What Operator Corresponds to the Observable Field-Strength in the Weinberg-Salam Model?

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It is investigated in canonical formalism what operator corresponds to the actually observable electro-magnetic field strength in the Weinberg-Salam model. It is pointed out that the naive definition
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon_{\mu \nu \alpha \beta} \partial_\alpha A_\beta \]
with
\[ A_\mu = (g^2 + g'^2)^{-1/2} \left( g W_{\mu}^a + g' W_{\mu}^{a'} \right) \]
is not BRS invariant and cannot meet the requirement of observability. A possible candidate operator is proposed.

Canonical operator formalism of non-Abelian gauge theories, formulated in a manifestly covariant way by Kugo and Ojima, made it possible to discuss many important problems directly related to the Heisenberg operators and the state vectors which was difficult to treat in the more familiar path-integral formalism. Ojima initiated the task to analyze the concept of observables in gauge theories in the framework of this formalism. By imposing on observables a weak condition called 'gauge independence', he proved that any local observable \( O \) is BRS-invariant:
\[ \partial (O) \equiv [iQ, O] = 0. \] (1)

Recently, Kugo and the author clarified the general form of BRS-invariant local operators.

Electro-magnetic fields are observables actually measurable in classical theory or quantum optics. In Quantum Electrodynamics (QED) it simply means that the field strength \( F_{\mu \nu} \) is a gauge-invariant (BRS-invariant) operator and a (strictly) local observable:
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial (F_{\mu \nu}) = 0. \] (2)

In the Weinberg-Salam (W-S) model (or in other unified models based on non-Abelian gauge groups), however, the situation becomes more complicated. In fact the photon field \( A_\mu \) in the W-S model is usually supposed to be a linear combination of the third-component of the \( SU(2) \) gauge fields \( W_{\alpha} \) and the \( U(1) \) gauge field \( W_\mu^0 \):
\[ A_\mu = (g^2 + g'^2)^{-1/2} \left( g W_{\mu}^a + g' W_{\mu}^{a'} \right). \]

The BRS-transformations of these gauge fields \( W_{\alpha} \) and \( W_\mu^0 \) are given by
\[ \partial (W_{\mu}^a) = \partial_\mu c^a + g t^{a \beta} W_\beta^b c^b, \]
\[ \partial (W_\mu^0) = \partial_\mu c^0, \]
where \( c^a \) and \( c^0 \) are \( SU(2) \) and \( U(1) \) FP-ghost fields, respectively. Notice that the BRS-transformation of the field \( A_\mu \) in Eq. (3) is not identical to that of the usual QED \((U(1))\) gauge field:
\[ \partial (A_\mu) = (g^2 + g'^2)^{-1/2} \left[ \partial_\mu (g c^a + g' c^0) + g' g (W_\mu^a c^0 - W_\mu^0 c^a) \right]. \] (6)

This fact causes a trouble; that is, if we naively define the electro-magnetic field strength \( F_{\mu \nu} \) as
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
we meet a difficulty that \( F_{\mu \nu} \) becomes BRS non-invariant and hence becomes unobservable:
\[ \partial (F_{\mu \nu}) \neq 0. \] (8)

In this short note we discuss how we...
should construct the observable operator \( G_{\mu \nu} \) corresponding to the actual electromagnetic field strength.

We require the following natural conditions for the operator \( G_{\mu \nu} \):

i) \( G_{\mu \nu} \) is the antisymmetric 2nd-rank Lorentz tensor and has mass dimension two.

ii) \( G_{\mu \nu} \) is a strictly local observable:

\[
\hat{\delta}(G_{\mu \nu}) = 0.
\]

iii) \( G_{\mu \nu} \) has zero FP-ghost number:

\[
[iQ_{\mu}, G_{\mu \nu}] = 0.
\]

iv) \( G_{\mu \nu} \) satisfies the first pair of Maxwell's equation

\[
\epsilon^{\mu \nu \rho \sigma} \partial_{\rho} G_{\sigma \lambda} = 0,
\]

where \( G_{\mu \nu} = \tilde{G}_{\mu \nu} + \hat{\delta} \) (some operator) and \( \tilde{G}_{\mu \nu} \) stands for the part not expressible as the BRS-transform of any operators. We now show that these conditions restrict the form of the operator \( G_{\mu \nu} \) to a considerable extent. First, from Eq. (11) \( \tilde{G}_{\mu \nu} \) can be written as a rotation form of some vector operator \( \hat{\gamma} X_{\mu} \) by virtue of the converse of Poincaré's lemma:

\[
\tilde{G}_{\mu \nu} = \partial_{\mu} X_{\nu} - \partial_{\nu} X_{\mu}.
\]

Since the observability condition (9) and \( G_{\mu \nu} = \tilde{G}_{\mu \nu} + \hat{\delta} \) (some operator) imply

\[
\hat{\delta}(G_{\mu \nu}) = \partial_{\nu} \hat{\delta}(X_{\mu}) - \partial_{\mu} \hat{\delta}(X_{\nu}) = 0,
\]

\[
\hat{\delta}(X_{\mu}) \text{ can be written as a gradient form of some scalar operator } \hat{\gamma} \phi \text{ by CBL:}
\]

\[
\hat{\delta}(X_{\mu}) = \partial_{\mu} \phi.
\]

Further Eq. (14) implies

\[
\hat{\delta}(\phi) = 0.
\]

On the other hand, since \( \tilde{G}_{\mu \nu} \) is BRS-invariant and not expressible as the BRS-transform of any operators, \( \tilde{G}_{\mu \nu} \) should be group-singlet, and so should be \( X_{\mu} \) and \( \phi \). Taking into account that \( \phi \) is a BRS-invariant scalar operator having FP-ghost number one, we can write \( \phi \) as

\[
\phi = c^0 \cdot H + \hat{\delta}(M),
\]

where \( H \) is a BRS-invariant scalar operator and not expressible as the BRS-transform of any operators. Hence from Eqs. (5) and (16) we have

\[
\partial_{\mu} \phi = \hat{\delta}(\omega_{\mu} H + \partial_{\mu} M) + c^0 \cdot \partial_{\mu} H.
\]

This and Eq. (14) imply \( H = \text{const} = c^0 \). Accordingly we have

\[
\hat{\delta}(X_{\mu} - \alpha \omega_{\mu} - \partial_{\mu} M) = 0,
\]

which implies

\[
X_{\mu} = \alpha \omega_{\mu} + \partial_{\mu} M + 3H_{\mu},
\]

where \( H_{\mu} \) is a BRS-invariant vector operator and composed solely of gauge fields and matter ones. Moreover if \( H_{\mu} \) contains the gauge fields \( W_{\mu}^a \) and \( \omega_{\mu}^a \), they must appear only in the following forms:

\[
\begin{align*}
F_{\rho \sigma} &= \partial_{\rho} W_{\sigma}^a - \partial_{\sigma} W_{\rho}^a + g \epsilon^{abc} W_{\rho}^b W_{\sigma}^c, \\
D_{\mu}^a &= \epsilon^{abc} \partial_{\rho} W_{\rho}^b + g \epsilon^{abc} W_{\mu}^b, \\
\partial_{\mu} &= \partial_{\rho} - i W_{\rho}^a \cdot (g T)^a,
\end{align*}
\]

where \( T^a \) is the generators of the group, and we have adopted compact notations such as \( \langle \omega_{\mu}^a \rangle_{\sigma} = \omega_{\rho}^a \) and \( \epsilon^{abc} = 0 \). Consequently \( G_{\mu \nu} \) is given as

\[
G_{\mu \nu} = \alpha (\partial_{\mu} \omega_{\nu}^a - \partial_{\nu} \omega_{\mu}^a) + \partial_{\mu} H_{\nu} - \partial_{\nu} H_{\mu},
\]

where we have omitted the unphysical term \( \hat{\delta} \) (some operator). Here some remarks are in order. (i) Equation (23) shows that the field strength \( G_{\mu \nu} \) should be given as a rotation of 'physical photon' field.
which itself is group-singlet. So, since
$F_\mu^a$ presented in Eq. (7) is given by $A_\mu^{a+}$ containing explicitly the third-component
of the $SU(2)$ gauge fields, it cannot correspond to the physical field strength $G_\mu^a$
in Eq. (23). (ii) Although Eq. (23) is given in the W-S model, i.e., the $SU(2) \times U(1)$
gauge theory, similar equation holds in any unified gauge theory based
on a compact Lie group $G$. In that case $w_\mu^a$ is interpreted as gauge fields belonging
to the $U(1)$ subgroups in $G$ and $H_\mu^a$ is group $G$-singlet. Especially when $G$ is
simple or semi-simple, instead of the first term of r.h.s. holds since no $U(1)$
gauge fields $w_\mu^a$ are there. (iii) In such models as the W-S model in which no
BRSt-invariant fundamental vector field operators exist, $H_\mu^a$ has to be a composite
operator. (iv) Similar model as the W-S model in which no
BRST-invariant fundamental vector field operators exist, $H_\mu^a$ has to be a composite
operator. Next we give an explicit form of $A_\mu^a$
or $H_\mu^a$ as a candidate for the physical photon field in the W-S model on the
basis of some physical consideration. Recall that the usually adopted form Eq. (3)
\begin{equation}
A_\mu^0 = \alpha_0 w_\mu^0 + H_\mu^0
\end{equation}
in unitary gauge really has all the properties of the physical photon field except
for the BRSS singletness property; in fact, in unitary gauge, $A_\mu^0$ represents the massless field which couples to (almost) minimal
$U(1)$ current. Hence it is natural to suppose that the BRSS singlet $A_\mu^a$ becomes identical with apparently BRSS non-singlet
$A_\mu^{a+}$ in unitary gauge. This is actually possible. Recalling that the $SU(2)$ doublet
Higgs field $\Phi$ takes the form
\begin{equation}
\Phi = \left( \begin{array}{c} 0 \\ v + \phi \end{array} \right),
\end{equation}
we notice that
\begin{equation}
\phi = \frac{\Phi}{(\Phi^\dagger \Phi)^{1/2}}
\end{equation}
becomes \begin{pmatrix} 0 \\ 1 \end{pmatrix} in unitary gauge. Thus the following choice for $A_\mu^a$
coincides with $A_\mu^{a+}$ in unitary gauge:
\begin{equation}
A_\mu^a = \alpha_0 w_\mu^0 + i \alpha_0 \phi^\dagger \phi,
\end{equation}
where
\begin{equation}
\alpha_0 = (g^2 + g'^2)^{1/2} / g,
\end{equation}
\begin{equation}
\alpha_1 = -2g' / (g^2 + g'^2)^{1/2},
\end{equation}
\begin{equation}
\phi = \theta \mu^0 - \frac{ig'}{2} w_\mu^0 - \frac{ig}{2} \phi^\dagger W_\mu^a.
\end{equation}
Then we give
\begin{equation}
G_\mu^a = \alpha_0 \left( \partial_\mu w_\mu^0 - \partial_\mu w_\mu^a \right)
+ i \alpha_0 \left( \partial_\mu \phi^\dagger \phi - \partial_\mu (\phi^\dagger \phi) \right).
\end{equation}
From Eq. (30) the second pair of Maxwell's equations\footnote{\textsuperscript{11}} is given
\begin{equation}
\partial_\gamma G_\mu^a = J_\gamma^a + \sigma (M_\gamma^a),
\end{equation}
where
\begin{equation}
J_\mu^a = -\alpha_0 j_\mu^a + \langle \alpha_0 g'^2 / 2 \rangle j_\mu^0
- \langle \alpha_0 g'^2 / 2 \rangle \left( \phi^\dagger \phi \right) j_\mu^a
+ \alpha_0 \left( \langle \partial_\mu \phi^\dagger \phi \rangle \right) j_\mu^0
- \left( \langle \partial_\mu \phi \rangle \phi^\dagger \phi \right) j_\mu^a
+ \left( \langle \partial_\mu \phi \rangle \phi \phi^\dagger \phi \right) j_\mu^0
- \left( \langle \partial_\mu \phi \rangle \phi \phi^\dagger \phi \right) j_\mu^a,
\end{equation}
\begin{equation}
M_\mu^a = -i \alpha_0 \partial_\mu \phi^\dagger - \langle \alpha_0 g'^2 / 2 \rangle \partial_\mu \phi^\dagger
- i \alpha_0 g' \left( \phi^\dagger \phi \right) \partial_\mu \phi^\dagger
- i \alpha_0 g' \left( \phi \phi^\dagger \phi \right) \partial_\mu \phi^\dagger.
\end{equation}
$J_\mu^a$ and $J_\mu^0$ are respectively the $SU(2)$ and
$U(1)$ subgroup matter currents:
\begin{equation}
2i j_\mu^a = \partial_\mu \phi^\dagger \phi - \phi^\dagger \phi \partial_\mu \phi
+ i g' \left( \phi \phi^\dagger \phi \phi^\dagger \phi \right)
+ \sum \text{fermions} f_L \tilde{f}_L f_L,
\end{equation}
\begin{equation}
2i j_\mu^0 = \partial_\mu \phi^\dagger \phi - \phi^\dagger \phi \partial_\mu \phi
+ i g' \left( \phi \phi^\dagger \phi \phi^\dagger \phi \right)
+ \sum \text{fermions} y_L \tilde{y}_L y_L,
\end{equation}
Notice the $J_\mu^a$ in Eq. (31) is gauge invarient and in unitary gauge it becomes
almost minimal electro-magnetic current:
\[ J_{\mu}^{\text{uni}} = \frac{g'g}{\sqrt{g^2 + g'^2}} \sum_{\text{fermions}} Q_f \tilde{f}_\mu f \]

\[ = \frac{g'g}{2\sqrt{g^2 + g'^2}} (2W_{\mu} + W_{-\mu} - \frac{1}{2}W_{\mu}^2 W_{-\mu}) \]

\[ = \frac{igg'}{2\sqrt{g^2 + g'^2}} (W_{-\mu}^\dagger \partial_\mu W_{\mu} - \frac{1}{2}W_{\mu}^2 W_{-\mu}) \]

\[ = \frac{1}{2} \left( W_{-\mu}^\dagger \partial_\mu W_{\mu} - \frac{1}{2}W_{\mu}^2 W_{-\mu} \right) - \frac{1}{2} \left( W_{\mu}^\dagger \partial_\mu W_{-\mu} - \frac{1}{2}W_{-\mu}^2 W_{\mu} \right) \]

\[ = \frac{1}{2} \left( W_{-\mu}^\dagger \partial_\mu W_{\mu} - \frac{1}{2}W_{\mu}^2 W_{-\mu} \right) \]

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where

\[ W_{\mu} = 2^{-1/2} (W_{\mu}^\dagger + iW_{\mu}) \]

\[ Q_f = \text{[the value of } I_1 + Y/2 \text{ of fermion]} \]

From Eqs. (31) and (36), we may regard Eq. (30) as a satisfactory candidate for the actual field strength. One, however, might find some problem in Eq. (25), i.e., in a point that we have used non-polynomial operator \( \hat{d} \) to give \( G_{\mu\nu} \) in Eq. (30). If we took \( G_{\mu\nu} \) in the following form to avoid such non-polynomial operator,

\[ G_{\mu\nu} = \alpha_1 \left( \partial_\mu \omega_{\nu} - \partial_\nu \omega_{\mu} \right) \]

\[ + i\alpha_2 \left( \epsilon_{\mu\nu\rho\sigma} \partial_\rho \omega_{\sigma} - \partial_\sigma \omega_{\rho} \right) \]

with suitable coefficients \( \alpha_1 \) and \( \alpha_2 \), we would have obtained an equation similar to Eq. (30) but we could not have obtained such a desirable equation as Eq. (36).

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