Another BRS Transformation

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In the covariant canonical formalism of Yang-Mills theory and of the internal-Lorentz part of vierbein formalism, a new invariance is found under “another BRS transformation” satisfying the nilpotency property. In this transformation, the roles of the Faddeev-Popov ghost $c$ in the BRS transformation are played by the Faddeev-Popov anti-ghost $\bar{c}$. Some implications of this new symmetry including the Ward-Takahashi identities following from it are discussed.

§ 1. Introduction

In the local covariant operator formalism of non-Abelian gauge theories, the BRS symmetry has played a principal role by virtue of its nilpotency property. The BRS transformation is, roughly speaking, a local gauge transformation with the gauge function (transformation parameter) replaced by the Faddeev-Popov (FP) ghost field $c(x)$. Here the roles of the FP ghost $c$ and anti-ghost $\bar{c}$ are very asymmetric. In contrast to the FP ghost $c$, the FP anti-ghost $\bar{c}$ seems to play only a secondary role in this formalism: The FP ghost $c$ behaves in the “active voice” in the BRS transformation as the “gauge function” and its behaviour [(2·5a)~(2·5c)] can be understood in an intrinsic manner from the viewpoint of the cohomology of Lie algebra. On the other hand, the FP anti-ghost $\bar{c}$ as well as the auxiliary field $B$ plays a “passive” role as the Lagrange multiplier introduced in the Lagrangian $\mathcal{L}_{\text{GF}}+\mathcal{L}_{\text{FP}}$

$$\mathcal{L}_{\text{GF}}+\mathcal{L}_{\text{FP}} \sim B(\partial^\mu A_\mu) + i\bar{c}(\partial^\mu D_\mu c), \quad (1·1)$$

corresponding, respectively, to the gauge condition

$$\partial^\mu A_\mu = 0 \quad (1·2)$$

and to the condition to preserve the above (1·2) under the BRS transformation

$$\delta(\partial^\mu A_\mu) = \partial^\mu D_\mu c = 0, \quad (1·3)$$

and their intrinsic meanings in the BRS transformation [(2·5d), (2·5e)] are not so clear. Further, as stressed in Ref. 1), the FP anti-ghost $\bar{c}$ is not hermitean conjugate of the FP ghost $c$ and obeys an equation of motion different from that of $c$. Hence, it does not seem to be able to take the place of $c$.

Nevertheless, we can find another symmetry transformation, with the FP anti-
ghost $\varepsilon$ as its “gauge function”, satisfying the nilpotency property similarly to BRS transformation—this is “another BRS transformation”, which will be presented in § 2 for the Yang-Mills theory and for the vierbein formalism of quantum gravity. Its interesting features consist in the similarity of its form to the global gauge transformations. In view of this similarity, the relations among BRS, another BRS and global gauge transformations are clarified in § 3, where some other implications of this symmetry such as a new set of Ward-Takahashi identities derived from it will be discussed.

§ 2. Another BRS transformation

(A) The case of Yang-Mills theory

We first discuss the case of the Yang-Mills theory. With the notational conventions of Ref. 1), the Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_S(A, \varphi) + \mathcal{L}_{GF} + \mathcal{L}_{FP},$$

$$\mathcal{L}_S(A, \varphi) = - F_{\mu \nu} F^{\mu \nu}/4 + \mathcal{L}_{\text{matter}} (\varphi, \mathcal{D} \varphi),$$

$$\mathcal{L}_{GF} = - A_\mu \cdot \partial^\mu B + (\alpha/2) B \cdot B,$$

$$\mathcal{L}_{FP} = - i \partial \varepsilon \cdot D_x \varepsilon.$$

While the original BRS transformation is defined by

$$\delta A = D_x \varepsilon,$$

$$\delta \varphi = i \varepsilon \cdot g T \varphi,$$

$$\delta \varepsilon = - g \varepsilon \times c / 2,$$

$$\delta B = - g \varepsilon \times c / 2,$$

$$\delta B = 0,$$

"another BRS transformation" is given by Eqs. (2·5a) ~ (2·5c) with $c$ replaced by $\varepsilon$ and by modifying (2·5d) and (2·5e) in the following way:

$$\delta' A_\mu = D_x \varepsilon = - g \varepsilon \times A_\mu + \partial_x \varepsilon,$$

$$\delta' \varphi = i \varepsilon \cdot g T \varphi,$$

$$\delta' \varepsilon = - g \varepsilon \times \varepsilon / 2,$$

$$\delta' c = - g \varepsilon \times c - i B,$$

$$\delta' B = - g \varepsilon \times B.$$

Similarly to $\delta$, the definition of $\delta'$ is extended as an anti-derivation on the polynomial algebra $\varepsilon$ generated by field operators:

$$\delta' (XY) = (\delta' X) Y + (-)^{\mu \nu} X \delta' Y.$$
\[ \delta' (\eta Y + \zeta Z) = \gamma \delta' Y + \delta' Z \]  

(2.8)

for \( \eta, \zeta \in \mathcal{C} \), \( Y, Z \in \mathcal{Z} \) and for a monomial \( X \in \mathcal{Z} \) of degree \( \rho_X \) in \( c \) and \( \bar{c} \). It is interesting to note that the form of “another BRS transformation” (2.6) is very akin to that of the (infinitesimal) gauge transformation of the first kind:

\[
\begin{align*}
\delta A &= \delta \epsilon \times A, \\
\delta \varphi &= -i \delta \epsilon \cdot T \varphi, \\
\delta \bar{c} &= \delta \epsilon \times \bar{c}, \\
\delta c &= \delta \epsilon \times c, \\
\delta B &= \delta \epsilon \times B,
\end{align*}
\]

(2.9a - 2.9e)

by the correspondence \( \delta \epsilon \leftrightarrow -g \bar{c} \). In other words, it is *almost* the (infinitesimal) local gauge transformation with an “infinitesimal” gauge function \(-g \bar{c}(x)\) just aside from the term \(-iB\) in (2.6d) and the factor \(1/2\) in (2.6c). This should be compared with the case of BRS transformation (2.5), where the transformation properties (2.5d) and (2.5e) of \( \bar{c} \) and \( B \) have no analogy with (2.9c) and (2.9e) in the global gauge transformation. Thus, we can say that symmetry properties of field operators related to the gauge group are reflected more directly in “another BRS transformation” than in BRS transformation. (This observation will play an important role in the next section.)

In a similar manner to the case of BRS transformation, we can verify the nilpotency property of “another BRS transformation” from the definition (2.6) \( \sim (2.8) \):

\[ \delta' \delta' = 0. \]  

(2.10)

The equalities \( \delta' \delta' A = \delta' \delta' \varphi = \delta' \delta' \bar{c} = 0 \) follow immediately from the corresponding ones for \( \delta \) by interchanging \( c \) with \( \bar{c} \). The equalities \( \delta' \delta' c = \delta' \delta' B = 0 \) can be checked directly by using the Jacobi identity for the structure constant of the gauge group:

\[
\begin{align*}
\delta' \delta' c &= -g \delta' \bar{c} \times c + g \bar{c} \times \delta' c - \delta' B \\
&= -g (-g \bar{c} \times \bar{c}/2) \times c + g \bar{c} \times (-g \bar{c} \times c - iB) + ig \bar{c} \times B \\
&= g \left\{ (\bar{c} \times \bar{c}) / 2 + \bar{c} \times \bar{c} \times c \right\} + ig (-\bar{c} \times B + \bar{c} \times B) \\
&= 0, \quad (2.11a)
\end{align*}
\]

\[
\begin{align*}
\delta' \delta' B &= -g \delta' \bar{c} \times B + g \bar{c} \times \delta' B \\
&= -g (-g \bar{c} \times \bar{c}/2) \times B + g \bar{c} \times (-g \bar{c} \times B) \\
&= g \left\{ (\bar{c} \times \bar{c}) / 2 - \bar{c} \times \bar{c} \times B \right\} = 0. \quad (2.11b)
\end{align*}
\]

In addition to the nilpotency, \( \delta' \) satisfies the interesting anticommutativity with
the BRS-\(\delta\):
\[
\delta'\delta = -\delta\delta'.
\] (2.12)

which can be verified by using (2.5) and (2.6).

Next, we check the invariance of the Lagrangian under \(\delta'\), which trivially holds for the local-gauge-invariant part \(\mathcal{L}_S(A, \varphi)\) \([2.2]\):
\[
\delta'\mathcal{L}_S = 0,
\] (2.13a)

owing to the forms of (2.6a) and (2.6b). The only problem is, therefore, about the remaining part, \(\mathcal{L}_{GF} + \mathcal{L}_{FP}\), which can be written as
\[
\mathcal{L}_{GF} + \mathcal{L}_{FP} = i\delta' (\varphi^\alpha \cdot A^\alpha) + \alpha B \cdot B/2 = i\delta' (\varphi^\alpha \cdot A^\alpha - \alpha \varphi^\alpha \cdot B/2).
\] (2.14)

By using the equalities
\[
\begin{align*}
\varphi^\alpha \cdot A^\alpha &= D^\alpha \varphi \cdot A^\alpha - g A^\alpha \times \varphi \cdot A^\alpha = \delta' (A^\alpha \cdot A^\alpha)/2, \\
B \cdot B &= B \cdot (i\delta' \varphi + ig \varphi \times \varphi) = i\delta' (B \cdot B),
\end{align*}
\] (2.15a \& 15b)

together with (2.12), Eq. (2.14) can be cast into the form
\[
\mathcal{L}_{GF} + \mathcal{L}_{FP} = i\delta' (A^\alpha \cdot A^\alpha)/2 + i\alpha \delta' (B \cdot B)/2 = -i\delta' (\varphi^\alpha \cdot A^\alpha - \alpha \varphi^\alpha \cdot B/2),
\] (2.14\')

which clearly shows the invariance of \(\mathcal{L}_{GF} + \mathcal{L}_{FP}\) by virtue of the nilpotency (2.10):
\[
\delta' (\mathcal{L}_{GF} + \mathcal{L}_{FP}) = 0. \quad (2.13b)
\]

Thus, “another BRS transformation” turns out to be one more symmetry transformation of the system (2.1) with the nilpotency property.

On account of this new symmetry, we have its Noether current:
\[
J^\mu_\nu = D^\varphi \varphi \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} + i(\varphi \cdot gT \varphi) \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} - \frac{1}{2} g \varphi \times \varphi \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} + (\varphi \times \varphi - iB \cdot \varphi) \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} - g \varphi \times B \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} + \delta' J^\varphi \varphi \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} - ig \varphi \cdot \varphi \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)},
\] (2.16)

which can be rewritten as
\[
J^\varphi \varphi = B \cdot D^\varphi \varphi - \partial_\mu B \cdot \varphi + (i/2) g (\varphi \times \varphi) \cdot D_\varphi \varphi - \partial_\mu (\varphi \cdot F_\varphi)
\]
\[
= \delta' J^\varphi \varphi \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)}, \quad (2.17)
\]

by using the equation of motion
\[
D^\varphi F_\varphi = \partial_\mu B + ig (T \varphi) \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)} - ig (\varphi \varphi \cdot \varphi) \cdot \frac{\delta \mathcal{L}}{\delta (\varphi^\mu \varphi^\nu)},
\] (2.18)
Since $J_\alpha^e$ is the conserved current of the FP ghost charge $Q_e$
\[
J_\alpha^e = i (\bar{\epsilon} \cdot D \epsilon - \partial \bar{\epsilon} \cdot c),
\]
\[
Q_e = \int d^2 x J_\alpha^e = i \int d^2 x (\bar{\epsilon} \cdot D \epsilon - \partial \bar{\epsilon} \cdot c),
\]
Eq. (2.17) clearly shows the conservation of $J_\alpha^e$:
\[
\partial^\mu J_\mu^e = 0.
\]
The corresponding conserved charge $Q_e'$ generates "another BRS transformation" (2.6)~(2.8):
\[
Q_{e'} = \int d^2 x J_\alpha^{e'} = \int d^2 x (B \cdot D \epsilon - \partial \bar{\epsilon} + (i/2) \bar{\epsilon} \times \bar{\epsilon} \cdot D \epsilon),
\]
\[
[iQ_{e'}, \phi] = \delta' \phi,
\]
where commutator (anti-commutator) should be taken for $\phi$ with even (odd) powers of FP ghosts.
Corresponding to the nilpotency (2.10) of $\delta'$ and to Eq. (2.17), the charge $Q_{e'}$ is nilpotent:
\[
Q_{e'}^2 = (Q_{e'}, Q_{e'})/2 = -(i/2) \int d^2 x \delta' (B \cdot D \epsilon - \partial \bar{\epsilon} + ig \bar{\epsilon} \times \bar{\epsilon} \cdot D \epsilon/2)
\]
\[
= 0
\]
and satisfies the relation
\[
[iQ_{e'}, Q_e] = \delta' Q_e = Q_{e'}.
\]
Further, we can check the relation
\[
\{Q_{e}, Q_{e'}\} = -iQ_{e'} = -i \int d^2 x (B \cdot D \epsilon - \partial \bar{\epsilon} + ig \bar{\epsilon} \times \bar{\epsilon} \cdot D \epsilon/2)
\]
\[
= 0,
\]
which corresponds to the relation (2.12) for $\delta$ and $\delta'$. Thus, we obtain the following diagram.
It may be instructive to compare the above consequences with those for the BRS symmetry:

\[
Q_B = \int d^2 x \left( B \cdot D \bar{e} - \bar{\theta}_a B \cdot e + i g \bar{\theta}_a e \cdot e / 2 \right)
\]

\[
\leftrightarrow Q_B' = \int d^2 x \left( B \cdot D \bar{e} - \bar{\theta}_a B \cdot e + i g \bar{\theta}_a e \cdot e / 2 \right),
\]

\[
[i Q_B, Q_B] = Q_B \leftrightarrow [i Q_B, Q_B'] = -Q_B'.
\]

\[
Q_B' = Q_B''' = \langle Q_B, Q_B' \rangle = 0.
\]

(B) The internal-Lorentz part of the vierbein formalism

The covariant canonical formulation of quantum gravity in the vierbein formalism is achieved by Nakanishi in a very elegant fashion. In what follows, we adopt his notations with slight modifications:

\[
\mathcal{L}_{\text{int}} = \mathcal{L} + \mathcal{L}_D + \mathcal{L}_{\text{IL}}.
\]

\(\mathcal{L}\) consists of the matter Lagrangian density of tensor fields (with integral spins) coupled general-covariantly to gravity and of the Einstein Lagrangian density of \(g_{m} = h_{s}^a \gamma_m h_{s}\) together with its corresponding gauge-fixing and FP terms; \(\mathcal{L}_D\) is the matter Lagrangian density of spinor fields (with half-integral spins) coupled to gravity via the internal-Lorentz (IL)**-covariant derivatives. \(\mathcal{L}_{\text{IL}} = \mathcal{L}_{\text{ILGF}} + \mathcal{L}_{\text{ILFP}}\) is composed of the Lagrangian density \(\mathcal{L}_{\text{ILGF}}\) to fix the gauge of the internal-Lorentz part (i.e., spin-affine connection part)

\[
\mathcal{L}_{\text{ILGF}} = h g^{\alpha \beta} \Gamma^{s \alpha}_{s} \partial_s s_{0} + \alpha h_{s a} e^a
\]

and of the corresponding FP ghost term \(\mathcal{L}_{\text{ILFP}}\) given by

\[
\mathcal{L}_{\text{ILFP}} = -i h g^{\alpha \beta} \partial_s \bar{\psi}_{s a} \cdot (\bar{\psi}_s \Gamma^{s \alpha}_{s} - \bar{\psi}_s \Gamma^{s \alpha}_{s} + \partial_t \bar{\psi}_s)\),
\]

where the spin-affine connection (with the minus sign) \(\Gamma^{s \alpha}_{s}\) is defined by

\[
\Gamma^{s \alpha}_{s} = (\partial_s h_{s}^a - \Gamma^{s \alpha}_{s} h_{s}^a) h_{s}^a
\]

in terms of the vierbein components \(h_{s}^a\). The IL-BRS transformation is given** by

\[
\delta_{\text{IL}} (h_{s m}) = -t^a h_{s m} ,
\]

\[
\delta_{\text{IL}} (s_{0}) = 0 ,
\]

*) The author is indebted to Prof. N. Nakanishi for his suggestion to examine the possibility of "another BRS transformation" in the vierbein formalism.

**) Instead of the term "local-Lorentz (LL)" used in Ref. 5) as well as in most of recent papers, we adopt here the term "internal-Lorentz (IL)", because both local and global versions of "local-Lorentz" transformations come up for discussion in the following.
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\[ \delta_{BRS}(t_{ab}) = -t_{ab} t_{ab}, \]
\[ \delta_{BRS}(i_{ab}) = i_{ab}, \]
\[ \delta_{BRS}(I_{s}^a) = -t_{s} I_{s}^a + t_{s} I_{s}^a - \partial_s t_{ab}. \]

and
\[ \delta_{BRS}(\phi) = -t_{ab} \hat{\phi}_{ab}/2, \]
\[ \delta_{BRS}(\overline{\psi}) = t_{ab} \overline{\psi} \hat{\phi}_{ab}/2. \]

for instance, for the case of Dirac spinor \( \psi \). Such quantities constituting \( \mathcal{L} \) as
\[ g_{as}, \ h = \sqrt{-\det(g_{as})}, \ b_s, \ c_s, \ \overline{c}_s \] are invariant under \( \delta_{BRS} \):
\[ \delta_{BRS}(g_{as}) = \delta_{BRS}(h) = \delta_{BRS}(b_s) = \delta_{BRS}(c_s) = \delta_{BRS}(\overline{c}_s) = 0. \]

Now, making some notational changes which facilitate the correspondence between the present case and the case of Yang-Mills theory as
\[ \mathcal{L} = \mathcal{L}_{BRS}(\mathcal{L}) \]
we rewrite the IL-BRS transformation (2.30) and \( \mathcal{L}_{BRS} \) as follows:
\[ \delta_{BRS}(h_{ab}) = -c_a h_{ab} = -(c \cdot h)_a b, \]
\[ \delta_{BRS}(b_{ab}) = (\partial_s c + [b_s, c])^a b = (D_s c)^a b, \]
\[ \delta_{BRS}(c_{ab}) = 0, \]
\[ \delta_{BRS}(b_{ab}) = -c_a c_{ab} = -[c, c]_{ab}/2, \]
\[ \delta_{BRS}(c_{ab}) = i B_{ab}, \]
\[ \delta_{BRS}(\phi) = -c_{ab} \phi_{ab}/2, \]
\[ \delta_{BRS}(\overline{\psi}) = \epsilon_{ab} \overline{\psi} \phi_{ab}/2; \]
\[ \mathcal{L}_{BRS} = h ( - g^a \phi B_{ab} c^b + c B_{ab} B^b - i g^a \phi c_{ab} (D_s c)^a b ) \]
\[ = \delta_{BRS} (i h g^a \phi \partial_s c_{ab} c^b - ic \phi c_{ab} B^b). \]

In (2.33), we have used the antisymmetry of Latin indices, for instance,
\[ \omega_{ab} = \omega_{ba}, \]
\[ c_{ab} = -c_{ba}, \]
\[ c^{a b} = \overline{c}_{ab} c^{a b} = -\overline{c}_{b a} c^{a b} = -c^{a b}, \]
and so on. Introducing the following notations:
\[ [\hat{g}_{ab}, \hat{g}_{cd}] = f^{ef}_{(ab), (cd)} \hat{g}_{ef}, \]
\[ f^{ef}_{(ab), (cd)} = \left( (\hat{g}_{ab} \hat{g}_{cd} - \hat{g}_{ab} \hat{g}_{cd} \hat{g}_{de} + \hat{g}_{ab} \hat{g}_{cd} - \hat{g}_{ab} \hat{g}_{cd}) - (c \leftrightarrow f) \right)/4, \]
we have such relations as
\[ [\varphi_a, c]_{ab} = f_{(ab)}^{(ac)} \varphi_b c_{cb} = (\omega_a \times c)^{ab}, \]  
\[ [c, c]_{ab} = f_{(ab)}^{(cd)} \varphi^c c_{ab} = (c \times c)^{ab}, \]  
which show more clearly the similarity to the case of Yang-Mills theories.

Thus, we can completely reproduce for this case the consequences obtained in the preceding subsection. "Another BRS transformation" is defined by
\[ \delta'_{IL}(h_{ab}) = - (\bar{\epsilon} \cdot h_{ab}) \cdot \epsilon_a = - \bar{\epsilon}_a h_{ab}, \]  
\[ \delta'_{IL}(\varphi_{ab}) = (D_a \varphi)^{ab} = - (\bar{\epsilon} \times \omega_a)^{ab} + \partial_a \bar{\epsilon}^{ab}, \]  
\[ \delta'_{IL}(\bar{\epsilon}_{ab}) = - (\bar{\epsilon} \times \bar{\epsilon})_{ab}/2 = - \bar{\epsilon}_a \bar{\epsilon}_b, \]  
\[ \delta'_{IL}(\varphi_{ab}) = - (\bar{\epsilon} \times \epsilon)_{ab} - iB_{ab} = - \bar{\epsilon}_a \epsilon_b + \bar{\epsilon}_b \epsilon_a - iB_{ab}, \]  
\[ \delta'_{IL}(B_{ab}) = - (\bar{\epsilon} \times B)_{ab} = - \bar{\epsilon}_a B_b + \bar{\epsilon}_b B_a, \]  
\[ \delta'_{IL}(\psi) = - \bar{\epsilon}^{ab} \varphi_{ab}/2, \quad \delta'_{IL}(\bar{\psi}) = \bar{\epsilon}^{ab} \bar{\varphi}_{ab}/2, \]  
and satisfies the relations
\[ \delta'_{IL}(\bar{\varphi} + \bar{\varphi}) = \delta'_{IL}(\bar{\varphi}) = 0, \]  
\[ (\delta'_{IL})^2 = 0, \]  
\[ \delta'_{IL} \delta_{IL} = - \delta_{IL} \delta'_{IL}. \]  
The corresponding conserved Noether current is given by
\[ J_{\nu}^{s} = h g^{\omega} (B_{ab} (D_a \varphi)^{ab} - \partial_a B_{ab} - \bar{\epsilon}^{ab} + i(\bar{\epsilon} \times \epsilon)^{ab} (D_a \varphi)) \]  
\[ = \delta'_{IL}(J_{\nu}^{s}) \]  
\[ = 0, \]  
where \( J_{\nu}^{s} \) is the conserved current of the IL-FP ghost charge, namely,
\[ J_{\nu}^{s} = i h g^{\omega} (\bar{\varphi}_{ab} (D_a \varphi)^{ab} - \partial_a \bar{\varphi}_{ab} - \epsilon^{ab}) \]  
\[ = J_{\nu}^{s}, \]  
\[ Q_{\nu} = \int d^3 x J_{\nu}^{s} ( = Q_{\nu}). \]  
The current \( J_{\nu}^{s} \) (2.42) much resembles the IL-BRS current given by
\[ J_{\nu}^{s} = - h g^{\omega} (B_{ab} (D_a \varphi)^{ab} - \partial_a B_{ab} - \bar{\epsilon}^{ab} + i(\bar{\epsilon} \times \epsilon)^{ab} / 2) \]  
\[ = - \delta_{IL}(J_{\nu}^{s}) = - J_{\nu}^{s}. \]  
The relations for the IL-BRS charge \( Q_{\nu} = \int d^3 x J_{\nu}^{s} ( = Q_{\nu}) \) and the charge of "another IL-BRS transformation" given by
\[ Q_{\nu} = \int d^3 x J_{\nu}^{s} \]
are completely identical to those in (2·22) ~ (2·25) for the case of the Yang-Mills theory. The remark made in the preceding subsection about the similarity of "another BRS transformation" to global and local gauge transformations also applies straightforwardly to the present case.

§ 3. Some implications of "another BRS symmetry"

The two types of gauge theories of the Yang-Mills field and of the spin-affine connection in the vierbein formalism differ considerably in their characters and origins—for instance, while the gauge group of the former case is compact, the latter is a gauge theory of the non-compact group \( SL(2, \mathbf{C}) \), and its gauge field (= spin-affine connection) \( \omega_\mu^a \) is a secondary quantity determined by the vierbein components \( h_\mu^a \) which are the fundamental variables of the theory [see, (2·29)]. On the contrary, the fundamental variables in the Yang-Mills theory are the components of the Yang-Mills field \( A_\mu^a \) themselves. As seen in the preceding section, however, both systems as gauge theories of internal symmetries have very similar gauge-fixing and FP ghost terms [cf. (2·3), (2·4) and (2·27), (2·28)]. "Another BRS transformations", (2·6) and (2·28), of these two cases are completely identical in their forms, which resemble the global (and local) gauge transformations. These facts lead to the following interesting observation.

The "Maxwell" equation\(^0\) of the Yang-Mills theory,

\[
\partial^\mu F_\mu^a + g J_\mu^a = \{Q_\theta, (D_\mu \vec{e})^a\}
\]

(3·1)

with the Noether current \( J_\mu^a \) of the global gauge transformation,

\[
J_\mu^a = -i (T^a\varphi)_\mu + (A^a \times F_\mu)^a + (A_\mu \times B)^a
\]

\[
- i (\vec{e} \times D_\mu \vec{e})^a + i (\vec{e} \times B)^a
\]

(3·2)

tells us that the conserved charge \( N^a \) defined by

\[
N^a = \int d^3x \{Q_\theta, (D_\mu \vec{e})^a\} = -i \int d^3x \delta ((D_\mu \vec{e})^a)
\]

(3·3)

behaves as the generator of the global gauge transformation in the commutators with the local field polynomials:

\[
gQ^a := \int d^3x g J_\mu^a \cdot N^a
\]

(3·4)

\[
[iN^a, \Phi^b] = [igQ_\theta^a, \Phi^b] = -g f^{abc} \Phi^c
\]

(3·5a)

\[
[iN^a, \psi_t] = [igQ_\theta^a, \psi_t] = ig (T^a \varphi)_t
\]

(3·5b)

where \( \Phi^a = A^a \), \( B^a \), \( \varphi^a \), \( \vec{e}^a \). Although the vierbein formalism has no equation corresponding to (3·1), similar results hold; the conserved charge \( N^{ab} \) defined by
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\[ N^a = \int d^3x \partial^a \{ Q_a, D_x \}^a = -i \int d^3x \partial^a \{ h g^{ab} (D_x)^{ab} \} \]

(3.6)

coinsides with the Noether charge \( M_{\text{NL}}^{ab} \) of the global internal-Lorentz transformation obtained by Nakanishi:

\[ M_{\text{NL}}^{ab} = 2 \int d^3x g^{ab} (\partial \epsilon_{\rho}^{ab} - \Gamma_{\rho}^{\alpha \beta} \epsilon_{\beta}^{ab} + \Gamma_{\rho}^{\beta \alpha} \epsilon_{\alpha}^{ab} - i (\epsilon^{ab} (\partial \epsilon_{\rho}^{ab} + \Gamma_{\rho}^{\alpha \beta} \epsilon_{\beta}^{ab} - \Gamma_{\rho}^{\beta \alpha} \epsilon_{\alpha}^{ab} - a \leftrightarrow b))) = 2 N^{ab}, \]

(3.7)

\[ [2 i N^{ab}, \Phi^d] = [i M_{\text{NL}}^{ab}, \Phi^d] = (\gamma^a \Phi^d - \gamma^d \Phi^a + \gamma^{cd} \Phi^c - \gamma^{ac} \Phi^d), \tag{3.8a} \]

\[ [2 i N^{ab}, \phi] = [i M_{\text{NL}}^{ab}, \phi] = -\delta^{ac} \phi, \quad [2 i N^{ab}, \tilde{\phi}] = [i M_{\text{NL}}^{ab}, \tilde{\phi}] = \tilde{\phi} \delta^{ac}, \tag{3.8b} \]

where \( \Phi^d = \omega^d, B^d, c^d, e^d \).

These circumstances, which seem somewhat accidental, can be easily understood by the help of “another BRS transformation”. Noting the following expression of \( N^a \) derived from (2.6a) and (2.12):

\[ N^a = i \int d^3x \partial^a \{ Q_a, \Pi^a \} = -i \int d^3x \{ Q_a, \Pi^a \} \]

(3.9)

with the canonical momentum variable \( \Pi^a = (\partial/\partial \tilde{e}^a) L = -(D_x)^a \) satisfying

\[ [i \Pi^a (x), \tilde{e}^b (y)]_{x-y} = \delta^{ab} \tilde{e}^b (x-y), \]

(3.10)

we obtain

\[ [i N^a, X] = \int d^3x [\{ Q_a, \Pi^a (x) \}, X] \]

\[ = \int d^3x (\{ \Pi^a (x), [Q_a, X] \} + [Q_a, \{ \Pi^a (x), X \}]_x) \]

\[ = -\int d^3x [i \Pi^a (x), \delta^a (X)]_x = \delta^a (\int d^3x [i \Pi^a (x), X]_x). \]

(3.11)

Equations (3.9) ~ (3.11) and their analogues in the vierbein formalism, together with the characteristic forms of “another BRS transformation” (2.6) and (2.38) mimicking the gauge transformations, explain, in both theories, the behaviour (3.5) and (3.8) of \( N^a \) and \( N^{ab} \) as the generators of the global gauge transformations.

Now we note that the parallelism found in the above between the Yang-Mills theory and the vierbein formalism is based upon the common feature as gauge theories of internal symmetries. On the contrary, the proper gravitational part \((g_{\mu\nu}\)-part) of quantum gravity is a gauge theory of an external symmetry whose “gauge group” is the group of general coordinate transformations, and we cannot find here “another BRS transformation” in such a form as those obtained in the preceding section. The precise reason for this matter has been clarified by
Another BRS Transformation

Nakanishi\(^{a}\) as follows. Although there exists a conserved nilpotent charge
\[ M(b_t, \tilde{c}_t) = \int \mathcal{D} x^\alpha \mathcal{D} \xi^a \partial_\alpha \xi^a \partial_t \xi^a - \partial_\alpha \xi^a \partial_t \xi^a, \]
generating the following transformation very akin to “another BRS transformation”:
\[ [iM(b_t, \tilde{c}_t), g_{\alpha}] = -\varepsilon (\partial_\alpha \tilde{c} \cdot g_{\alpha} + \partial_\alpha c \cdot g_\alpha + \tilde{c} \partial_\alpha g_{\alpha}), \quad (3.12a) \]
\[ [iM(b_t, \tilde{c}_t), b_\alpha] = -\varepsilon \tilde{c} \partial_\alpha b_\alpha, \quad (3.12b) \]
\[ [iM(b_t, \tilde{c}_t), c^\nu] = -\kappa \varepsilon \partial_\nu c^\nu + i\tilde{\nu} b_\nu, \quad (3.12c) \]
\[ [iM(b_t, \tilde{c}_t), \tilde{c}_\nu] = -\kappa \varepsilon \partial_\nu \tilde{c}_\nu, \quad (3.12d) \]
we cannot construct a general-linear-invariant charge from \( M(b_t, \tilde{c}_t) \). This
\( M(b_t, \tilde{c}_t) \) is a member of Nakanishi's Poincaré-like superalgebra\(^{b}\) in a 16-di-

mensional superspace. (In the cases of the Yang-Mills theory and of the vierbein
formalism also, we can find, in the Landau gauge, the corresponding but rather
different superalgebra of conserved charges including \( Q_3 \) and \( Q_3' \). This will be
discussed elsewhere.)

In the Landau gauge where the FP ghost \( c \) and the antighost \( \tilde{c} \) are almost
interchangeable, “another BRS transformation” can be decomposed as follows. The
Lagrangians (2.1) and (2.34) with \( \alpha = 0 \) are invariant, respectively, under the transforma-
sions \( \mathcal{E}_{FP} \) and \( \mathcal{E}_{FP}' \) interchanging \( c \) and \( \tilde{c} \):
\[
\begin{align*}
\mathcal{E}_{FP} : & \\
A_\nu \rightarrow A_\nu , \\
\phi \rightarrow \phi , \\
B \rightarrow B - iGc \times c , \\
c \rightarrow \pm \tilde{c} , \\
\tilde{c} \rightarrow \mp c , \\
h_{a\alpha} \rightarrow h_{a\alpha} , \\
\psi \rightarrow \psi , \\
\mathcal{E}_{FP}' : & \\
B_{ab} \rightarrow B_{ab} - i(\tilde{c}_a \epsilon c_b + \tilde{c}_b \epsilon c_a) = (B - i\epsilon \times c)_{ab} , \\
c_{ab} \rightarrow \pm \tilde{c}_{ab} , \\
\tilde{c}_{ab} \rightarrow \mp c_{ab} ,
\end{align*}
\]
\[
(3.13)
\]
in terms of which “another BRS transformations” \( \delta' \) and \( \delta'_{IL} \) can be written,
respectively, as
\[ \delta' = \mp \mathcal{E}_{FP} \mathcal{E}_{FP}' , \quad (3.15) \]
\[ \delta'_{IL} = \mp (\mathcal{E}_{FP}')^{-1} \mathcal{E}_{FP} \mathcal{E}_{FP}' . \quad (3.16) \]

In non-Landau gauges, however, \( \mathcal{E}_{FP}(\mathcal{E}_{FP}')^{-1} \) is no longer a symmetry of the

\(^{a}\) \( \mathcal{E}_{FP} \) is an automorphism of the polynomial algebra \( \mathcal{Z} \) of fields and satisfies the property:
\[ \mathcal{E}_{FP} \phi = e^{\pm i\mathcal{Q}} \phi e^{\mp i\mathcal{Q}} , \quad \text{for } \phi \in \mathcal{Z} . \]
system, whereas 'another BRS transformation' still remains a symmetry transformation. As a consequence of "another BRS symmetry", we obtain a new set of Ward-Takahashi identities, whose validity has been partially checked perturbatively. In the case of Yang-Mills theory, they are explicitly written as follows:

\[ \langle 0 | T (\delta' \phi_t) \phi_t \cdots \phi_t A_{a_1}^x B_{i}^p \cdots B_{i}^p e^x \cdots e^x | 0 \rangle + \langle 0 | T \phi_t (\delta' \phi_t) \cdots \phi_t A_{a_1}^x B_{i}^p \cdots B_{i}^p e^x | 0 \rangle + \cdots + \langle 0 | T \phi_t (\delta' \phi_t) \cdots A_{a_1}^x B_{i}^p \cdots B_{i}^p e^x | 0 \rangle + \cdots \]

\[ = \langle 0 | \{ iQ_n, T (\phi_t \cdots \phi_t A_{a_1}^x \cdots A_{a_1}^x B_{i}^p \cdots B_{i}^p e^x \cdots e^x ) \} | 0 \rangle = 0. \tag{3.17} \]

It will be convenient to express Eq. (3.17) in the diagrammatical fashion à la 't Hooft-Veltman by using the relation \( iB = -g \bar{c} \times c - \delta' c. \)

As simple examples, we obtain

\[ \langle 0 | T ((\bar{c} \times c)^x (x) B^p (y)) | 0 \rangle = \langle 0 | T (c^x (x) (\bar{c} \times B)^x (y)) | 0 \rangle, \tag{3.18} \]
Another BRS Transformation

\[ iB^c \rightarrow B^a = 0 \]

\[ = -g(\varepsilon \times c) \stackrel{\varepsilon}{\rightarrow} B^b + g \stackrel{\varepsilon}{\rightarrow} (\varepsilon \times B)^b. \]

Fig. 2. Diagrammatic explanation of Eq. (3.18).

and

\[ \langle 0| T(c^a(x)(D_a \varepsilon)^b(y)) |0\rangle = -\delta_{ab}\partial_\nu D_\nu(x-y) - \langle 0| T(g(\varepsilon \times c)^a(x) A^b_a(y)) |0\rangle. \]

(3.19)

\[ iB_x^a \rightarrow A^b_x = -g(\varepsilon \times c) \stackrel{\varepsilon}{\rightarrow} A^b_y = \varepsilon^a_x \rightarrow (\varepsilon \times c)^b_y = \delta_{ab}\partial_\nu D_\nu(x-y). \]

Fig. 3. Diagrammatic explanation of Eq. (3.19).

Finally, we add a comment upon the roles played by “another BRS symmetry” in extracting physical contents of the theory. Since “another BRS charge” \( Q_{b}' \) is nilpotent [(2.23)] and has the relation (2.24) with \( Q_b \) similarly to \( Q_b' \), the quartet mechanism \(^{10}\) works in completely the same way as the case of BRS charge \( Q_b \) and the physical contents of the theory will remain unchanged at the level of asymptotic states and fields. In view of this remark and of the anticommutativity (2.25) of \( Q_{b}' \) with \( Q_{b} \), the subsidiary condition

\[ Q_{b}'|\text{phys}\rangle = 0 \]

(3.20)

can be imposed additionally to \( Q_{b}|\text{phys}\rangle = 0 \) without any inconsistency and without any change in the physical contents of the theory. However, combined with the BRS symmetry, the condition (3.20) and the new Ward-Takahashi identities (3.17) might facilitate our insight into the structure of gauge theories in some such respects as the problem of observables, owing to the similarity of “another BRS transformation” to the (global and local) gauge transformation.

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References

6) T. Kugo, private communication.

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