Progress of Theoretical Physics, Vol. 64, No. 6, December 1980

Renormalization Groups of Gell-Mann and Low and of Callan and Symanzik

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(Received July 8, 1980)

The renormalization group of Gell-Mann and Low and that of Callan and Symanzik are similar in form but conceptually quite different. Introduction of a reinterpretation of the parameters in the Callan-Symanzik equations enables us to relate Green's functions in this approach to those of Gell-Mann and Low.

§ 1. Introduction

The idea of the renormalization group is rather old. Its existence was first suggested by Stueckelberg and Petermann,[1] and then its explicit formulation was given by Gell-Mann and Low.[2] Its mathematical formulation was further elucidated by Bogoliubov and Shirkov.[3]

When we formulate field theory in terms of Green's functions the physical content of the theory does not depend on where the renormalization point is chosen. The explicit expressions of Green's functions depend, however, on the renormalization point. Thus we introduce scale transformations of the renormalization point which is of the dimension of mass and study the response of Green's functions. This group of scale transformations is then called the renormalization group (RG). In the next section we shall briefly recapitulate the formulation of the RG.

Recently a different version of the RG was introduced by Callan[4] and Symanzik[5] (CS). They have studied the response of Green's functions under an infinitesimal variation of the bare mass by keeping the bare coupling constant and the cut-off momentum fixed. These two approaches are conceptually different in that the physical mass is held fixed in the former while it is varied in the latter. Furthermore, the Callan-Symanzik equations are inhomogeneous because of the presence of the mass-insertion term in a sharp contrast to the homogeneous Gell-Mann-Low (GML) equations. In § 3 we shall show how we can transform the CS equations into a form akin to the GML equations by introducing a new interpretation of the basic parameters appearing in the RG.

In the last section we shall find an explicit transformation connecting Green's functions of these two approaches.

§ 2. The renormalization group of Gell-Mann and Low

In this section we shall briefly recapitulate the formulation of the RG à la Gell-Mann and Low for the purpose of fixing the notation. As a model we shall consider the neutral scalar theory with the ϕ^4 coupling.

In order to fix the normalization of Green's functions we have to give a set of prescriptions. First of all we introduce a parameter μ of the dimension of mass and properly normalize the propagator at $p^2+\mu^2=0$. The coupling constant $g(\mu)$ is then introduced as the value of the one-particle-irreducible four-point Green's function for $p_ip_j=(\mu^2/3)(1-4\delta_{ij})$, $(i, j=1, 2, 3, 4)$. These prescriptions are sufficient to determine all the Green's functions uniquely. The n-point Green's function is denoted by

$$G^{(n)}(p_i; \mu, g(\mu)). \quad (i=1, 2, \cdots, n) \tag{2·1}$$

When the parameter μ is equal to the physical mass m_P, we have the standard mass-shell renormalization and we define

$$g_P=g(m_P). \tag{2·2}$$

The RG argument yields the following relationship:

$$G^{(n)}(p_i; m_P, g_P)=Z(\mu)^{-n/2}G^{(n)}(p_i; \mu, g(\mu)). \tag{2·3}$$

The RG is defined as the group of scale transformations of the parameter μ. Equation (2·3) exhibits the characteristic features of the RG that the change of the scale of μ simply induces the change of the coupling constant and that of the overall normalization of Green's functions.

In order to express the scale change of μ we shall introduce a new parameter ρ by

$$\rho=\ln\left(\frac{\mu}{m_P}\right), \tag{2·4}$$

and rewrite Eq. (2·3) as

$$G^{(n)}(p_i; m_P, g_P)=Z(\rho)^{-n/2}G^{(n)}(p_i; \mu(\rho), g(\rho)). \tag{2·5}$$

The left-hand side of this equation is independent of ρ so that the derivative of the right-hand side with respect to ρ must vanish. As functions of ρ, $\mu(\rho)$, $g(\rho)$ and $Z(\rho)$ apparently satisfy the boundary conditions

$$\mu(0)=m_P, \quad g(0)=g_P, \quad Z(0)=1. \tag{2·6}$$

Now g is a function of ρ and g_P, so that its derivative with respect to ρ is also a function of ρ and g_P. Since g_P can be expressed in terms of ρ and g, $\partial g/\partial\rho$ can be regarded as a function of μ/m_P and g.

$$\frac{\partial}{\partial\rho}g=\beta\left(g,\ \frac{\mu}{m_P}\right). \tag{2·7}$$

Similarly we have

$$\frac{\partial}{\partial\rho}\ln z^{-1}=2\gamma_\phi\left(g,\ \frac{\mu}{m_P}\right). \tag{2·8}$$

With the help of these functions the vanishing of the derivative of Eq. (2·5) with respect to ρ yields

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\left(g,\ \frac{\mu}{m_P}\right)\frac{\partial}{\partial g}+n\gamma_\phi\left(g,\ \frac{\mu}{m_P}\right)\right)G^{(n)}(p_i;\ \mu,g)=0\,. \tag{2·9}$$

We define a differential operator $\mathscr{D}$ by

$$\mathscr{D}=\mu\frac{\partial}{\partial\mu}+\beta\left(g,\ \frac{\mu}{m_P}\right)\frac{\partial}{\partial g}, \tag{2·10}$$

then the infinitesimal change of Q, a function of g and μ, under the RG is given by

$$\delta Q=(\mathscr{D}Q)\,\delta\rho, \tag{2·11}$$

where $\delta\rho$ denotes the infinitesimal parameter of the RG.

§ 3. The renormalization group of Callan and Symanzik

In the RG mentioned in the preceding section the physical mass and coupling constant, m_P and g_P, are held fixed and only the renormalization point μ is varied. On the contrary, both the physical mass and coupling constant are varied in the CS equations, and for the reason to be mentioned later we shall simply denote them by m and g, respectively.

The CS equation for the n-point Green's function reads

$$\begin{aligned}\left(m\frac{\partial}{\partial m}+\beta(g)\frac{\partial}{\partial g}+n\gamma_\phi(g)\right)&G^{(n)}(p_i;\ m,g)\\ &=\varDelta G^{(n)}(p_i;\ m,g)\,.\end{aligned} \tag{3·1}$$

The right-hand side denotes the so-called mass-insertion term obtained by inserting $m^2\phi^2$ into the n-point Green's function, and we can generalize it to the j-fold mass-insertion term

$$\varDelta^j G^{(n)}(p_i;\ m,\ g)\,. \tag{3·2}$$

The generalized CS equation is given by

$$\begin{aligned}\left(m\frac{\partial}{\partial m}+\beta(g)\frac{\partial}{\partial g}+n\gamma_\phi(g)-j(2+\gamma_s(g))\right)&\varDelta^j G^{(n)}(p_i;\ m,g)\\ &=\varDelta^{j+1}G^{(n)}(p_i;\ m,g)\,.\end{aligned} \tag{3·3}$$

The RG equations of Gell-Mann and Low, (2·9), are homogeneous, but the corresponding CS equations (3·3) are inhomogeneous. In order to remedy this defect we unify the infinite set of inhomogeneous equations (3·3) by introducing a generating function of the multiple mass-insertion terms.

$$G^{(n)}(p_i; m, g, K) = \sum_{j=0}^{\infty} \frac{K^j}{j!} \Delta^j G^{(n)}(p_i; m, g). \tag{3·4}$$

This generating function satisfies a homogeneous equation[6)]

$$\left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} + n\gamma_\phi(g) - \{(2+\gamma_s(g))K+1\}\frac{\partial}{\partial K}\right)G^{(n)}(p_i; m, g, K)$$
$$=0. \tag{3·5}$$

It is true that this equation is homogeneous, but we have introduced three para meters m, g and K as compared with the two parameters μ and g in Eq. (2·9).

Next we define a differential operator $\overline{\mathscr{D}}$ by

$$\overline{\mathscr{D}} = m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \{(2+\gamma_s(g))K+1\}\frac{\partial}{\partial K}, \tag{3·6}$$

then the infinitesimal transformation of the RG of the CS type induces the following change of Q, a function of the parameters m, g and K:

$$\delta Q = (\overline{\mathscr{D}}Q)\,\delta\rho. \tag{3·7}$$

For example, we have

$$\begin{aligned} \delta m &= m\delta\rho, \\ \delta g &= \beta(g)\,\delta\rho, \\ \delta K &= -\{(2+\gamma_s(g))K+1\}\,\delta\rho. \end{aligned} \tag{3·8}$$

The solutions of Eqs. (3·8) under the initial conditions

$$\bar{g}(0) = g, \quad \overline{K}(0) = K \tag{3·9}$$

are denoted by me^ρ, $\bar{g}(\rho)$ and $\overline{K}(\rho)$, respectively. The equation corresponding to Eq. (2·3) in the present approach reads

$$G^{(n)}(p_i; m, g, K)$$
$$= \exp\left(n\int_0^\rho \gamma_\phi(\bar{g}(\rho'))\,d\rho'\right)G^{(n)}(p_i; me^\rho, \bar{g}(\rho), \overline{K}(\rho)). \tag{3·10}$$

We are going to choose a special value of ρ, denoted by $\bar{\rho}$ and defined by

$$\overline{K}(\bar{\rho}) = 0. \tag{3·11}$$

Then $\bar{\rho}$ is a function of g and K, and the propagator for this choice of ρ turns out

to be given by

$$G^{(2)}(p_i;\, me^{\bar\rho}, \bar g(\bar\rho), 0),$$

so that it has a pole at

$$-p^2 = m^2 e^{2\bar\rho}. \tag{3·12}$$

The single particle mass $me^{\bar\rho}$ is a function of m, g and K. So far we have considered that the physical mass is originally given by m but is shifted to $me^{\bar\rho}$ by the introduction of an external source K.

From now on we shall consider, instead, that

$$m_P = me^{\bar\rho} \tag{3·13}$$

denotes the fixed physical mass, and m represents a movable renormalization point. This interpretation is supported by

$$\bar{\mathscr{D}} m_P = 0, \tag{3·14}$$

which is a consequence of the fact that m_P is the pole of the propagator[7] as indicated by Eq. (3·12).

Substitution of Eq. (3·13) in (3·14) gives

$$\bar{\mathscr{D}} \bar\rho = -1. \tag{3·15}$$

As a matter of fact, the function $\bar\rho$ is determined by this equation along with the boundary condition that $\bar\rho=0$ for $K=0$. Integration of the second equation in Eq. (3·8) with the boundary condition (3·9) yields

$$\bar\rho = \int_g^{\bar g(\bar\rho)} \frac{dx}{\beta(x)}. \tag{3·16}$$

Apply $\bar{\mathscr{D}}$ to the above equation and use Eq. (3·15) to find

$$\begin{aligned}-1 &= \bar{\mathscr{D}}\bar\rho \\ &= \frac{1}{\beta(\bar g(\bar\rho))}\bar{\mathscr{D}}\bar g(\bar\rho) - \frac{1}{\beta(g)}\bar{\mathscr{D}} g \\ &= \frac{1}{\beta(\bar g(\bar\rho))}\bar{\mathscr{D}}\bar g(\bar\rho) - 1.\end{aligned}$$

Hence we obtain

$$\bar{\mathscr{D}} \bar g(\bar\rho) = 0. \tag{3·17}$$

Namely, the on-shell coupling constant $g_P = \bar g(\bar\rho)$ is also an invariant of the RG. The next important step is to regard K as a function of g and $\bar\rho$, or rather a function of g and m_P/m. Then Green's functions can be expressed in favor of the

parameters p_i, m, g and m_P rather than in p_i, m, g and K. We put

$$
\begin{aligned}
&G^{(n)}\left(p_i;\, m, g, K\left(g, \frac{m}{m_P}\right)\right)\\
&\qquad \equiv G^{(n)}(p_i;\, m, g, m_P)\,.
\end{aligned}
\tag{3·18}
$$

In view of Eq. (3·14) the Green's function above can be shown to satisfy the following equation:

$$
\begin{aligned}
&(\bar{\mathscr{D}} + n\gamma_\phi(g))\,G^{(n)}(p_i;\, m, g, m_P)\\
&= \left((\bar{\mathscr{D}}m)\left(\frac{\partial}{\partial m}\right)_{g, m_P} + (\bar{\mathscr{D}}g)\left(\frac{\partial}{\partial g}\right)_{m, m_P} + (\bar{\mathscr{D}}m_P)\left(\frac{\partial}{\partial m_P}\right)_{g, m}\right.\\
&\qquad \left. + n\gamma_\phi(g)\right)G^{(n)}(p_i;\, m, g, m_P)\\
&= \left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} + n\gamma_\phi(g)\right)G^{(n)}(p_i;\, m, g, m_P)\\
&= 0\,.
\end{aligned}
\tag{3·19}
$$

Thanks to the reinterpretation of the basic set of parameters, this new RG equation involves only two parameters m and g just as in Eq. (2·9).

Finally, it might be instructive to give K as an explicit function of g and $\bar{\rho}$. Combination of Eqs. (3·8), (3·9) and (3·11) immediately leads us to

$$
K = \int_0^{\bar{\rho}} d\rho\, \exp\left(\int_0^{\rho} d\rho\,(2 + \gamma_s(\bar{g}(\rho)))\right),
\tag{3·20}
$$

where $\bar{g}(\rho)$ is defined by

$$
\rho = \int_g^{\bar{g}(\rho)} \frac{dg}{\beta(g)}\,.
\tag{3·21}
$$

§ 4. The relationship between the two renormalization groups

As we have shown in the preceding section the CS equations have been brought into a form very close to the GML equations (2·9), and we are ready to discuss their relationship.[8] Since Green's functions in these two approaches depend differently on the parameters, we shall distinguish between them by affixing subscripts GL and CS, respectively. Then the RG equations are

$$
\left(m\frac{\partial}{\partial m} + \beta\left(g, \frac{m}{m_P}\right)\frac{\partial}{\partial g} + n\gamma_\phi\left(g, \frac{m}{m_P}\right)\right)G^{(n)}_{\mathrm{GL}}(p_i;\, m, g) = 0\,,
\tag{2·9}
$$

$$
\left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} + n\gamma_\phi(g)\right)G^{(n)}_{\mathrm{CS}}(p_i;\, m, g) = 0\,.
\tag{3·19}
$$

In Eq. (2·9) above we have replaced μ by m, and in Eq. (3·19) above we have suppressed the parameter m_P in the Green's function. These two sets of Green's functions can be made to be related to each other by making the following ansatz:

$$G_{\mathrm{GL}}^{(n)}(p_i;\, m,\, g) = Z_2^{n/2} G_{\mathrm{CS}}^{(n)}(p_i;\, m,\, Z_1^{-1} Z_2^2 g)\,. \tag{4·1}$$

The factors Z_1 and Z_2 are functions of g and m/m_P, and when $m = m_P$ both sides of Eq. (4·1) reduce to the Green's functions renormalized on the mass-shell. Thus we have

$$Z_1 = Z_2 = 1\,, \qquad \text{for} \quad m = m_P\,. \tag{4·2}$$

In what follows we shall put

$$g' = Z_1^{-1} Z_2^2 g\,, \tag{4·3}$$

and define

$$\mathscr{D} = m\left(\frac{\partial}{\partial m}\right)_g + \beta\left(g,\, \frac{m}{m_P}\right)\left(\frac{\partial}{\partial g}\right)_m, \tag{4·4}$$

$$\bar{\mathscr{D}}' = m\left(\frac{\partial}{\partial m}\right)_{g'} + \beta(g')\left(\frac{\partial}{\partial g'}\right)_m. \tag{4·5}$$

The transformation of the parameters represented by Eq. (4·3) can be studied by assuming the following equality:

$$\mathscr{D} = \bar{\mathscr{D}}'. \tag{4·6}$$

We first observe that

$$\frac{\partial}{\partial g'} = \left(\frac{\partial g'}{\partial g}\right)^{-1} \cdot \frac{\partial}{\partial g}\,,$$

$$m\left(\frac{\partial}{\partial m}\right)_{g'} = m\left(\frac{\partial}{\partial m}\right)_g - m\left(\frac{\partial g'}{\partial m}\right)_g \left(\frac{\partial g'}{\partial g}\right)^{-1} \frac{\partial}{\partial g}\,.$$

Thus we have

$$\bar{\mathscr{D}}' = m\left(\frac{\partial}{\partial m}\right)_g + \left(\frac{\partial g'}{\partial g}\right)^{-1}\left(-m\frac{\partial g'}{\partial m} + \beta(g')\right)\frac{\partial}{\partial g}\,, \tag{4·7}$$

and the ansatz (4·6) leads to

$$\left[m\frac{\partial}{\partial m} + \beta\left(g,\, \frac{m}{m_P}\right)\frac{\partial}{\partial g}\right] g' = \beta(g') \tag{4·8}$$

or

$$\left[m\frac{\partial}{\partial m} + \beta\left(g,\, \frac{m}{m_P}\right)\frac{\partial}{\partial g}\right] F(g') = 1\,, \tag{4·9}$$

where

$$F(g')=\int^{g'}\frac{dx}{\beta(x)}\,. \tag{4·10}$$

These equations determine g' as a function of g and m/m_P. The boundary condition is given by

$$g'=g \qquad \text{for} \qquad m=m_P\,, \tag{4·11}$$

which is a consequence of Eqs. (4·2) and (4·3).

Then, by combining Eqs. (2·9), (3·19), (4·1) and (4·6), we finally arrive at the equation to determine Z_2:

$$\left[m\frac{\partial}{\partial m}+\beta\left(g,\frac{m}{m_P}\right)\frac{\partial}{\partial g}-2\gamma_\phi(g')+2\gamma_\phi\left(g,\frac{m}{m_P}\right)\right]Z_2=0\,. \tag{4·12}$$

Once $g'=Z_1^{-1}Z_2^2g$ and Z_2 are known, the relationship between the two sets of Green's functions is completely settled by Eq. (4·1).

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