S}_q^y (t) \rangle e^{-\omega t} = \frac{(Jz)^2}{N} \sum_{k} \sum_{k'} e^{ik\cdot\hat{k}t} \int_0^\infty dt' e^{-\omega t'} \left\{ \langle S_{p+k}\cdot S_{-k}, S_{q+k}\cdot S_{-k'} (t) \rangle \right\} 
\]

+ \( \langle S_{p+k}\cdot S_{-k'}, S_{q+k}\cdot S_{-k'} (t) \rangle \) \( + \langle S_{k}\cdot S_{p-k}, S_{q+k}\cdot S_{-k'} (t) \rangle \) \( + \langle S_{k}\cdot S_{p-k}, S_{q+k}\cdot S_{-k} (t) \rangle \) \( = I_{n}^x + i_{n}^x + I_{n}^y + I_{n}^z \),

(5.17)

where \( I_{n}^x \) to \( I_{n}^z \) correspond to the first to fourth terms in the integrand, respectively. By making use of (4.1) and (4.3) and retaining the correlation functions of up to sixth order with \( a_{q'p}^* \) or \( a_{q'p} \), we can calculate \( I_{n}^y \) and so forth. The thermal expectation values of the first and the third order correlations vanish and the second and the fourth order correlations also do not contribute to the clamping of \( S^x_q \). The sixth order correlation function \( C_6^x \) is important and is expressed as

\[ C_6^x = \frac{(Jz)^2}{4N^2} \sum_{k} \sum_{k'} \sum_{q} \sum_{q'} \langle \hat{S}_{k}\cdot\hat{S}_{k'} \rangle \int_0^\infty dt' \int_0^\infty dt'' \langle a_{q'}^* a_{q''} (a_{p+k}^* a_{-p-k}) \rangle \times e^{i\omega (t''+t')} (a_{q''} a_{q'+k}). \]  

(5.18)

By applying the canonical transformation (3.9) to Eq. (5.18), we get as the most dominant part in the limit of \( q \) vanishing

\[ C_6^x = -\frac{\pi (Jz)^2}{2N^2} \sum_{k} \sum_{k'} \left\{ 2 \hat{\kappa} \hat{\kappa} \cdot \langle U_{k'}-V_{k'} \rangle \langle U_{k'-k'}+V_{k+k} \rangle \right\} \]

\[ \times \left( \langle U_{k+k'} \rangle \langle U_{k+k'} \rangle - \frac{1}{2} \hat{\kappa}^2 \langle U_{k-k'} \rangle \langle U_{k+k'} \rangle \right). \]
Here we have discarded the summation under which the factor \( \delta(E_{k+k'} + E_k + E_{k'}) \) is included. This summation really can have no contribution. In the case \( k_0 T / Jz \ll 1 \), the contribution from small \( k \) and \( k' \) is important, where

\[
x_k = \frac{1}{\sqrt{\alpha k}}, \quad U_k - V_k \sim \sqrt{\alpha k},
\]

\[
U_{k+k'} U_k + V_{k+k'} V_k \sim \frac{1}{2\alpha \sqrt{k|k+k'|}} (1 + \alpha^2 k|k+k'|),
\]

\[
U_{k+k'} V_k + V_{k+k'} U_k \sim \frac{1}{2\alpha \sqrt{k|k+k'|}} (1 - \alpha^2 k|k+k'|).
\]

Thus, after replacing the summations by the integrations and by making use of the relation \( \gamma_q = \gamma_{-q} \), we obtain

\[
C_q^r = \frac{9(Jz)^2 \alpha^3}{128\pi^3} h^2 \int_0^\infty dk \int_0^\infty dk' (kk')^2 (k + k') \Phi_{n_z+n_{k'} n_{k''}} \int_{-1}^1 dx
\]

\[
\times \delta \left[ Jz \alpha (\sqrt{k^2 + k'^2} + 2kk' x - k - k') \right] \delta_{p,q} = \frac{9JzA}{256\pi^3 \alpha^4 (Jz)^3} \left( \frac{k_0 T}{Jz} \right)^4 \delta_{p,q},
\]

where \( A \) is defined by

\[
A = \int_0^\infty dx dx dy \frac{x^2 y^2 (x+y)^2}{(e^x - 1) (e^y - 1) (e^{x+y} - 1)}.
\]

For each of \( I_p^r \), \( I_q^r \) and \( I_0^r \), we also get the same result. Thus, in result \( I_{p,q}^r \) is obtained as

\[
I_{p,q}^r = D^r q_0^p \delta_{p,q},
\]

\[
D^r = \frac{9A(k_0 T)^4}{32\pi^3 \alpha^4 (Jz)^4},
\]

under the condition

\[
\frac{k_0 T}{Jz} \ll 1 \quad \text{and} \quad q \to 0.
\]

In the next place, by making use of (5.4), we can obtain
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\[ \int_0^\infty dt (\tilde{S}_{p} \cdot \tilde{S}_{q} (t)) e^{-it} = \frac{3}{N} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \int_0^\infty dt e^{-it} \times \left[ \langle \tilde{S}_{p-k} \cdot \tilde{S}_{k} \rangle \langle \tilde{S}_{q-k} \cdot \tilde{S}_{k} \rangle - \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \right] \]

\[ = I_1 + I_2 + I_3 + I_4, \quad (5.26) \]

in which, e.g., \( I_1 \) is given by

\[ I_1 = \frac{1}{N} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \int_0^\infty dt \int_0^\beta d\epsilon e^{-it} \left[ \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle - \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \right]. \]

Similarly to the case of \( T_{x,y} \), the sixth order correlation term \( C_6 \), which only contributes to \( I_1 \), is obtained as

\[ C_6 = \frac{1}{N} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \int_0^\infty dt \int_0^\beta d\epsilon e^{-it} \left[ \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle - \langle \tilde{S}_{p-k} \cdot \tilde{S}_{q-k} \rangle \langle \tilde{S}_{q+k} \cdot \tilde{S}_{p-k} \rangle \right]. \]

By carrying out the canonical transformation (3.9) and retaining the most dominant part in the limit of \( q \) vanishing, we get in the limit \( \epsilon \to 0 \)

\[ C_6 = \pi (Jz)^2 \frac{1}{2N} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \left\{ \frac{1}{4} \langle \mathbf{k} \cdot q \rangle \langle \mathbf{k}' \cdot q \rangle \left[ (U_{k+h} + V_{k}) (U_{k'} + V_{k'}) \right] \sum_{n=0}^{\infty} \delta \left( E_{k+h} - E_{k} - E_{k'} \right) \delta_{p,q} \right\}. \]

By the use of (5.20), (5.29) can be calculated as

\[ I_1 = C_6 = \frac{3}{2N} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \left\{ \frac{1}{4} \langle \mathbf{k} \cdot q \rangle \langle \mathbf{k}' \cdot q \rangle \left[ (U_{k+h} + V_{k}) (U_{k'} + V_{k'}) \right] \sum_{n=0}^{\infty} \delta \left( E_{k+h} - E_{k} - E_{k'} \right) \delta_{p,q} \right\}. \]

Each of the contributions \( I_1 \), \( I_2 \), and \( I_3 \) in Eq. (5.26) also contributes in the
same way as $I_r$. Consequently, $\Gamma_{\rho,q}^q$ is obtained under the condition (5.25) as
\[ \Gamma_{\rho,q}^q = D_{\rho,q}^q \frac{3A}{8\pi^2 \alpha \hbar (J_z)^2}. \] (5.31)
\[ D_{\rho,q}^q = \frac{3A}{8\pi^2 \alpha \hbar (J_z)^2}. \] (5.32)

§ 6. Hydrodynamic equations

In the previous section, we have calculated the damping constants of the longitudinal spin component $S_q^z$ and the transverse ones $S_q^x, S_q^y$ at very low temperatures and have found that the former obeys the diffusion equation, where the diffusion constant is proportional to the inverse temperature. In this section we shall make the hydrodynamic study on the transverse spin components.

By inserting (4.12), (4.14), (5.23) and (5.31) into (2.7) and (2.8), we obtain, under the condition (5.25),
\[ \frac{d}{dt} S_q^z(t) = [2\langle S^x \rangle J_z^z] S_q^x(t) - D_{\rho,q}^q S_q^z(t), \] (6.1)
\[ \frac{d}{dt} S_q^y(t) = -[2\langle S^x \rangle J_z^z] q^x S_q^x(t) - D_{\rho,q}^q S_q^y(t). \] (6.2)

By solving these coupled equations, we get a hydrodynamic spin wave mode with the frequency spectrum
\[ \omega (q) = \pm 2\langle S^x \rangle J_z q^z - \frac{1}{2} \frac{i}{2} (D_{\rho}^q + D_{\rho}^p) q^z, \] (6.3)
where we have neglected the terms of order $q^3$ and higher.

It should be noted that the hydrodynamic spin wave mode differs from the spin wave excitation (3.16) which is derived by diagonalizing the Hamiltonian; the former occurs as a result of the complicated interaction among the excitations. At the vanishing temperature, where $\langle S^x \rangle = 1/2$ is substituted, Eq. (6.3) is reduced to the energy spectrum (3.16), as it should be.

Our result agrees with the frequency spectrum of macroscopic spin wave which was derived by Halperin and Hohenberg on the basis of hydrodynamic consideration. Equations (6.1) and (6.2) are rewritten in the real space in the form
\[ \frac{\partial}{\partial t} \left[ F S^x (r, t) \right] = 2\langle S^x \rangle J_z F S^x (r, t) + D_{\rho} F \left[ F \cdot F S^x (r, t) \right], \] (6.4)
\[ \frac{\partial}{\partial t} S^y (r, t) = -2\langle S^x \rangle J_z q^x F \cdot F S^x (r, t) + D_{\rho}^p S^y (r, t), \] (6.5)
where the spin density $S^x (r, t)$ is defined by
We introduce $\phi(r, t)$ as an angle of the spin fluctuation occurring in the $xy$-plane with the $x$-axis. By assuming $\phi(r, t)$ to be small, $S^x(r, t)$ can be approximated as

$$S^x(r, t) \approx \langle S^x \rangle \phi(r, t).$$

Furthermore, by introducing $v(r, t) = \dot{\phi}(r, t)$, $m_z(r, t) = S^z(r, t)$, Eqs. (6·4) and (6·5) can be rewritten down as

$$\frac{\partial}{\partial t} v(r, t) = 2Jz m_z(r, t) + D \mathbf{F} \cdot [\mathbf{F} \cdot v(r, t)],$$

$$\frac{\partial}{\partial t} m_z(r, t) = -2\langle S^x \rangle Jz \alpha F \cdot v(r, t) + Dm_z(r, t).$$

These are equivalent to the hydrodynamic equations derived by Halperin and Hohenberg.3 Thus we can give the microscopic expressions for the thermodynamic constants and the transport-coefficients, which appear in the hydrodynamic equations (2·47b), (2·47c) in their paper; viz.,

$$\rho_s = 2\langle S^x \rangle Jz \alpha^2,$$

$$\chi^{-1} = 2Jz,$$

$$K_n = \frac{3A}{16\pi \rho^2 \alpha^2} \left( \frac{k_B T}{Jz} \right)^\gamma,$$

$$K_m = \frac{9A}{16\pi \rho^2 \alpha^2} \left( \frac{k_B T}{Jz} \right)^\gamma,$$

By applying the spin wave approximation, the temperature dependence of the thermal average $\langle S^x \rangle$ can be easily calculated as

$$\langle S^x \rangle = \langle S^x \rangle_0 - \frac{1}{24\rho^2 \alpha^2} \left( \frac{k_B T}{Jz} \right)^2,$$

where $\langle S^x \rangle_0$ denotes the average of spin magnitude per site at $T=0$, which is regarded to be a little smaller than 1/2 owing to the zero-point motion of the spin.

§ 7. Concluding remarks

We have found that the motions of orthogonal components $S^x_q$, $S^y_q$ and $S^z_q$ of spin in the Fourier decomposition are infinitely slow in the limit of long wave
length $q \rightarrow 0$, which justify the use of the approximation (5.1) in the calculation of damping constants.

By taking into account the large difference between the temperature dependences of the longitudinal and transverse dampings, we can consider that $S_q^z$ always remains at the local equilibrium value at temperatures where the hydrodynamic Goldstone mode is essential in determining the dynamical behaviour of the system. In deriving the hydrodynamic equations, Halperin and Hohenberg assumed a wave number-independent microscopic relaxation of $S_q^z$ to a local equilibrium value. According to the present investigation, this is not the case; that is, we do not have such a relaxation time but get a macroscopic relaxation time dependent on the wave number. The magnitude of that time, however, is much smaller for $S^z$ than for $S^x$ and $S^y$. On account of this, the hydrodynamic motion of $S_q^x$ is not important compared to that of the transverse components $S_q^y$, $S_q^z$ and can be neglected in the same way as done by Halperin and Hohenberg. The reason in our case, however, is not the same as in their case.

It is worth while noting that the motion of $S_q^z$ is a diffusive one in the case $aq \ll k_B T/J \ll 1$, but not in the case $k_B T/J \ll aq \ll 1$, as is shown in the Appendix. We have also derived the hydrodynamic equations for $S_q^y$ and $S_q^x$ under the condition that only (5.25) is satisfied. No other type of motion of transverse spin components obeying a different type of equation of motion under any other condition has been found so far.

It is believed that the hydrodynamic theory for the Quantum Lattice Model is analogous to the one for superfluid helium with damped normal fluid, such as helium in fine pores, for the reason that the law of momentum conservation is regarded as unsatisfied in the lattice system so that only a mode analogous to the fourth sound appears as a hydrodynamic mode. However, it should be pointed out that the lattice structure is averaged out in the hydrodynamic scale of length in which we are interested. In this respect, in the long wavelength limit, there is a possibility of the existence of the normal mode corresponding to the first sound, which does not vanish at the critical temperature $T_c$. In fact, by taking account of nonlinear terms with spin operators, we can find the mode which survives even at $T_c$, which will be discussed in a subsequent paper.

The computation was performed at the Computation Center of Nagoya University.

**Appendix**

—Calculation of $\tau_{\nu, \alpha}^{\nu, \alpha}$ in the Case $k_B T/J \ll aq \ll 1$—

In this case, Eq. (4.7) is approximated as

$$
\tau_{\nu, \alpha}^{\nu, \alpha} \approx \frac{2}{\cal N} \sum_p [(U_{q_p} + V_{q_p})^2 + (U_{p_q} + V_{p_q})^2] \frac{1}{E_{q_p} - \nu_p}
$$
\[ + \frac{2}{N} \sum_{p} (U_{p} V_{p+q} + U_{p+q} V_{p}) \left( \frac{1}{E_{p+q} + E_{p}} \right) , \tag{A.1} \]

where in the first summation, the relation \( q \gg p \) has been assumed by taking account of the factor \( n_{q} \). By performing the first summation, we get \( \langle k_{B} T \rangle \times [12 \pi^{4} \times \rho(Jz)^{2} q^{2}] \). The second summation in \( \langle A.1 \rangle \) is approximated as

\[ \frac{1}{2Jz \pi^{4} N} \sum_{p} \rho |p+q| \left| \frac{1}{p+q} \right| \]

which dominates the first summation in the present case. By making use of these results, we obtain

\[ \langle \zeta_{q} \rangle = \frac{1}{8 \pi^{4} \pi^{4} \pi^{4} Jz} \ln \frac{1}{q} . \tag{A.2} \]

The substitution of \( \langle A.2 \rangle \) into Eq. (5.12) gives

\[ \Gamma_{q}^{z} = \frac{\pi k_{B} T}{\hbar \ln \frac{1}{q}} , \tag{A.3} \]

References

5) D. Forster, Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions (W. A. Benjamin, 1975).
7) T. Holstein and H. Primakoff, Phys. Rev. 58 (1940), 1098.