Gravity from Poincaré Gauge Theory
of the Fundamental Particles. III

—Weak Field Approximation—

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We apply the weak field approximation to the most general gravitational field equations

in Poincaré gauge theory. The weak gravitational field \( h_\mu \) is a multimass field obeying a

fourth-order field equation. In the Newtonian approximation we show that there are two

routes to arrive at the Newtonian potential. The torsion field is decomposed into six irre­

ducible building blocks with spin-parity, \( 2^+, 2^-, 1^+, 1^-, 0^+, \) and \( 0^- \), each of which obeys the

Klein-Gordon equation. Finally, we construct a possible candidate for the massless graviton

field which obeys the linearized Einstein equation.

§ 1. Introduction

In previous papers called I and II, respectively,\(^1\) we studied Poincaré gauge

theory with linear and quadratic Lagrangian densities in the translation and Lo­

rentz gauge field strengths with ten free parameters included in the gravity action.

The most general gravitational field equations were derived by means of the action

principle with independent variations of the translation and Lorentz gauge fields.

Here in this paper we shall choose the conventional method that the Lorentz gauge field \( A \) can be decomposed into the Ricci rotation coefficient \( \mathbf{J} \) (which is given by first derivatives of the tetrad field) and the contorsion field \( \mathbf{K} \).

\[
A = \mathbf{J} + \mathbf{K},
\]

and then apply the weak field approximation to the most general gravitational field equations. (In the Riemann-Cartan space-time the geometrical method also gives rise to the conventional method.) So independent variables are the line­
arized gravitational field \( h_\mu \) and the contorsion field \( K_{\mu\nu} \), the former of which satisfies a fourth-order differential field equation just because of the conventional method formulated by (1·1). Therefore, \( h_\mu \) describes a multimass field which splits into a graviton field with vanishing mass, a particle with mass \( m_2 \) and spin 2 and a particle with mass \( m_0 \) and spin 0. However, the redefinition of a multimass field, which incorporates second derivative terms, will give us a fairly good candidate for a massless graviton field with spin 2. This redefined field \( \phi_{\mu\nu} \),
obeys the usual wave equation of second order, whose source term is exactly the symmetrized energy-momentum tensor appearing in the linearized Einstein equation.

The arrangement of the present paper is as follows. In the next section we shall apply the weak field approximation method to the most general gravitational field equations derived in a previous paper. In § 3 the Newtonian approximation will be carried on, showing that there are two different routes in arriving at the Newtonian potential: Denoting \( a_i \) (\( i=1, 2, \ldots, 6 \)) as coefficients multiplied by Lagrangian densities quadratic in the Lorentz gauge field strength, one is to take all \( a_i \) as finite, and the other is to take all \( a_i \) as infinite, the latter of which is just the result of New General Relativity. In § 4 the torsion field is decomposed into its irreducible building blocks, which have spinparity as \( 2^+ \), \( 2^- \), \( 1^+ \), \( 1^- \), \( 0^+ \) and \( 0^- \) and propagate in vacuum. In § 5 the massless graviton field \( g_{\mu\nu} \) is redefined and satisfies the usual wave equation whose source term is given by the symmetrized energy-momentum tensor occurring in the linearized Einstein equation. The final section will be devoted to conclusion.

§ 2. The linearized gravitational field equations

We shall apply the most general gravitational field equations to weak field situations, where the translation gauge field \( a_\mu \) and the Lorentz gauge field \( A_{\mu\nu} \) are both so weak that it is sufficient to keep only those terms which are linear in \( a_\mu \) and \( A_{\mu\nu} \). In this approximation we need not distinguish Latin indices from Greek ones; so we shall use Greek indices throughout this paper with the understanding that they are raised and lowered with the Minkowski metric \( \eta_{\mu\nu} \).

The metric tensor \( g_{\mu\nu} \) is written as

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

with

\[
h_{\mu\nu} = a_\mu a_\nu + A_{\mu\nu}.
\]

Following the conventional method mentioned in the Introduction, we shall express the linearized gravitational field equations in terms of \( a_\mu \) and the contorsion field \( K_{\mu\nu} \). The torsion field is given in the weak field approximation by

\[
T_{\mu\nu} = A_{\mu\nu} - \partial_\mu a_\nu + \partial_\nu a_\mu.
\]

The contorsion field is related to the three irreducible parts of the torsion field, \( \ell_{\mu\nu} \), \( v_\mu \), and \( a_\mu \), by

\[
K_{\mu\nu} = \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu}),
\]

We denote symmetrization and antisymmetrization of tensor indices by round brackets ( ) and square brackets \([ \ ]\), respectively. For example,

\[
V_{[\mu\nu\rho\sigma]} = \frac{1}{4} (V_{\mu\nu\rho\sigma} - V_{\mu\nu\sigma\rho}).
\]
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\( -\frac{4}{3} t_{\nu_1 \mu 1} - \frac{2}{3} \eta_{\nu_1 \nu_2} - \frac{1}{2} \xi_{\nu_1 \nu_2} a^\nu \) \tag{2.4}

or conversely

\( t_{\nu_1 \mu} = K_{\nu_1 \mu} + \frac{1}{3} \eta_{\nu_1 \nu_2} - \frac{1}{3} \eta_{\nu_1 \mu} v_\nu \) \tag{2.5a}

\( v_\mu = K_{\mu} \) \tag{2.5b}

\( a_\mu = \frac{1}{3} \xi_{\mu_1 \mu} K_{\mu_1} \) \tag{2.5c}

Applying the weak field approximation to the alternative form of the gravitational field equation (I.5.12), we have

\( 2a_\mu G_{\mu\nu}^{(1)} + 2\partial^\nu F_{\mu\nu}^{(1)} = T_{\mu\nu} \) \tag{2.6}

where \( F_{\mu_1}^{(1)} \) is the linearized expression for \( F_{\mu_1} \) of (I.5.15),

\[ F_{\mu_1}^{(1)} = \left( \alpha + \frac{2a}{3} \right) (t_{\mu_1 \mu} - t_{\mu_\mu}) + \left( \beta - \frac{2a}{3} \right) (\eta_{\mu_1 \nu} v_\nu - \eta_{\mu_1 \mu} v_\nu) \]

\[ - \frac{1}{3} \left( r + \frac{3a}{2} \right) \xi_{\mu_1 \mu} a_\nu \] \tag{2.7}

and \( G_{\mu\nu}^{(1)} \) is the linearized Einstein tensor,

\[ G_{\mu\nu}^{(1)} = -\frac{1}{2} \left[ \Box h_{\mu\nu} - \eta^\nu (\partial_\mu h_{\nu\nu} + \partial_\nu h_{\mu\nu} + \eta_{\mu\nu} \partial^\mu h_{\rho\rho}) \right] \] \tag{2.8}

with

\[ \Box h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad h = \eta^\nu h_{\mu\nu}. \] \tag{2.9}

Here the d’Alembertian \( \Box \) is defined by \( \Box = \partial^\nu \partial_\nu \).

Equation (I.5.18) takes the following form in the weak field approximation:

\[ P_{\mu\nu} = -\frac{1}{2} S_{\mu\nu} \] \tag{2.10}

with

\[ P_{\mu\nu} = (3a_3 + 2a_1) \partial_\nu G_{\mu\rho}^{(1)} + (a_1 + a_3 + 4a_\xi) \eta_{\mu\nu} \partial_\xi G_{\xi\rho}^{(1)} \]

\[ - 2\partial^\nu J_{\mu_1 \mu_2 \nu_1 \nu_2}^{(1)} (K) - H_{\mu_1 \nu_2}^{(1)} \] \tag{2.11}

where \( J_{\mu_1 \mu_2 \nu_1 \nu_2}^{(1)} (K) \) and \( H_{\mu_1 \nu_2}^{(1)} \) are the linearized expressions for \( J_{\mu_1 \mu_2} (K) \) of (1.5.6) and \( H_{\mu_1 \nu_2} \) of (1.3.32), respectively. Here \( G_{\mu\nu}^{(1)} = \eta^\rho G_{\rho\nu}^{(1)} \). By a straightforward calculation we get
\[ P_{\mu\nu} = (3a^2 + 2a^3) \partial_\nu G^{(1)}_{\mu\nu} + (a^2 + a^3 + 4a^3) \eta_{\mu\nu} \delta_{\mu\nu} G^{(1)} \]

\[ -\frac{2}{3} (3a^2 + 4a^3) \left\{ \Box t_{\xi,\lambda} - \partial^\sigma t_{\xi,\lambda,\sigma} \right\}, \]

\[ + \frac{3}{2} \left( \partial^\sigma t_{\rho,\lambda,\mu} - (\lambda \nabla^\rho \mu) \right) + \frac{3}{4} \eta_{\xi,\lambda} \partial^\rho \partial^\sigma t_{\rho,\lambda,\sigma} \right\} \]

\[ + (3a^2 + 2a^3) \left\{ \frac{1}{3} \left( \partial^\sigma t_{\rho,\lambda,\mu} - (\lambda \nabla^\rho \mu) \right) + \partial^\rho t_{\rho,\lambda,\mu} - (\lambda \nabla^\rho \mu) \right\} \]

\[ -\frac{2}{3} (a^3 + a^3) \eta_{\xi,\lambda} \partial^\rho v_\lambda + a^3 \left( \epsilon_{\sigma\mu\nu} \Box a^\sigma - \epsilon_{\lambda \mu \nu} \partial^\rho a^\sigma \right) \]

\[ -\frac{1}{2} a^4 \left( \epsilon_{\sigma\mu\nu} \partial^\rho a^\sigma - (\lambda \nabla^\rho \mu) \right) + a^4 \epsilon_{\sigma\mu\nu} \partial^\rho a^\sigma \]

\[ + 2 \left( a^3 + 2a^3 \right) t_{\xi,\lambda,\mu} + 2 \left( a^3 - 2a^3 \right) \eta_{\xi,\lambda,\mu} + \frac{2}{3} \left( a^3 + 3a^3 \right) \epsilon_{\sigma\mu\nu} a^\sigma . \quad (2.12) \]

The last three terms come from \( H^{(1)}_{\mu\nu} \) and stand for the mass terms of the torsion field.

The energy-momentum tensor \( T_{\mu\nu} \) and the spin tensor \( S_{\mu\nu} \) are taken to lowest order in \( a_\mu \) and \( K_{\mu\nu} \); so they are independent of \( a_\mu \) and \( K_{\mu\nu} \) and satisfy the ordinary conservation law,

\[ \partial^\sigma T_{\sigma\mu} = 0 , \quad (2.13) \]

and the Tetrode formula in special relativity,

\[ 2T_{(\mu\rho)} = \partial^\sigma S_{\mu\rho \sigma} . \quad (2.14) \]

Since the translation gauge field \( a_{\mu\nu} \) appears in (2.6) and (2.10) only through \( G^{(1)}_{\mu\nu} \), the linearized gravitational field equations are invariant under the following gauge transformations:

\[ h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (2.15a) \]

\[ a_{\mu\nu} \rightarrow a'_{\mu\nu} = a_{\mu\nu} + \omega_{\mu\nu} - \omega_{\mu\nu} = 0 , \quad (2.15b) \]

where \( A_\mu \) and \( \omega_{\mu\nu} \) are small but otherwise arbitrary four and six functions, respectively. These are the linearized version of general coordinate and local Lorentz transformations, respectively. The transformation (2.15b) means that the antisymmetric part \( a_{(\mu\nu)} \) is devoid of physical significance. Also, (2.15a) allows us to put the harmonic condition,
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\[ \partial^\nu \hbar_{\mu\nu} = 0, \]  

(2.16)

which we assume henceforth. The linearized Einstein tensor is then given by

\[ G_{\mu\nu}^{(1)} = -\frac{1}{2} \Box \hbar_{\mu\nu}. \]  

(2.17)

It is convenient to decompose (2.6) into the symmetric and antisymmetric parts:

\[ 2 \alpha G_{\mu\nu}^{(1)} - 6 \left( \alpha + \frac{2 \alpha}{3} \right) \partial^\nu t_{\mu(\rho\sigma)} - 2 \left( \beta - \frac{2 \alpha}{3} \right) (\partial_\rho v_\sigma - \eta_\rho\sigma \partial^\nu v_\nu) = T_{\mu(\nu)}, \]  

(2.18)

\[ 2 \left( \alpha + \frac{2 \alpha}{3} \right) \partial^\nu t_{\mu(\rho\sigma)} + 2 \left( \beta - \frac{2 \alpha}{3} \right) \partial_\rho v_\sigma + \frac{2}{3} (\gamma + \frac{3 \alpha}{2}) \varepsilon_{\rho\sigma\nu\rho} \partial^\nu a^\sigma = - T_{\mu(\nu)}. \]  

(2.19)

Taking trace of (2.18), we have

\[ 2 \alpha G_{\mu\nu}^{(1)} + 6 \left( \beta - \frac{2 \alpha}{3} \right) \partial^\nu v_\nu = T, \]  

(2.20)

with \( T = \eta^{\mu\nu} T_{\mu\nu} \). Both sides of (2.6) are divergenceless with respect to \( \nu \) owing to (2.13), while the divergence of (2.10) with respect to \( \nu \) yields (2.19) by virtue of (2.14). Therefore, Eqs. (2.6) and (2.10) give \((16 + 24) - (4 + 6) = 30\) independent equations for \(6 + 24 = 30\) independent field variables, i.e., for \( \hbar_{\mu\nu} \) satisfying (2.16) and \( K_{\mu\nu} \).

As is seen from (2.6) and (2.10), six parameters \( a_i \) \((i = 1, 2, \ldots, 6)\) enter the linearized gravitational field equations through the six combinations, \(3a_1 + 4a_2, 3a_2 + 2a_3, a_4 + a_5, 2a_5 + a_6, a_5 + 12a_6\) and \( a_1 + a_3 \): These combinations are not independent, however, and the condition

\[ 3a_1 + 4a_2 + 2a_3 - a_4 + a_5 + 2a_5 + a_6 = a_5 + 12a_6 = a_1 + a_3 = 0, \]  

(2.21)

is satisfied when the parameters \( a_i \) are given by

\[ a_2 = \frac{4}{3} a_1, \quad a_3 = - a_1, \quad a_4 = - 2a_1, \quad a_5 = \frac{1}{6} a_1, \quad a_6 = a_1 \]  

(2.22)

with \( a_1 \) arbitrary. For this choice the quadratic Lagrangian density \( L_\rho \) of (I.3.23) becomes

\[ L_\rho = a_1 \left[ (A_{ijmn} A^{ijmn}) + \frac{4}{3} (B_{ijmn} B^{ijmn}) - (C_{ijmn} C^{ijmn}) \right. \]

\[ + 2 (E_{ij} E^{ij} - I_{ij} I^{ij}) + \frac{1}{6} F^2 \right] \]

\[ = a_1 (F_{ijmn} F^{mnij} - 4 F_{ij} F^{ij} + F^2). \]  

(2.23)

Thus, the variation of the action \( A_\rho \).
identically vanishes in the weak field approximation: Namely, we have

$$\delta A = 0 \quad (2.25)$$

in the weak field approximation. This property of the Lorentz gauge field strength $F_{\mu\nu}$ is to be compared to the Bach-Lanczos identity in the Riemann space-time,

$$\delta \int d^4x \left[ R_{\mu\nu\rho\sigma} (\{\}) R^{\rho\sigma\nu\mu} (\{\}) - 4R_{\mu\rho} (\{\}) R^\nu (\{\}) + R (\{\}) \right] = 0. \quad (2.26)$$

[Note added in proof: It can be shown that the identity (2.25) does hold exactly. See V of this series.]

§ 3. The Newtonian approximation

Using (2.18) and (2.20) in the divergence of (2.10),

$$\partial^\mu P_{\mu(\rho)} = -\frac{1}{2} \partial^\nu S_{\nu(\rho)} \quad (3.1)$$

we obtain the fourth-order field equation for $T_{\mu\nu}$,

$$2\alpha G_{\mu\nu} - 2\alpha \left( \frac{1}{m_0} \left[ \Box G_{\mu\nu} - \frac{1}{3} (\partial_\rho \partial_\rho) G_{\mu\nu} \right] \right) - 2\alpha \left( \frac{1}{m_0} \right) \left[ \partial_\rho \partial_\rho \right] G_{\mu\nu} = T^{(\text{eff})}_{\mu\nu} \quad (3.2)$$

where $m_0$ and $m_0$ are given by

$$m_0 = \left\{ \begin{array}{c} \frac{2a (\alpha + 2a/3)}{\alpha (\alpha + 2a/3)} \end{array} \right\}^{1/2} \quad (3.3a)$$

and $T^{(\text{eff})}_{\mu\nu}$ is defined by

$$T^{(\text{eff})}_{\mu\nu} = T^{(\text{sym})}_{\mu\nu} - \frac{a_0 + 12a_0}{9 (\beta - 2a/3)} (\partial_\rho \partial_\rho) T \left( \begin{array}{c} \frac{3a_1 + 2a_0}{3 (\alpha + 2a/3)} \left[ \Box T_{\rho\sigma} - \frac{1}{3} (\partial_\rho \partial_\rho) T - \partial^\rho \partial^\rho \partial_\rho \partial_\rho \right] \end{array} \right) \quad (3.4)$$

with $T^{(\text{sym})}_{\mu\nu}$ the symmetrized energy-momentum tensor in special relativity,

$$T^{(\text{sym})}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} \partial^\rho (S_{\rho\mu} + S_{\rho\nu} + S_{\rho\nu}). \quad (3.5)$$
In the harmonic gauge of (2.16), the field equation for $\overline{h}_{\mu\nu}$ given by (3.2) becomes

$$\begin{align*}
a \square \overline{h}_{\nu} - a \left( \frac{1}{m_{1}^{2}} \right) \left\{ \left( \square \overline{h}_{\nu} - \frac{1}{3} \left( \varphi_{\nu} \square - \partial_{\lambda} \partial_{\nu} \right) \overline{h} \right) \right\} \right.

&= - \frac{a}{3 \left( m_{1}^{2} \right)} \left( \varphi_{\nu} \square - \partial_{\lambda} \partial_{\nu} \right) \overline{h} = - T^{(\text{eff})}_{\nu},

\end{align*}$$

which contains the fourth-order differential operators like $\square^{2}$ and $\square \partial_{\lambda} \partial_{\nu}$, operated on $\overline{h}_{\nu}$. Here $\overline{h} = \gamma^{\nu} \overline{h}_{\nu}$.

Let us consider the gravitational field around a static, spinless source located at the origin, for which $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \begin{cases} M \delta^{\mu}(x), & \mu = \nu = 0, \\ 0, & \text{otherwise} \end{cases}$$

with $x = (x^{1}, x^{2}, x^{3})$ and $S_{\mu\nu}$ is vanishing. It then follows from (3.4) that

$$T^{(\text{eff})}_{\mu\nu} = M \delta^{\mu}(x) - \frac{a_{5} + 12 a_{4}}{9 (\beta - 2 a/3)} + \frac{2 (3 a_{4} + 2 a_{1})}{9 (\alpha + 2 a/3)} M \delta^{\mu}(x),$$

$$T^{(\text{eff})}_{\mu\nu} = 0,$$

$$T^{(\text{eff})}_{\mu\nu} = \left\{ \begin{array}{ll}
\frac{a_{5} + 12 a_{4}}{9 (3 \beta - 2 a/3)} - \frac{3 a_{5} + 2 a_{4}}{9 (\alpha + 2 a/3)} M \partial_{\mu} \partial_{\nu} - \partial_{\mu} \partial_{\nu} \partial^{\mu}(x), & \\
\frac{a_{5} + 12 a_{4}}{9 (3 \beta - 2 a/3)} - \frac{3 a_{5} + 2 a_{4}}{9 (\alpha + 2 a/3)} M \partial_{\mu} \partial_{\nu} - \partial_{\mu} \partial_{\nu} \partial^{\mu}(x), & \\
\frac{a_{5} + 12 a_{4}}{9 (3 \beta - 2 a/3)} - \frac{3 a_{5} + 2 a_{4}}{9 (\alpha + 2 a/3)} M \partial_{\mu} \partial_{\nu} - \partial_{\mu} \partial_{\nu} \partial^{\mu}(x), & \\
\frac{a_{5} + 12 a_{4}}{9 (3 \beta - 2 a/3)} - \frac{3 a_{5} + 2 a_{4}}{9 (\alpha + 2 a/3)} M \partial_{\mu} \partial_{\nu} - \partial_{\mu} \partial_{\nu} \partial^{\mu}(x), & \\
\end{array} \right.$$
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the \((1/r)\)-term of \(\phi\) coincides with the Newtonian potential, \(\phi = -GM/r\), and that the Yukawa potentials are of sufficiently short range. It is convenient to treat two cases separately according as the parameters, \(a_i\) \((i = 1, 2, \ldots, 6)\), are finite or not.

(i) **The case that the parameters \(a_i\) are all finite**

When \(m_2\) and \(m_3\) vanish, the strength of the Yukawa potentials vanishes because of the factors, \((m_2)^2\) and \((m_3)^2\), in \(R_2\) and \(R_3\). Therefore, the first term of \(\phi = -M/16\pi ar\), must coincide with the Newtonian potential, irrespectively of whether \(m_2\) and \(m_3\) are vanishing or not. We thus choose the parameter \(a\) as

\[ a = 1/2\kappa, \quad \kappa = 8\pi G. \quad (3.11) \]

The remaining two terms of \(\phi\) are possible corrections to the Newtonian potential, and they should be sufficiently small compared with the first term for macroscopic distances. So we choose the remaining parameters as follows: (1) The masses, \(m_2\) and \(m_3\), are large enough to ensure that the exponential factors of the Yukawa potentials rapidly die out, or (2) the coefficients, \(R_2\) and \(R_3\), are sufficiently small. In view of (3.10a and b), we see that the Yukawa potentials are attractive or repulsive according as \(-\alpha (a + 2a/3)\) and \(\beta (\beta - 2a/3)\) are positive or negative, respectively. It should be mentioned that, for example, the massive spin-2 particle with \(\alpha (a + 2a/3) > 0\) yields the repulsive force and becomes ghost.*

The case of \(a = 0\) deserves special care, since it is assumed in (3.9) that the parameter \(a\) is nonvanishing. By letting the parameter \(a\) tend to zero in (3.6) and then solving the resulting equation with respect to \(h_{00}\), we have the following gravitational potential,

\[ \phi = -\frac{1}{2} h_{00} = \left( \frac{1}{\alpha} + \frac{1}{4\beta} \right) \frac{M}{18\pi r} - \frac{1}{3a_2 + 2a_5} + \frac{1}{4(a_5 + 12a_6)} \frac{Mr}{12\pi}, \quad (3.12) \]

which contains the **linearly rising potential** in addition to the \((1/r)\)-potential.**

This linearly rising potential stands for the characteristic feature of the case of \(a = 0\) with \(a_i\) finite. As the basic postulate we demand that the space-time around a static localized source should be asymptotically flat. This requires that the linearly rising potential should be vanishing,

\[ \frac{1}{3a_2 + 2a_5} + \frac{1}{4(a_5 + 12a_6)} = 0. \quad (3.13) \]

Furthermore, since \(h = \eta^{00} h_{00}\) is given by

\[ \eta^{00} h_{00} = \frac{1}{3a_2 + 2a_5} + \frac{1}{4(a_5 + 12a_6)} = 0. \quad (3.13) \]

* The massive particles of spin 2 and spin 0 are normal particles with positive-definite energy and positive mass, when the parameters satisfy the conditions,

\[
\begin{align*}
3a_2 + 2a_5 > 0, & \quad \alpha (\alpha + 2a/3) < 0 \quad \text{for the spin-2 particle,} \\
a_5 + 12a_6 > 0, & \quad \beta (\beta - 2a/3) > 0 \quad \text{for the spin-0 particle.}
\end{align*}
\]

This will be shown in a next paper (see §4 of IV).
\[ h = \frac{M}{6\pi\beta r} - \frac{Mr}{4\pi(a_i + 12a_6)} \] (3.14)

we must put

\[ \frac{1}{a_i + 12a_6} = 0. \] (3.15)

Equations (3.13) and (3.15) contradict the assumption that the parameters \( a_i \) are finite numerical constants. Consequently, we conclude that when the parameters \( a_i \) are finite the case of \( a = 0 \) should be disregarded.\(^*\)

(ii) The case of \( a_i = \infty \) \((i = 1, 2, \ldots, 6)\)

This case is just New General Relativity\(^b\) as was shown in §4 of II. Since \( m_2 \) and \( m_6 \) are vanishing with \( R_2 \) and \( R_4 \) finite, all the three terms of the gravitational potential (3.9) behave like \( 1/r \), giving the following gravitational potential,

\[ \phi = \left( \frac{1}{\alpha} + \frac{1}{4\beta} \right) \frac{M}{18\pi r}, \] (3.16)

which coincides with the Newtonian potential if we put

\[ \alpha + 4\beta + 9\alpha_0\beta = 0. \] (3.17)

This is just the Newtonian approximation condition in New General Relativity. (See also (II. 4.22).)

Since we have studied the latter case in Ref. 2, we shall assume henceforth throughout this series (unless otherwise stated explicitly) that the parameter \( a \) is given by (3.11).

§ 4. Propagation of the torsion field

Let us now derive the field equations for the torsion field by using Eqs. (2.18) and (2.19) in the linearized field equation for the Lorentz gauge field (2.10) with (2.12). For this purpose, it is more convenient to rewrite (2.10) by decomposing \( P_{\mu\nu} \) into three irreducible parts, \( P_{\mu\nu}^{(i)} \), \( P_{\mu\nu}^{(e)} \) and \( P_{\mu\nu}^{(a)} \), in the same manner as the spin tensor (see the Appendix of II for the decomposition scheme of the spin tensor). Denoting the irreducible parts of the spin tensor as \( S_{\mu\nu}^{(i)} \), \( S_{\mu\nu}^{(e)} \) and \( S_{\mu\nu}^{(a)} \), we have

\[ \frac{1}{2} (3a_2 + 4a_6) \left( \partial^\rho t_{\rho\mu\nu} - (\partial^\rho t_{\mu(\rho)} + \partial^\rho t_{\nu(\rho)} - \partial^\rho t_{\rho(\mu)} + 2\partial^\rho t_{\rho(\nu)} + \partial^\rho t_{\rho(\mu)} + \partial^\rho t_{\rho(\nu)}) \right) \]
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\[ -\frac{2}{3} (\partial^\alpha t_{\rho\gamma})_{\ell} + \partial^\alpha t_{\rho\gamma}, \ell \]
\[ -\frac{1}{4} (\eta_{\alpha\beta} \partial^\alpha \partial^\beta t_{\rho\gamma} + \eta_{\alpha\beta} \partial^\alpha t_{\rho\gamma} \ell - 2 \eta_{\alpha\beta} \partial^\beta t_{\rho\gamma}) \]
\[ - (3a_2 + 2a_3) \left\{ \frac{1}{4} \left( G_{\rho\gamma}^{(1)} + \frac{1}{2} G_{\rho\gamma}^{(2)} \right) \right\} \]
\[ - \frac{1}{12} (\eta_{\alpha\beta} G^{(1)} + \eta_{\alpha\beta} \partial^\alpha \partial^\beta G^{(1)}) \]
\[ - \frac{1}{2} (\partial^\alpha t_{\rho\gamma}, \ell + \partial^\alpha t_{\rho\gamma}, \ell - 2 \partial^\beta t_{\rho\gamma}) \]
\[ + \frac{1}{6} (\partial^\alpha t_{\rho\gamma}, \ell + \partial^\alpha t_{\rho\gamma}, \ell) \]
\[ - \frac{1}{18} (a_4 + a_5) \left\{ \frac{1}{3} \left( \nabla_{\rho} - \frac{3}{2} \partial^\alpha \partial^\beta t_{\rho\gamma} \right) \right\} \]
\[ + \frac{1}{2} \left( \partial_{\rho\gamma} \partial_{\rho\gamma} \frac{3}{2} \partial^\alpha \partial^\beta t_{\rho\gamma} \right) \]
\[ - 6 \partial^{\alpha} t_{\rho\gamma}, \ell + \partial t_{\rho\gamma}, \ell \]
\[ - 3 (v_{\rho\gamma}, \rho + v_{\rho\gamma}, \rho - 2 v_{\rho\gamma}, \rho) \]
\[ + \frac{1}{4} (2a_3 + a_4) \left( \nabla_{\rho} - \frac{3}{2} \partial^\alpha \partial^\beta t_{\rho\gamma} \right) \frac{3}{2} \frac{a + 2a}{3} t_{\rho\gamma} \]
\[ = - \frac{1}{2} S_{\rho\gamma}^{(1)} \] (4.1)
\[ \frac{2}{3} (a_4 + a_5) \left( \nabla_{\rho} - \frac{3}{2} \partial^\alpha \partial^\beta t_{\rho\gamma} \right) \frac{3}{2} \frac{a + 2a}{3} t_{\rho\gamma} \]
\[ - (a_5 + 12a_3) \left( \frac{1}{2} \partial_{\rho} G^{(1)} - \partial_{\rho} \partial_{\rho} t_{\rho\gamma} \right) = - \frac{1}{2} S_{\rho\gamma}^{(1)} \] (4.2)
\[ \frac{2}{3} (2a_3 + a_4) \left( \nabla_{\rho} + \frac{2}{3} \nabla_{\rho} \partial_{\rho}, \partial_{\rho} t_{\rho\gamma} \right) \frac{2}{3} \frac{3a + a - a}{3} \partial_{\rho} \partial_{\rho} t_{\rho\gamma} \]
\[ \frac{4}{3} \left( \nabla_{\rho} + \frac{3a}{2} \right) a_3 = - \frac{1}{2} S_{\rho\gamma}^{(1)} \] (4.3)

A. Reduction of Eq. (4.2)

Using (2.20) in the divergence of (4.2), we get
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\[ \square - (m_s^2) \sigma = j^{(e)} \tag{4.4} \]

with \( \sigma = \overline{\sigma} \gamma_5 \) and

\[ m_0 = \left\{ \frac{2a}{\beta (a_3 + 12a_4)} \right\}^{1/2} \tag{4.5} \]

\[ j^{(e)} = \frac{1}{6 \beta} \Box T - \frac{a}{3 \beta (a_3 + 12a_4)} \partial^\mu S^{(e)}_\mu. \tag{4.6} \]

The field \( \sigma \) describes a massive particle having spinparity \( J^P = 0^+ \). The mass of the \( \sigma \) coincides with \( m_0 \) defined by (3.3b), showing that the Yukawa potential with the range \( 1/m_0 \) in the gravitational potential (3.9) is due to \( \sigma \)-exchange.

By virtue of (4.4) and the divergence of (2.18), Eq. (4.2) can be rewritten as

\[ \square - m_s^2 \overline{\nu}_\mu = j^{(r)}_\mu, \tag{4.7} \]

where \( \overline{\nu}_\mu, m_\nu \) and \( j^{(r)}_\mu \) are defined by

\[ \overline{\nu}_\mu = \nu_\mu - (m_\nu)^{-2} \partial_\mu \sigma, \tag{4.8} \]

\[ m_\nu = \left\{ \frac{9 (\alpha + 2a/3) (\beta - 2a/3)}{2 (\alpha + \beta) (a_3 + a_4)} \right\}^{1/2} \tag{4.9} \]

\[ j^{(r)}_\mu = - \frac{m_\nu^2}{6 (\beta - 2a/3)} S^{(e)}_\mu + \frac{1}{2 (\alpha + \beta)} \partial^\sigma \partial^\rho S^{(e)}_{\mu\sigma \rho} \]

\[ + \frac{1}{6 \beta} \frac{m_\nu^2}{m^4} \partial_\mu T - (m_\nu)^{-2} \partial^\rho j^{(e)} \tag{4.10} \]

The field \( \overline{\nu}_\mu \) is divergenceless in vacuum by virtue of (4.4) (see also the Appendix), so it describes a massive particle with \( J^P = 1^- \).

B. Reduction of Eq. (4.3)

The divergence of (4.3) gives

\[ \square - m_a^2 B = j^{(B)} \tag{4.11} \]

with \( B = \overline{\sigma} a_\mu \), and

\[ m_a = \left\{ \frac{2 (\gamma + 3a/2)}{3 (a_1 + a_3)} \right\}^{1/2} \tag{4.12} \]

\[ j^{(B)} = - \frac{1}{4 (a_1 + a_3)} \partial^\mu S^{(0)}_\mu. \tag{4.13} \]

The field \( B \) describes a massive particle with \( J^P = 0^- \).

Using (2.19) and (4.11) in (4.3), we have

\[ \square - m_a^2 \overline{a}_{\mu} = j^{(a)}_\mu \tag{4.14} \]

with \( \overline{a}_\mu, m_a \) and \( j^{(a)}_\mu \) defined by
The field $\tilde{\alpha}$ is divergenceless in vacuum (see the Appendix), and therefore it describes a massive particle with $J^0 = 1$.

C. Reduction of Eq. (4·1)

Use of (2·18) in the divergence of (4·1) with respect to $\varphi$ gives after a straightforward calculation,
\[ \Box \Box \varphi = (m_0)^2 \chi_{\varphi \varphi} = f^{(e)}_{\varphi \varphi}, \]  
where $\chi_{\varphi \varphi}$, $m_0$ and $f^{(e)}_{\varphi \varphi}$ are given by
\[ \chi_{\varphi \varphi} = (m_0)^2 \left\{ \Box \Box \varphi - \frac{\beta - 2a/3}{3(\alpha + 2a/3)} \partial_\mu \partial^\mu \varphi \right\}, \]  
\[ m_0 = \left\{ \frac{-2a(\alpha + 2a/3)}{\alpha(3a + 2a)} \right\}^{1/2}, \]  
\[ f^{(e)}_{\varphi \varphi} = \frac{(m_0)^2}{6(\alpha + 2a/3)} \left\{ \Box \Box S_{\varphi \varphi}^{(e)} - \frac{1}{3} \partial_\mu S_{\varphi \varphi}^{(e)} \right\} \frac{\delta \varphi}{\delta \varphi} T^{(e)} - \frac{2a}{3(\alpha + 2a/3)} \partial_\mu \partial^\mu S_{\varphi \varphi}^{(e)} \]  
\[ - \frac{\beta - 2a/3}{3(\alpha + 2a/3)} (m_0)^2 \frac{\Box \Box \varphi}{\Box \Box \varphi} f^{(e)}_{\varphi \varphi}. \]  

The field $\chi_{\varphi \varphi}$ is symmetric by definition, and furthermore it is traceless and divergenceless in vacuum (see the Appendix): Consequently, the $\chi_{\varphi \varphi}$ describes a massive particle with $J^0 = 2$. The mass of $\chi_{\varphi \varphi}$ coincides with $m_0$ defined by (3·3a), and so the Yukawa potential with the range $1/m_0$ in the gravitational potential (3·9) is due to $\chi$-exchange.

The field $\chi_{\varphi \varphi}$ is the "traceless" and "divergenceless" part of $\partial^\mu \varphi_{\varphi \varphi}$: As for the antisymmetric field $\vartheta t_{\varphi \varphi \varphi}$, Eq. (2·19) shows that it is represented by $\varphi_\varphi$ and $a_\varphi$, and so we need not be concerned with it.

Next, let us extract the "traceless" and "divergenceless" part of $t_{\varphi \varphi \varphi}$. This is performed by the following three steps: (i) Eliminate $G^{(e)}_{\varphi \varphi}$ and $G^{(e)}_{\varphi}$ in (4·1) by using (2·18), (ii) represent $\vartheta t_{\varphi \varphi \varphi}$ and $\vartheta t_{\varphi \varphi \varphi}$ in terms of $\chi_{\varphi \varphi}$, $\vartheta_\varphi$ and $\partial_\varphi$, and finally, (iii) rewrite a resulting expression for (4·1) by using the Klein-Gordon equations for $\chi_{\varphi \varphi}$, $\vartheta_\varphi$ and $\partial_\varphi$, so that it becomes a Klein-Gordon equation. As a result, we have
where \( \mathbf{I}_{\mu
u} \), \( m_1 \), and \( j_{i\mu}^{[\nu]} \) are defined by

\[
\mathbf{I}_{\mu\nu} = t_{\mu\nu} - (m_2)^{-1} (\chi_{\mu\nu} + \chi_{\nu\mu}) - 2 \chi_{[\mu\nu]},
\]

\[
+ \frac{\beta - 2a/3}{\alpha + 2a/3} \left\{ \frac{1}{6} (\eta_{\mu\nu} \mathbf{\nabla}_\rho + \eta_{\nu\rho} \mathbf{\nabla}_\mu - 2 \eta_{\rho\mu} \mathbf{\nabla}_\nu) + m_2^{-1} (\mathbf{\nabla}_\mu \mathbf{j}_\nu - \mathbf{\nabla}_\nu \mathbf{j}_\mu) \right\},
\]

\[
+ \frac{\gamma + 3a/2}{3(\alpha + 2a/3)} m_2^{-1} (\varepsilon_{\lambda\mu\nu} \partial_{\rho} + \varepsilon_{\rho\lambda\mu} \partial_{\nu}) \partial^2 \mathbf{A}^\nu,
\]

\[
m_1 = \left[ \frac{3(\alpha + 2a/3)}{3a_2 + 4a_3} \right]^{1/3}. \tag{4.24}
\]

\[
j_{i\mu}^{[\nu]} = \frac{1}{3a_2 + 4a_3} \left[ - S_{\mu\nu}^{[\rho]} + \frac{3a_2 + 2a_3}{4a} (T_{i(\mu)} + T_{i(\nu)}, \mathbf{j} - 2 T_{i(\rho)}, \mathbf{j}) \right.
\]

\[
- \frac{9a_2 + 8a_3 - 2a_4}{6(\alpha + 2a/3)} (T_{i(\mu)}, \mathbf{j} + T_{i(\nu)}, \mathbf{j}) \right.
\]

\[
- \frac{2a_5 + a_4 - 4a_6}{8a} (\eta_{\mu\nu} T_{i}, \mathbf{j} - 2 \eta_{\nu\mu} T_{i}, \mathbf{j})
\]

\[
- \frac{1}{12} (\eta_{\mu\nu} S_{\rho}^{(e)} + \eta_{\rho\mu} S_{\nu}^{(e)} - 2 \eta_{\nu\rho} S_{\mu}^{(e)})
\]

\[
+ \frac{1}{12(\alpha + 2a/3)} \partial^\rho \partial^\sigma (\eta_{\mu\nu} S_{\rho\sigma}^{e} + \eta_{\rho\nu} S_{\mu\sigma}^{e} - 2 \eta_{\rho\mu} S_{\nu\sigma}^{e})
\]

\[
- (m_2)^{-1} (j_{i\mu}^{[\nu]}, \mathbf{j} + j_{i\mu}^{[\nu]}, \mathbf{j} - 2 j_{i\rho}^{[\nu]}, \mathbf{j})
\]

\[
- \frac{\beta - 2a/3}{\alpha + 2a/3} \left\{ \frac{1}{m_2} (j_{i\mu}^{[\nu]}, \mathbf{j} + j_{i\mu}^{[\nu]}, \mathbf{j} + \frac{1}{6(\alpha + 2a/3)} (\eta_{\mu\rho} + \eta_{\rho\mu} j_{i\nu}^{(e)} + 2 \eta_{\nu\rho} j_{i\mu}^{(e)})) \right\}
\]

\[
+ \frac{\gamma + 3a/2}{3m_2^3(\alpha + 2a/3)} (\varepsilon_{\lambda\mu\nu} \partial_{\rho} + \varepsilon_{\rho\lambda\mu} \partial_{\nu}) \partial^2 j_{i\rho}^{(e)}; \tag{4.25}
\]

The field \( \mathbf{I}_{\mu\nu} \) is symmetric with respect to \( \lambda \) and \( \mu \), and satisfies the cyclic identity,

\[
\mathbf{I}_{\mu\nu} + \mathbf{I}_{\nu\lambda} + \mathbf{I}_{\lambda\mu} = 0. \tag{4.26}
\]

As for the trace and divergence, it is \textit{traceless and divergenceless in vacuum} (see the Appendix): Therefore, the field \( \mathbf{I}_{\mu\nu} \) involves five independent degrees of freedom, and describes a massive particle with \( J^\mu = 2 \). For a particle at rest, we have

\[
\mathbf{I}_{\mu\nu} = \mathbf{I}_{\mu\lambda} = \mathbf{I}_{\lambda\nu} = 0, \tag{4.27}
\]

and the space-components \( t_{\alpha\beta} \) \( (\alpha, \beta, \gamma = 1, 2, 3) \) are expressed in terms of five inde-
pendent quantities, for example, \( I_{13}, I_{88}, I_{19}, I_{31}, \) and \( I_{38,} \). We have thus seen that the torsion field can be decomposed into the six independent fields, \( I_{13}, \psi_{\alpha}, \phi_{\alpha}, \sigma = \partial^\alpha v_\alpha \), and \( B = \partial^\alpha a_\alpha \), all of which obey the Klein-Gordon equations. Each of these fields is irreducible in the sense that it is traceless and divergenceless in vacuum, thus describing a massive particle with definite spin-parity. It is still to be clarified, however, whether these six classes of the irreducible fields are normal (with positive mass and positive energy) or abnormal (with imaginary mass and/or negative energy). The energy of these irreducible fields will be calculated in a next paper of this series (see § 4 of IV).

§ 5. The massless graviton field

We shall now extract from \( H_{\mu} \) the massless graviton field. For this purpose, we write (2·18) in terms of the irreducible torsion fields \( \chi_\sigma \) and \( \sigma \),

\[
2aG^{(\gamma)}_{\mu} - 6 \left( \alpha + \frac{2a}{3} \right) \chi^{\alpha} + 2 \left( \beta - \frac{2a}{3} \right) \left( \eta_{\alpha} - (m^2) \right) \sigma = T^{(\alpha)}_{\mu}. \tag{5·1}
\]

Let us define \( \tilde{H}_{\mu} \) by

\[
\tilde{H}_{\mu} = H_{\mu} + \frac{6}{a(m^2)^2} \left( \frac{\alpha + 2a}{3} \right) \chi^{\alpha} - 2 \left( \frac{\beta - 2a}{3} \right) \left( \eta_{\alpha} - (m^2) \right) \sigma,
\]

and rewrite (5·1) by using \( \tilde{H}_{\mu} \) in place of \( H_{\mu} \). It follows from (2·8) and the above definition of \( \tilde{H}_{\mu} \) that the fields \( \chi_\sigma \) and \( \sigma \) appear in (5·1) in the forms, \( \{ \square - (m^2)^2 \} \chi_\sigma \), \( \{ \square - (m^2) \} \sigma \) and \( \partial^\alpha \chi_\sigma \), showing that \( \chi_\sigma \) and \( \sigma \) can be eliminated by means of their field equations, (4·4) and (4·18); The terms with \( \partial^\alpha \chi_\sigma \) are eliminated by using (A·8). We then get the following field equation for \( \tilde{H}_{\mu} \):

\[
\square \tilde{H}_{\mu} - \partial^\alpha \left( \partial_\mu \chi^{\alpha} + \partial_\alpha \tilde{H}^{\alpha}_{\mu} + \chi_\sigma \partial^\alpha \tilde{H}^{\sigma}_{\mu} \right) = \psi_\sigma \partial^\alpha \tilde{H}^{\sigma}_{\mu},
\]

where \( T^{(\psi_{\mu})}_{(\nu)} \) is given by (3·5). The source term of (5·3) can be simplified by introducing the new field variable \( \phi_{\mu} \) by

\[\text{For the spin component } J_4 \text{ along the 3rd axis, } J_4 = \pm 2 \text{ for } (I_{313} \pm (i/3)(I_{313} + I_{313})), \text{ and } J_4 = \pm 1 \text{ for } (I_{313} \pm iI_{313}) \text{ and } J_4 = 0 \text{ for } (I_{313} - I_{313}).\]
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\[ \phi_{\mu\nu} = \frac{\alpha}{2\alpha^2} \left( T_{\mu\nu} - \frac{1}{6} \eta_{\mu\nu} T \right) + \frac{a_5 + 12a_6}{12\alpha^2} \eta_{\mu\nu} T \]

\[ - \frac{\beta}{a(m_3)^2} \left[ \frac{1}{(m_3)^2} - \frac{1}{\eta_{\mu\nu} j^{(\sigma)}} \right] \]

\[ = T_{\mu\nu} - \frac{3\alpha(3a_5 + 2a_3)}{a_3^2} \chi^*_{\mu\nu} - \frac{\beta(a_5 + 12a_6)}{a_3^2} \left( \eta_{\mu\nu} - (m_3) T_{\mu\nu} j^{(\sigma)} \right) \]

with \( \chi^*_{\mu\nu} \) defined by

\[ \chi^*_{\mu\nu} = \frac{1}{6\alpha} \left( T_{\mu\nu} - \frac{1}{36\alpha} \left( 1 + \frac{a_5 + 12a_6}{3a_5 + 2a_3} \right) \eta_{\mu\nu} T \right) \]

\[ + \frac{\beta}{6(\alpha + 2a_3/3)(m_3)^2} \eta_{\mu\nu} j^{(\sigma)}. \]

In fact, it can be shown that the field \( \phi_{\mu\nu} \) obeys

\[ \Box \phi_{\mu\nu} - \eta^{\iota\sigma} (\partial_\iota \phi_{\mu\nu} + \partial_\iota \phi_{\mu\nu}) + \eta_{\mu\nu} \partial^\iota \phi_{\sigma\iota} = -\frac{1}{a} T_{\mu\nu}^{(\text{sym})} \]

\[ = -2\alpha T_{\mu\nu}^{(\text{sym})}, \]

where we have used (3.11) in the last term. This equation (5.6) is just the linearized Einstein equation, and so we can interpret the field \( \phi_{\mu\nu} \) as the massless graviton field of spin 2.

It follows from the field equation for \( \chi_{\mu\nu} \) that the field \( \chi^*_{\mu\nu} \) obeys the following Fierz-Pauli equation:

\[ \Box \chi^*_{\mu\nu} - \eta^{\iota\sigma} (\partial_\iota \chi^*_{\mu\nu} + \partial_\iota \chi^*_{\mu\nu}) + \eta_{\mu\nu} \partial^\iota \chi^*_{\sigma\iota} = \frac{1}{6\alpha} \left( T_{\mu\nu}^{(\text{sym})} + \frac{2\alpha}{3\alpha} T_{\mu\nu}^{(\text{sym})} \right) \]

\[ + \frac{(m_3)^2}{6(\alpha + 2a_3/3)(m_3)^2} \left( 1 + \frac{a_5 + 12a_6}{3a_5 + 2a_3} \right) \eta_{\mu\nu} T \]

\[ - \frac{\beta(a_5 + 12a_6)}{6\alpha(3a_5 + 2a_3)} \eta_{\mu\nu} j^{(\sigma)}. \]

in which the source term does not involve second-order derivatives of \( T_{\mu\nu} \) but only \( \Box T \) through \( j^{\iota\sigma} \). By rewriting the Klein-Gordon equation for \( \chi_{\mu\nu} \), (4.18), into a form similar to (5.7), on the other hand, we see that the source term of \( \chi_{\mu\nu} \) contains \( \Box T_{\mu\nu} \) as well as \( \Box T \). Since higher derivative terms in the source are dangerous in quantized theory, the field \( \chi^*_{\mu\nu} \) seems to be preferable to \( \chi_{\mu\nu} \) as the dynamical field variable.

The field equation (5.6) is invariant under gauge transformations,

\[ \phi_{\mu\nu} \rightarrow \phi'_{\mu\nu} = \phi_{\mu\nu} + \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} - \eta_{\mu\nu} \partial^\sigma A_{\sigma} \]

(5.8)
with $A_*$ small but otherwise arbitrary four functions. Owing to this gauge freedom, we can set the divergenceless condition,

$$\nabla^a \phi_{ab} = 0.$$  \hspace{1cm} (5.9)

Then Eq. (5.6) becomes the d'Alembert equation,

$$\Box \phi_{ab} = -2\kappa T_{ab}^{\text{sym}}.$$  \hspace{1cm} (5.10)

The weak gravitational field $\tilde{h}_{ab}$ is represented as a linear sum of the three classes of fields, $\phi_{ab}$, $\chi_{ab}$, and $\sigma$,

$$\tilde{h}_{ab} = \phi_{ab} + \frac{3a(3a^2 + 2a_3)}{a^2} \chi_{ab} + \frac{\beta(a_5 + 12a_9)}{a^2} (\eta_{ab} - (m_4 - 1) \partial_a \partial_b) \sigma$$

$$+ \frac{\beta - 2a/3}{a(m_4)^2} \eta_{ab} j^{(4)},$$  \hspace{1cm} (5.11)

showing that the field $\tilde{h}_{ab}$ is the multirass field. This is also the reason why $\tilde{h}_{ab}$ obeys the fourth-order field equation (3.6).

§ 6. Conclusion

We have applied the weak field approximation to the most general gravitational field equations derived in Poincaré gauge theory with the linear and quadratic Lagrangian densities. Adopting the conventional method, the linearized field equation for the linearized gravitational field $h_{ab}$ was of fourth order, showing that $h_{ab}$ is the multirass field, which is given by the graviton with $m = 0$, and the particles with $m = m_2$ and $m = m_0$. This fact was also viewed from the Newtonian approximation method, where there are two Yukawa potentials in addition to the famous Newtonian potential.

On the other hand, there are the six irreducible fields in the torsion field, which have $J^p = \text{spin-parity}$ as $2^-, 2^+, 1^-, 1^+, 0^+$ and $0^-$, each of which satisfies the Klein-Gordon equation. Thus, the previous multirass field can be redefined as the purely genuine field $\phi_{ab}$ of spin 2 and mass 0, obeying the usual wave equation of second order, whose source term is exactly the symmetrized energy-momentum tensor appearing in the linearized Einstein field equation: So we call it the massless graviton field of spin 2. The two additional fields are called $\chi_{ab}$ and $\sigma$, which have $J^p = 2^-$ and mass $m_3$ and $J^p = 0^+$ and mass $m_0$, respectively. We denote in Table I the massless graviton field in terms of $\tilde{h}_{ab}$, $\chi_{ab}$ and $\sigma$, and the six irreducible torsion fields in terms of the original irreducible torsion fields, $t_{ab}$, $\nu_{a}$ and $\alpha_{a}$. These irreducible torsion fields might be normal (i.e., the energy is positive-definite and the mass is positive) or abnormal, depending on parameters $a_1$ involved. This issue will be reported in a forthcoming paper of this series; IV. Mass and energy of particle spectrum.
Table I. Massless and Massive fields in Poincare gauge theory. In the definition of \( \phi_m \) the matter terms are omitted here (see (5.4)).

<table>
<thead>
<tr>
<th>Fields</th>
<th>(Mass) ( a(m_0)^2 )</th>
<th>( J^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_m = \phi_m + \frac{6(\alpha + 2a/3)}{a(m_0)^2} \chi_m )</td>
<td>0</td>
<td>2^*</td>
</tr>
<tr>
<td>( -2(\beta - 2a/3) a(m_0)^2 \left( \psi_m - \frac{\partial \phi_m}{a(m_0)^2} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma = \sigma \psi )</td>
<td>( (m_0)^2 = \frac{2a(\beta - 2a/3)}{\beta(a_1 + 12a)} )</td>
<td>0^*</td>
</tr>
<tr>
<td>( \psi \psi = \psi \psi - (m_0)^2 \partial \sigma )</td>
<td>( m_\psi = \frac{2(\alpha + 2a/3)(\beta - 2a/3)}{3(a + 2a/3)} )</td>
<td>1^*</td>
</tr>
<tr>
<td>( \chi_m = \chi_m = \frac{2 \alpha - 2a/3}{3(\alpha + 2a/3)} \psi \psi \psi \psi )</td>
<td>( m_\chi = \frac{-2a(\alpha + 2a/3)}{\alpha(3a_1 + 2a)} )</td>
<td>2^*</td>
</tr>
<tr>
<td>( B = B \psi )</td>
<td>( m_B = \frac{-2(\tau + 3a/2)}{3(a_1 + a)} )</td>
<td>0^*</td>
</tr>
<tr>
<td>( \alpha = \alpha - (m_0)^2 \partial \beta )</td>
<td>( m_\alpha = \frac{2(\alpha + 2a/3)(\tau + 3a/2)}{3(a_1 + 4a)} )</td>
<td>1^*</td>
</tr>
<tr>
<td>( \kappa = k \psi \psi - \frac{2}{(m_0)^2} (\partial \psi \psi - \partial \chi_m) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \partial \chi_m = \frac{-2a - 2a/3}{\alpha + 2a/3} \left( \psi \psi - \psi \psi \psi \psi \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \partial \beta \psi \psi - \psi \psi \psi \psi )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \partial \psi \psi = \frac{2(\tau + 3a/2)}{3(\alpha + 2a/3)m_\psi} \beta \psi \psi \psi \psi )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Appendix**

---

**Divergence and Trace of the Irreducible Fields, \( \psi, \chi, \alpha, \beta \) and \( \kappa \)**

(i) The fields \( \psi \) and \( \alpha \):

Taking the divergence of (4.10) and (4.17), we get

\[
\delta^\mu j_\mu^{(\psi)} = - \left[ \partial - m_\psi \right] j^{(\psi)}/(m_0)^2, \tag{A.1}
\]

\[
\delta^\mu j_\mu^{(\alpha)} = - \left[ \partial - m_\alpha \right] j^{(\alpha)}/m_\alpha^2. \tag{A.2}
\]

Therefore, \( \delta^\mu \psi \) and \( \delta^\mu \alpha \) are frozen at the place of matter, given by

\[
\delta^\mu \psi = - j^{(\psi)}/(m_0)^2, \tag{A.3}
\]

\[
\delta^\mu \alpha = - j^{(\alpha)}/m_\alpha^2. \tag{A.4}
\]

(ii) The field \( \chi \):

It follows from (4.21) that

\[
\delta^\mu j^{(\chi)} = - \left[ \partial - (m_2)^2 \right] \frac{(\beta - 2a/3) j^{(\psi)}}{3(\alpha + 2a/3)(m_0)^2}. \tag{A.5}
\]
and
\[
\partial^\pi j_{\mu\nu}^{(\pi)} = \left[ \Box - (m^2) \right] \left\{ \frac{1}{12(\alpha + 2a/3)} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{(\beta - 2a/3)}{3(\alpha + 2a/3)} (m^2) j^{(\pi)} \right\},
\]
(A-6)

showing that \( \eta_{\mu\nu} \partial^\mu \partial^\nu S_{\mu\nu} \) cannot propagate in vacuum:
\[
\eta_{\mu\nu} \partial^\mu \partial^\nu S_{\mu\nu} = - \frac{(\beta - 2a/3)}{3(\alpha + 2a/3)} (m^2) j^{(\pi)},
\]
(A-7)
\[
\partial^\pi j_{\mu\nu}^{(\pi)} = \frac{1}{12(\alpha + 2a/3)} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{(\beta - 2a/3)}{3(\alpha + 2a/3)} (m^2) j^{(\pi)}.
\]
(A-8)

(iii) The field \( I_{\mu\nu} \):
From (4.25) we have
\[
\rho^{\pi/\alpha} j_{\mu\nu}^{(\pi)} = - 2\rho^{\pi/\alpha} j_{\mu\nu}^{(\pi)}
\]
\[
= \Box - m_i \left\{ \left( \beta - 2a \right) \left( j^{(\pi)} + \frac{\partial^\pi j^{(\pi)} - 1}{6 (m_i)^2} \partial^\pi \partial^\mu S_{\mu\nu} \right) \right\},
\]
(A-9)
\[
\partial^\pi j_{\mu\nu}^{(\pi)} = \Box - m_i \left\{ \frac{1}{12(\alpha + 2a/3)} \left\{ \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - 6 T_{\mu\nu} \right\}
\right. \]
\[
+ \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
+ \frac{18}{m_i^2} \partial^\pi \partial^\rho S_{\rho\nu} - \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\rho S_{\rho\nu} - \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
+ \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{2}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{2}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
\left. \left\{ \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - 6 T_{\mu\nu} \right\} \right\}
\]
(A-10)
\[
\partial^\pi j_{\mu\nu}^{(\pi)} = \Box - m_i \left\{ \frac{1}{12(\alpha + 2a/3)} \left\{ \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - 6 T_{\mu\nu} \right\}
\right. \]
\[
+ \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
+ \frac{18}{m_i^2} \partial^\pi \partial^\rho S_{\rho\nu} - \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\rho S_{\rho\nu} - \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
+ \frac{6}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{2}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - \frac{2}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu}
\]
\[
\left. \left\{ \partial^\pi \partial^\mu S_{\mu\nu} - \frac{18}{m_i^2} \partial^\pi \partial^\mu S_{\mu\nu} - 6 T_{\mu\nu} \right\} \right\}
\]
(A-11)

Since \( \Box - m_i^2 \) is factored out in these expressions, the trace and the divergence of \( I_{\mu\nu} \), i.e., \( \partial^\pi I_{\mu\nu} \), \( \partial^\pi I_{\mu\nu} \), and \( \partial^\pi I_{\mu\nu} \), are frozen at the place of matter: The expressions for them can easily be derived from (A-9) ~ (A-11).

References

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