On a New Family of Spinning Mass Solutions

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A new family of stationary axisymmetric, asymptotically flat exact solutions to the Einstein vacuum fields, having positive number distortion parameter $\delta$, mass parameter $m$, and rotation-reflection parameter $\lambda$, is presented in a closed form. When $\delta=1$, the present solution reduces to the Kerr solution. The present family of solutions contains no event horizon except the $\delta=1$ solution.

§ 1. Introduction

A new family of spinning mass solutions with arbitrary positive number distortion parameter $\delta$, i.e., a new family of stationary axisymmetric, asymptotically flat exact solutions to the Einstein vacuum field equations, is presented in a closed form. The present family of solutions has three parameters, the positive number distortion parameter $\delta$ labeling the member of the family, the mass parameter $m$, and the rotation-reflection parameter $\lambda$. The angular momentum $J$ about the symmetry axis ($z$ axis) is

$$ J = 2m^2 \frac{\delta}{(\delta^2 + 4\lambda^2)^{3/2}} (2\delta - 1) + 4\lambda^2. \quad (1) $$

When the rotation-reflection parameter does not vanish, the solutions with arbitrary distortion parameter $\delta$ (except $\delta=1$) are asymmetric (but symmetric only when $\delta=1$) with respect to the reflection at the equatorial plane. The solutions with arbitrary distortion parameter $\delta$ contain no event horizon except the $\delta=1$ solution. The $\delta=1$ member solution of the present family reduces to the Kerr solution, and the $\delta=2$ member solution reduces to the special case ($m$, $\alpha=\beta(=\lambda)$, and $q=0$ in their notation) of four-parameter $\delta=2$ Kinnersley-Chitre solution. When the rotation-reflection parameter $\lambda$ vanishes, the present family of solutions reduces to the Weyl-Zipoy-Voorhees family of static solutions.

In the latter half of this section the notations will be explained through the way to give convenient expressions of the gravitational field equations for sources with axial symmetry and angular momentum. In § 2 the closed form of the present family of spinning mass solutions will be given. Finally, in § 3 the properties of the solutions obtained will be studied.

The line element is

$$ ds^2 = f^{-1}(e^\rho (dz^2 + d\phi^2) + \rho d\rho^2) - f(dt + \omega d\phi)^2, \quad (2) $$
where \( z, \rho \) and \( \phi \) are the Weyl-Papapetrou coordinates and three metric functions \( f, \omega \) and \( \gamma \) are functions of \( z \) and \( \rho \) only. Prolate spheroidal coordinates \( x \) and \( y \) are introduced as
\[
\rho = \kappa (x^2 - 1)^{1/2} (1 - y^2)^{1/2} \quad \text{and} \quad z = \kappa xy, 
\]
where the unit of distance \( \kappa \) is
\[
\kappa = \frac{m}{\sqrt{\beta^2 + 4\lambda^2}}. 
\] (3)
The notations \( a = x^2 - 1 \) and \( b = y^2 - 1 \) are also introduced. The Einstein vacuum field equations are
\[
\Gamma (f^2 \rho \omega + \omega f^2 \rho \omega) = 0, 
\] (4)
\[
\Gamma (f^2 \rho \omega) = 0, 
\] (5)
\[
y_\gamma - 2 \gamma^{-1} f^{-2} \rho (f_x f_y + \rho^2 \omega_y) = 0, 
\] (6)
\[
y_\gamma + 4 \gamma^{-4} f^{-4} \rho ((f_x)^2 + (f_y)^2 - (f_x)^2 - (f_y)^2) = 0, 
\] (7)
where Eqs. (8) are used in advance and \( \gamma = \partial \gamma / \partial z \). It follows from Eqs. (5) and (4) that there are potentials \( \Omega \) and \( P \), respectively, which satisfy
\[
\rho^{-2} f_\rho^2 \omega_\rho = -\kappa^{-1} b^{-1} \Omega_y, \quad \rho^{-2} f_\rho^2 \omega_\rho = -\kappa^{-1} a^{-1} \Omega_x, 
\] (8)
\[
f^{-1} f_x + \rho^{-2} \omega f^2 \omega_x = -\kappa^{-1} a^{-1} (\omega \Omega)_y + a^{-1} F_y 
\]
and
\[
f^{-1} f_y + \rho^{-2} \omega f^2 \omega_y = -\kappa^{-1} b^{-1} (\omega \Omega)_x + b^{-1} P_x. 
\] (9)

\( A, B, H, I \) and \( G \) are defined by the relations
\[
f = A / B, \quad \Omega = 2I / B, \quad B = A + 2H + 2G 
\]
and
\[
H^2 + I^2 = AG + G^2. 
\]

Then Eqs. (8) and (9) become
\[
2a A^{-4} (H(A + 2G)_x - (A + 2G) H_x) = P_y, 
\] (10)
\[
2b A^{-4} (H(A + 2G)_y - (A + 2G) H_y) = P_x, 
\]
\[
2a A^{-4} (I(A + 2G)_x - (A + 2G) I_x) = Q_y, 
\] (11)
\[
2b A^{-4} (I(A + 2G)_y - (A + 2G) I_y) = Q_x, 
\]
\[
4a A^{-4} (HI_x - HI_x) = R_y, 
\]
\[
4b A^{-4} (HI_y - HI_y) = R_x 
\] (12)
and
\[
\omega = -\kappa (Q + R) + \text{constant}. 
\] (13)
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The form $P^2 + Q^2 - R^2$ is invariant of the SU(1,1) Ehlers group. Under the operation of one of three generators of the group SU(1,1) the relations

$$A' = A, \quad H' = H \cos \tau + I \sin \tau, \quad I' = -H \sin \tau + I \cos \tau,$$
$$G' = G, \quad P' = P \cos \tau + Q \sin \tau, \quad Q' = -P \sin \tau + Q \cos \tau \quad \text{and} \quad R' = R \quad (14)$$

hold. This is the NUT-Geroch transformation with respect to the timelike Killing vector, which is used in Eqs. (31) in order to obtain asymptotically flat solutions. Equations (10)~(12) are the gravitational field equations for sources with axial symmetry and angular momentum.

§ 2. New family of solutions

The present new family of spinning mass solutions with positive number distortion parameter $\delta$, mass parameter $m$ and rotation-reflection parameter $\lambda$, which satisfies the gravitational field Eqs. (10)~(12), is

$$f = \frac{A}{B},$$
$$\omega = -\kappa (Q + R - 4\lambda),$$
$$\kappa = m (\delta^2 + 4\lambda^2)^{-1},$$
$$\varepsilon^\tau = \frac{A}{(a-b)^{\delta^2}},$$
$$A = \alpha^\nu + 4\lambda^2 (x-y)^{4\delta-1}a^{4\nu-1}b,$$
$$B = A + 2H + 2G,$$
$$H = \frac{\tilde{H}}{\cos \tau + \tilde{I} \sin \tau}, \quad I = -\frac{\tilde{H}}{\sin \tau + \tilde{I} \cos \tau},$$
$$\tan \tau = 2\lambda / \delta,$$
$$\tilde{H} = \frac{1}{4} a^{4\nu-1} \left\{ (x+1)^{4\delta} - (x-1)^{4\delta} \right\}$$
$$+ \frac{1}{4} a^{4\nu-1} \left\{ (x-y)^{4\delta-1} \right\},$$
$$\tilde{I} = 2\lambda (x-y)^{4\nu-1} a^{4\nu-1},$$
$$G = \frac{1}{4} a^{4\nu-1} \left\{ (x+1)^{4\delta} - (x-1)^{4\delta} \right\},$$
$$P = \frac{\tilde{P}}{\cos \tau + \tilde{Q} \sin \tau}, \quad Q = -\frac{\tilde{P}}{\sin \tau + \tilde{Q} \cos \tau},$$
$$\tilde{P} A = 2\delta \gamma a^{4\delta} + 8\delta^2 (x+ \delta - 1) (x-y)^{4\nu-1} a^{4\nu-1}b,$$
$$\tilde{Q} A = 2\lambda (x-y)^{4\nu-1} a^{4\nu-1} \left\{ (y+1) (x+1)^{4\delta-1} + (y-1) (x-1)^{4\delta-1} \right\},$$
and
$$RA = 2\lambda (x-y)^{4\nu-1} a^{4\nu-1} \left\{ (y+1) (x+1)^{4\delta-1} - (y-1) (x-1)^{4\delta-1} \right\}. \quad (29)$$
§ 3. Properties of solutions

Equations (19), (23)‒(25), and (27)‒(29) show that, when \( x \) tends to 
infinity, the leading behaviors of potentials \( A, H, I, G, P, Q \) and \( R \) are
\[
A \sim x^{2\eta}, \\
H \sim 2\lambda x^{2\nu-1}, \\
I \sim 2\alpha x^{2\nu-1}, \\
G \sim (\beta^2 + 4\lambda^2) x^{2\nu-2}, \\
P \sim 2\delta x^2 + 0 x^{2\nu} + 8\beta^2 b x^{2\nu-1}, \\
Q \sim 4\beta x^2 + 0 x^{2\nu} - 4 (2\delta - 1) \lambda b x^{2\nu-1}, \\
R \sim 0 y x^{2\nu} + 4\lambda x^{2\nu} + 0 x^{2\nu-1}.
\]

In order to be \( \omega \rightarrow 0 \) when \( x \rightarrow \infty \), the NUT-Geroch transformation given in Eqs. 
(14) is used to eliminate \( y x^{2\nu} \) term from \( QA \), and the integration constant in 
Eq. (13) on the metric function \( \omega \) is taken to eliminate \( x^{2\nu} \) term from \( RA \). Then 
one gets \( \tau, Q \) and \( \omega \) given in Eqs. (22), (26) and (16), respectively. The unit 
of distance \( \kappa \) is defined to be the inverse of the coefficient of the term \( -2m/x \) 
in the leading behavior \( f \sim 1 - (2m/\kappa x) \) for \( x \rightarrow \infty \). Then \( \kappa \) given in Eq. (3) 
comes from Eqs. (30). The angular momentum \( J \) about the symmetry axis, \( z \) 
axis, is defined to be the coefficient of the term \( 2\beta/\kappa x \) in the leading behavior 
\( \omega \sim J2\beta/\kappa x \) for \( x \rightarrow \infty \). Then \( J \) given in Eq. (1) comes from Eqs. (31).

Equations (15)‒(29) given in § 2 show that the present family of solutions 
is asymmetric (but symmetric only when \( \delta = 1 \)) with respect to the reflection at 
the equatorial plane \( y = 0 \) in contrast with the symmetric Kerr-Tomimatsu-Sato 
family of spinning mass solutions. The parameter \( \lambda \) denotes the extent of this 
asymmetry (when \( \delta \neq 1 \)). From this property and from Eq. (1) on the angular 
momentum \( J \) about the symmetry \( z \) axis the parameter \( \lambda \) is named as the rotation-
reflection parameter.

The proper area \( \Sigma \) of the surface \( x = 1 \) is
\[
\Sigma = 4\pi \kappa \int_0^1 (\zeta^2 \phi^2)^{1/2} dy.
\]
It follows that \( \Sigma = 0 \) when \( \delta \neq 1 \) and \( \Sigma = 8\pi m^2 (1 + (1/(1 + 4\lambda^2))^{1/2}) \) when \( \delta = 1 \). 
Thus the present family of spinning mass solutions contains no event horizon 
except the \( \delta = 1 \) solution.

When the distortion parameter \( \delta \) is one, various potentials are
\[
A = a + 4\lambda b, \quad B = (x + (1 + 4\lambda^2)^{1/2})^2 + (2\lambda y)^2, \quad H = x + 4\lambda^2 y, \\
I = 2\lambda (x - y), \quad G = 1 + 4\lambda^2, \quad P = 2y a + 8\lambda^2 x b, \\
Q = 4\lambda (x - y) (xy + 1), \quad R = 4\lambda (x - y) (x + y),
\]
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\[ H = (1 + 4\lambda^2)^{1/2}x, \quad I = -2\lambda (1 + 4\lambda^2)^{1/2}y, \]
\[ PA = 2(1 + 4\lambda^2)^{1/2}ya, \quad QA = -4\lambda (1 + 4\lambda^2)^{1/2}xb, \]
\[ \tan \tau = 2\lambda, \quad \kappa = m/(1 + 4\lambda^2)^{1/2} \quad \text{and} \quad J = 2m^2\lambda/(1 + 4\lambda^2)^{1/2}. \]

(33)

When the Tomimatsu-Sato rotation parameters \( p \) and \( q \) \((p^2 + q^2 = 1)\) are identified with the present parameter \( \lambda \) as

\[ p = 1/(1 + 4\lambda^2)^{1/2} \quad \text{and} \quad q = 2\lambda/(1 + 4\lambda^2)^{1/2}, \]

(34)

Eqs. (33) show that the present \( \delta = 1 \) solution is the Kerr solution.\(^7\) Also the present \( \delta = 2 \) solution is the special case \((\alpha = \beta (= \lambda), q = 0 \text{ and } m \text{ in their notation})\) of four-parameter \( \delta = 2 \) Kinnersley-Chitre solution.\(^8\) The present paper is a first step to obtain the Kerr-Kinnersley-Chitre family of spinning mass solutions with arbitrary integer distortion parameter \( \delta \) and with \( 2\delta \) parameters.

References