Gravity from Poincaré Gauge Theory of the Fundamental Particles. IV

---Mass and Energy of Particle Spectrum---

Kenji HAYASHI and Takeshi SHIRAFUJI*

Institute of Physics, Tokyo University, Komaba, Tokyo 153
*Physics Department, Saitama University, Urawa, Saitama 338

(Received June 28, 1980)

Particle spectrum of Poincaré gauge theory is investigated in detail, in particular particle's mass and energy. Besides usual graviton, the torsion field gives rise to massive $2^+$, $2^-$, $1^+$, $1^-$, $0^+$ and $0^-$ particles, each of which obeys the Klein-Gordon equation. We find that there are at most three normal particles, defined by having positive mass and positive energy: namely, only nine possible triplets, $(2^+, 1^+, 0^-)$, $(2^+, 1^-, 0^+)$, $(2^-, 0^+, 0^+)$ and their parity conjugate, and $(1^+, 0^+, 0^-)$ and its parity conjugate and finally $(1^+, 1^+, 0^-)$. Further reduction to two and one particles is also possible provided $2^+$ and $2^-$ cannot coexist. Conditions of nine parameters, $\alpha, \beta, \gamma, a_1, a_2, \ldots, a_9$, for these normal particles to exist are also obtained.

§ 1. Introduction

In a previous paper called I, the most general gravitational field equations are derived from the Poincaré gauge-invariant action, which is linear in the Lorentz gauge field strength and quadratic in the translation and Lorentz gauge field strengths, with ten parameters, $a, \alpha, \beta, \gamma, a_1, a_2, \ldots, a_9$. In a succeeding paper called II, equations of motion for test bodies in Poincaré gauge theory are considered, together with various limits of these parameters, including General Relativity and New General relativity.

In a third paper called III, the weak field approximation is applied to the most general gravitational field equations, using the conventional method that the Lorentz gauge field $A$ is decomposed into the Ricci rotation coefficient $\mathcal{A}$ (which is given by first derivatives of the tetrad field) and the contorsion field $\mathcal{K}$.

$$A = \mathcal{A} + \mathcal{K}.$$ 

The fundamental fields are then the linearized gravitational field $h_{\mu\nu}$ and the contorsion field $K_{\mu
u}$. The former part, though modified by the presence of the torsion field, gives rise to the usual graviton which is massless and spin-parity is given by $2^+$. The latter part yields six massive particles of $2^+, 2^-, 1^+, 1^-, 0^+$ and $0^-$. It is, however, not known that these particles are normal; that is, particles obey the Klein-Gordon equation, having positive mass and positive energy. In III only the first point that these particles satisfy the Klein-Gordon equation was insured.
Here in the present paper we shall give conditions for massive particles of $2^+$, $2^-$, $1^+$, $1^-$, $0^+$ and $0^-$ to obtain that some of these particles acquire positive mass and positive energy.

In § 2 the usual procedure of defining the energy-momentum complex of matter and gravitation will be applied to the Poincaré gauge theory, following Weinberg and Wheeler et al. In § 3 the energy-momentum complex thus obtained is not symmetric, and hence its symmetrization is carried on. In § 4 conditions for these particles to have positive mass together with positive energy are then obtained, and some conclusions are also drawn. A final section will be devoted to conclusion.

§ 2. The energy-momentum complex

Let us construct the energy-momentum complex of the present theory from the most general gravitational field equations derived in I, generalizing the ordinary procedure employed in General Relativity. From the invariance of the gravitational action under the group of general coordinate transformations, follows the identity,

$$\partial_s (\varepsilon P_{\mu}^s + \varepsilon A_{\mu \nu} Q^{\mu \nu} - e P_{\mu}^s \partial_s e^a_\mu - e Q^{\mu \nu} \partial_s A_{\mu \nu} = 0)$$

(2.1)

with $\varepsilon = \det (e^a_\mu)$, where $P_{\mu}^s$ and $Q^{\mu \nu}$ are defined by

$$e P_{\mu}^s = e e^k_\mu P_k^s = - e^k_\mu (\partial e L_0 / \partial e^k_\mu),$$

(2.2a)

$$e Q^{\mu \nu} = - \partial e L_0 / \partial A_{\mu \nu}.$$  

(2.2b)

Here $L_0$ denotes the gravitational Lagrangian density, and $e^a_\mu$ and $A_{\mu \nu}$ are the tetrad and the Lorentz gauge fields, respectively.

We assume that the translation gauge field $a^s_\mu = e^s_\mu - \partial^s_\mu$ and the Lorentz gauge field are both vanishing at great distances from the finite system under study. (However, $a^s_\mu$ and $A_{\mu \nu}$ are not assumed to be small everywhere.) Let $P_{\mu \nu}^{(n)}$ denote that part of $P_{\mu \nu}^s$ which is $n$-th order in $a^s_\mu$ and $A_{\mu \nu}$: We shall use this notation also for any tensor formed of $a^s_\mu$ and $A_{\mu \nu}$. For example, the first-order part of the metric tensor, which we denote as $h_\mu \nu$, is given by

$$h_\mu \nu = g_\mu \nu = a_\mu a_\nu$$

(2.3)

with $a_\mu = \gamma_\mu a^s_\mu$. Indices on the first-order part of tensors can be raised and lowered with the Minkowski metric $\gamma$’s, whereas indices on true tensors must be raised and lowered with the metric $g$’s: Namely, $P_{\mu \nu}^{(1)} = \eta_\mu \eta P^{(1)}_{\mu \nu}$ and $P_{\mu \nu} = g_{\mu \nu} P_{\mu \nu}$. However, indices on the second- and higher-order parts should be raised and lowered neither with $\gamma$’s nor with $g$’s: For example,

$$P_{\mu \nu}^{(2)} = (g_{\mu \nu} P^{(1)}_{\mu \nu})^{(2)} = \eta_\mu \eta P^{(2)}_{\mu \nu} + h_\mu h_{\nu} P^{(2)}_{\mu \nu}.$$  

(2.4)

Since the identity (2.1) is valid to any order in $a^s_\mu$ and $A_{\mu \nu}$, the first-order
part $P^{(i)}_{\mu}$ obeys the identity,

$$\partial_\alpha P^{(i)}_{\mu} = 0.$$  (2.5)

We write the exact field equation for the translation gauge field as

$$P_{\mu}^{(i)} = T_{\mu} + t_{\mu},$$  (2.6)

where $T_{\mu}$ is the energy-momentum tensor of matter, appearing in the gravitational field equation (I.3.29), and the energy-momentum complex of gravitation, $t_{\mu}$, is defined by

$$t_{\mu} = - (P_{\mu} - P_{\mu}^{(i)}).$$  (2.7)

Then it follows from the identity (2.5) that the quantity defined by

$$\tau^\mu = \eta^{\mu\nu\rho\sigma} (T_{\nu\rho} + t_{\nu\rho})$$  (2.8)

satisfies the ordinary conservation law,

$$\partial_\nu \tau^\nu = 0,$$  (2.9)

which we can interpret as representing the conservation of the total energy and momentum of matter and gravitation. The $\tau^\mu$ is not a tensor, and it is not symmetric in $\mu$ and $\nu$, either. We shall call $\tau^\mu$ the total energy-momentum complex of matter and gravitation.

We have constructed in I the most general gravitational Lagrangian density $L_0$ with ten parameters, $\alpha, \beta, \gamma, a_1, a_2, \cdots, a_6$,

$$L_0 = a F + L_T + L_\psi,$$  (2.10)

where $F$ is the invariant linear in the Lorentz gauge field strength $F_{\lambda\mu\nu}$, $L_T$ is the quadratic forms of the translation gauge field strength $T_{\lambda\mu}$,

$$L_T = \alpha (t_{\lambda\mu} e^{\lambda\mu}) + \beta (v_i e^i) + \gamma (a_i a_i'),$$  (2.11)

with $t_{\lambda\mu}, v_i$ and $a_i$ the three irreducible parts of $T_{\lambda\mu}$, and $L_\psi$ is the quadratic forms of $F_{\lambda\mu\nu}$ with six parameters, $a_1, a_2, \cdots, a_6$ (see Appendix A for $L_\psi$). The $P_{\mu}$ and $Q^{\mu\nu} = e^\kappa Q_k^{\mu\nu}$ are then given by

$$P_{\mu} = 2 a F_{\lambda\mu} + 2 F_{\lambda\mu\nu} F_{\nu \kappa \lambda} + 2 D^\lambda F_{\nu \kappa \lambda} + 2 e^\lambda F_{\nu \kappa \lambda} + 2 H_{\mu} - 2 \gamma L_0,$$  (2.12)

$$Q^{\mu\nu} = - 2 D_{\mu} J^{(i)(x\kappa)} + (T_{\lambda\mu\nu} - 2 \eta_{\kappa} v_{\kappa}) J^{(i)(x\kappa)} - H_{\lambda\nu},$$  (2.13)

where $F_{\lambda\mu} = F_{\lambda\mu}^{(i)}$, and $F_{\lambda\mu}, H_{\lambda\mu}$ and $H_{\lambda\mu}$ are defined by

$$F_{\lambda\mu} = \beta (t_{\lambda\mu} - t_{\lambda\mu}) + \beta (\eta_{\kappa} v_{\kappa}) - \frac{r}{3} \varepsilon_{\lambda\mu\nu} a_{\nu}.$$  (2.14)
Gravity from Poincaré Gauge Theory of the Fundamental Particles 2225

\[ H_{ij} = T_{mn} F_{i}^{mn} - \frac{1}{2} T_{mn} F_{j}^{mn}, \]  \hspace{1cm} (2.15)

\[ H_{ij} = - (\alpha + 2\alpha/3) (t_{kl} - t_{lk}) - (\beta - 2\alpha/3) (\eta_{kl} v_{j} - \eta_{kj} v_{l}) \]
\[ - \frac{2}{3} (\gamma + 3\alpha/2) \varepsilon_{ijmn} a_{m}, \]  \hspace{1cm} (2.16)

while the tensor \( J_{[ij]kl} \) is given in Appendix A. Here \( D_{c} = e_{c}^{j} D_{j} \) denotes the Poincaré gauge covariant derivative introduced in I. By using the conventional method, namely, the decomposition of the Lorentz gauge field into the Ricci rotation coefficient \( \mathcal{A} \) and the contorsion tensor \( \mathcal{K} \),

\[ A_{ij} = J_{ij} + K_{ij}, \]  \hspace{1cm} (2.17)

in the first term of the right-hand side of (2.12), we obtain the following expression for \( P_{\mu\nu} \):

\[ P_{\mu\nu} = 2a G_{\mu\nu} (\{ \}) + 2 F_{(x)\mu} F_{(x)\nu} + 2 F_{\mu} F_{\nu} + 2 K_{\mu\nu} F_{(x)\rho\sigma} \]
\[ - g_{\mu\nu} (L_{\lambda} + L_{\rho}), \]  \hspace{1cm} (2.18)

where \( G_{\mu\nu} (\{ \}) \) is the Einstein tensor, and \( F_{(x)\mu} \) and \( L_{\lambda} \) are obtained from \( F_{\mu\nu} \) and \( L_{\rho} \) by replacing \( \alpha \), \( \beta \) and \( \gamma \) by \( (\alpha + 2\alpha/3) \), \( (\beta - 2\alpha/3) \) and \( (\gamma + 3\alpha/2) \), respectively. Here \( F_{\mu} = g_{\mu\rho} F_{\rho} \) is the covariant derivative with respect to the Christoffel symbol. The tensor \( F_{(x)\mu\nu} \) is related to \( H_{mn} \) of (2.16) by

\[ F_{(x)\mu\nu} = \frac{1}{2} (H_{\mu\nu} - H_{\nu\mu} - H_{\rho\sigma}), \]  \hspace{1cm} (2.19a)

or conversely

\[ H_{\mu\nu} = F_{(x)\mu\nu} - F_{(x)\nu\mu}. \]  \hspace{1cm} (2.19b)

The first-order part \( P_{(1)\mu\nu} \) is then given by

\[ P_{(1)\mu\nu} = 2a G_{(1)\mu\nu} (\{ \}) + 2 \vartheta^{\rho} F_{(1)\rho\mu\nu} \]  \hspace{1cm} (2.20)

with \( \vartheta^{\rho} = \eta^{\rho \sigma} \vartheta_{\sigma} \), where \( F_{(1)\rho\mu\nu} \) is the first-order part of \( F_{\rho\mu\nu} \). The first-order part of the Einstein tensor \( G_{(1)\mu\nu} (\{ \}) \) can be written as

\[ G_{(1)\mu\nu} (\{ \}) = \frac{1}{2} \vartheta^{\rho} \vartheta^{\sigma} H_{\rho\sigma\mu\nu}, \]  \hspace{1cm} (2.21)

where \( H_{\mu\nu\rho\sigma} \) is defined by

\[ H_{\mu\nu\rho\sigma} = - (\eta_{\mu\nu} \overline{F}_{\rho\sigma} - (\eta_{\mu\rho} \overline{F}_{\nu\sigma} + \eta_{\mu\sigma} \overline{F}_{\nu\rho}) + \eta_{\nu\rho} \overline{F}_{\mu\sigma}) \]  \hspace{1cm} (2.22)

with
As is seen from its definition, \( H_{\mu\nu\rho\sigma} \) has the same symmetry properties as the Riemann-Christoffel curvature tensor. Therefore, the total energy-momentum complex \( \tau^{\mu} \) is represented as

\[
\tau^{\mu} = \partial_{\nu} \phi^{\nu \mu}.
\]

where \( \phi^{\mu \nu} \) denotes the superpotential defined by

\[
\phi^{\mu \nu} = a \partial_{\xi} H^{\mu \nu \rho \sigma} + 2 F^{(\mu \nu \rho \sigma)},
\]

and is antisymmetric in \( \nu \) and \( \rho \). Here the indices on \( H^{\mu \nu \rho \sigma} \) was raised with the Minkowski metric \( \eta^{\nu \rho} \).

We can now calculate the total energy and momentum of the finite system under study;

\[
P^\mu = \lim_{r \to \infty} d^3x \tau^{\mu 0} = \lim_{r \to \infty} d^3x \phi^{\mu 0} (\alpha = 1, 2, 3)
\]

\[
= \lim_{r \to \infty} \frac{1}{16\pi G} d^3S_a \partial_{\nu} H^{\rho \nu 0 a}, (a = 1/2, \epsilon = 1/16\pi G)
\]

where the surface integral is taken over a large sphere of radius \( r \), and the surface integral of \( F^{(\mu \nu \rho \sigma)} \) is dropped because \( F^{(\mu \nu \rho \sigma)} \) dies out exponentially as \( r \to \infty \).

Thus, the total energy and momentum of an isolated finite system is given by the same formula as in General Relativity. In particular, it has the following properties:  

(i) In order to calculate the total energy and momentum of an arbitrary finite system, it is only necessary to know the asymptotic behavior of \( h^{\nu \rho} \). 

(ii) Although \( \tau^{\mu} \) is not a tensor, the total energy and momentum, \( P^\mu \), is a Lorentz four-vector in the sense that it transforms as a four-vector under any coordinate transformation which reduces to a Lorentz transformation at infinity and in the sense that it is invariant under any local Lorentz transformation which reduces to the identity at infinity.

§ 3. The total angular momentum and symmetrization of the energy-momentum complex

The exact field equation for the Lorentz gauge field, \((1 \cdot 3.30)\), can be rewritten as

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]
Gravity from Poincaré Gauge Theory of the Fundamental Particles

\[ Q^{\mu \nu} = -\frac{1}{2} (S^{\mu \nu} + \delta^{\mu \nu}), \]  

(3.1)

where \( S^{\mu \nu} \) is the intrinsic spin angular-momentum tensor of matter defined by (I-3.33), and \( \delta^{\mu \nu} \) is

\[ \delta^{\mu \nu} = 2 (Q^{\mu \nu} - Q^{\mu \nu}). \]  

(3.2)

Here \( Q^{\mu \nu} \) denotes the first-order part of \( Q^{\mu \nu} \), given by

\[ Q^{\mu \nu} = -2 \partial_\mu J^{\nu [\mu \nu]} - H^{\mu \nu} \]  

(3.3)

with \( J^{\nu [\mu \nu]} \) and \( H^{\mu \nu} \) the first-order part of \( J^{\nu [\mu \nu]} \) and \( H^{\mu \nu} \), respectively. If the right-hand side were absent, Eq. (3.1) could be reduced to the free Klein-Gordon equations for the irreducible torsion fields as has been shown in III (see §§ 4 and 5 of III). Accordingly, we can say that the torsion field is generated by the total densities and fluxes of the intrinsic spin angular momentum and that \( \delta^{\mu \nu} \) is the intrinsic spin angular-momentum complex of the translation and Lorentz gauge fields themselves. That is, we interpret the quantity

\[ \sigma^{\mu \nu} = S^{\mu \nu} + \delta^{\mu \nu} \]  

(3.4a)

as the total spin angular-momentum complex of matter and gravitation. By virtue of (3.1), \( \sigma^{\mu \nu} \) can be represented as

\[ \sigma^{\mu \nu} = -2Q^{\mu \nu} = 4\partial_\mu J^{\nu [\mu \nu]} + 2H^{\mu \nu}. \]  

(3.4b)

Since the gravitational action is Lorentz gauge invariant, \( P^{\mu} \) and \( Q^{\mu \nu} \) of (2.2a) and (2.2b) satisfy the identity,\(^{11}\)

\[ P^{\mu (\nu)} + \frac{1}{e} D_\nu (eQ^{\mu \nu}) = 0, \]  

(3.5)

where the covariant derivative acts only on Latin indices.\(^{k1}\) Taking the first-order part of (3.5), we have the identity,

\[ P^{\mu (\nu)} + \partial_\nu Q^{\mu \nu} = 0. \]  

(3.6)

This identity also follows directly from (2.19), (2.20) and (3.3), because Latin indices on the first-order part of any tensor can be converted into Greek ones with the Kronecker symbol, and vice versa; for example, \( P^{\mu \nu} = \delta^{\mu}_\lambda \delta^{\nu}_\mu P^{\mu \nu} \).

With the help of the gravitational field equations written as (2.6) and (3.1), the identity (3.6) gives

\[ 2\sigma^{\mu \nu} = \partial_\lambda \sigma^{\mu \nu}. \]  

(3.7)

\(^{k1}\) Namely, we define \( D_\mu (eQ^{\mu \nu}) \) by

\[ D_\mu (eQ^{\mu \nu}) = \delta_\mu (eQ^{\mu \nu}) + \varepsilon (A'_\mu Q^{\nu} + A'_\nu Q^{\mu}). \]
This relation is a generalization of the Tetrode formula in special relativity (see (III·2·14)), and allows us to define the *symmetrized energy-momentum complex* by

\[ \tau^{(\text{sym})\mu\nu} = \tau^{\mu\nu} - \frac{1}{2} \partial_\rho (\sigma^{\mu\rho\nu} + \sigma^{\nu\rho\mu} + \sigma^{\rho\mu\nu}), \]  

(3·9)

which is symmetric because of (3·7), and is conserved,

\[ \partial_\nu \tau^{(\text{sym})\mu\nu} = 0, \]  

(3·10)

due to (2·9). Using (2·24) and (3·4b) in (3·9), we get

\[ \tau^{(\text{sym})\mu\nu} = \partial_\rho \phi^{(\text{sym})\mu\rho\nu}, \]  

(3·11)

with the superpotential \( \phi^{(\text{sym})\mu\rho\nu} \) given by

\[ \phi^{(\text{sym})\mu\rho\nu} = J^{\mu\rho\nu} + Q^{\mu\rho\nu} + Q^{\nu\rho\mu} + Q^{\rho\mu\nu} \]

\[ = \alpha \partial_\sigma H^{\sigma\mu\rho\nu} - 2 \partial_\rho \left( J^{\mu\nu}[\sigma] - J^{\nu\sigma}[\mu] \right), \]  

(3·12)

which is antisymmetric in \( \nu \) and \( \rho \). Here we used the relation (2·19a). At great distances from the finite system under study, the field equation (3·1) can be approximated by the linearized field equation in vacuum,

\[ Q^{(1)\mu\rho\nu} = -2 \partial_\sigma J^{(1)\sigma\mu\rho\nu} - H^{(1)\mu\rho\nu} = 0, \]  

(3·13)

and so the superpotential \( \phi^{(\text{sym})\mu\rho\nu} \) coincides with the superpotential \( \phi^{\mu\rho\nu} \) of (2·25) there. Accordingly, the total energy and momentum of (2·26) can be written as

\[ P^\mu = \int d^4x \tau^{(\text{sym})\mu\nu}, \]  

(3·14)

Owing to Eq. (3·7), the *total angular-momentum complex* \( \mathcal{M}^{\mu\nu} \),

\[ \mathcal{M}^{\mu\nu} = (x^\mu x_{\rho} - x^\rho x_{\nu}) + g^{\mu\nu}, \]  

(3·15)

is conserved,

\[ \partial_\mu \mathcal{M}^{\mu\nu} = 0. \]  

(3·16)

Using (2·24) and (3·4b), we can rewrite (3·15) as

\[ \mathcal{M}^{\mu\nu} = \partial_\nu \xi^{\mu\nu}, \]  

(3·17)

where \( \xi^{\mu\nu} \) is defined by
Here the relations (2.19b) and (2.25) were employed. It should be noticed that $\xi^{\mu\nu}$ is antisymmetric in $\lambda$ and $\mu$ and in $\nu$ and $\rho$,

$$\xi^{\lambda\mu} = -\xi^{\mu\lambda} = -\xi^{\nu\rho}.$$  \hspace{1cm} (3·19)

Since $F_{\nu\mu}$ falls off exponentially at large distances from the localized source (see the footnote of page 2226), the total angular momentum $J^{\mu}$ is given by

$$J^{\lambda} = \int d^2x M^{\lambda\rho} = \lim_{r \to \infty} \int d^{2}S_{\alpha} \phi^{\lambda\rho\mu}$$

$$= \lim_{r \to \infty} \int d^{2}S_{\alpha}\{a \langle x^1 \partial_{x^2} H^{\rho\mu} - x^2 \partial_{x^1} H^{\rho\mu} \rangle$$

$$+ (H^{\lambda\mu\nu} - H^{\rho\lambda\mu}) \} + 4J^{\nu}[x \in \{\infty\}],$$  \hspace{1cm} (3·20)

where the surface integral is carried out over a large sphere of radius $r$. Therefore, the total angular momentum $J^{\mu}$ transforms as an antisymmetric second-rank tensor under any coordinate transformation which reduces to a Lorentz transformation at infinity. Also, $J^{\mu}$ is invariant under any local Lorentz transformation which reduces to the identity at infinity.

The term with $\phi_{\nu\rho\mu}^{\mu}$ in (3·20) represents a possible correction to the familiar expression for the total angular momentum in General Relativity. If the torsion field is massive, this term is vanishing because $J^{\nu}[x \in \{\infty\}]$ falls off at large distances from the localized source like a second-order derivative of the gravitational field $h_{\mu\nu}$, namely like $1/r^2$. If there exist some massless components in the torsion field which tail out like $1/r$ as $r \to \infty$, on the other hand, this term with $J^{\nu}[x \in \{\infty\}]$ may give rise to a finite correction to the total angular momentum, because this time $J^{\nu}[x \in \{\infty\}]$ will fall off like $1/r^2$ in general.

§ 4. ENERGY AND MOMENTUM OF THE GRAVITATIONAL AND TORSION FIELDS IN THE WEAK FIELD APPROXIMATION

We shall now calculate the energy and momentum of the gravitational field $h_{\mu\nu}$ and the torsion field $T_{\mu\nu}$ in the weak field approximation, supposing that these weak fields obey the linearized gravitational field equations in vacuo, which read

$$P^0_{\nu} = 0,$$  \hspace{1cm} (4·1)

$$Q^0_{\nu} = 0$$  \hspace{1cm} (4·2)

with $P^0_{\nu}$ and $Q^0_{\nu}$ the linear approximation of $P_{\nu}$ and $Q_{\nu}$ introduced in § 2, respectively. Throughout this section, all the tensor indices [except for those on $P^0_{\nu}$ and $Q^0_{\nu}$ below] are raised and lowered with the Minkowski metric $\gamma$'s. In this weak field approximation the energy-momentum complex $t_{\mu}$ of (2·7)
becomes a tensor, and can be approximated by the second-order terms,
\[ t_{\nu} = - P_{\nu}^{(2)} . \]  
(4·3a)

The indices on \( P_{\nu}^{(2)} \) should be raised according to the rule (2·4), and so we have
\[ t_{\nu} = \eta^{\gamma}_{\nu} t_{\nu}^{} = - P_{\nu}^{(2)} - \kappa^{\gamma}_{\nu} P_{\nu}^{(1)} , \]  
(4·3b)

where \( P_{\nu}^{(2)} \) is the second-order part of \( P_{\nu}^{(1)} = g^\nu_\mu P_{\mu} \).

Using (2·18) in (4·3a), we obtain
\[ t_{\nu} = - \left[ 2 a G_{\mu}^{(2)} \{ \} + 2 F_{\mu\nu}^{(1)} \right] J_{[\mu][\nu]}^{(1)} \]
\[ + 2 \left( \{ \rho \}^{(0)} + K_{\mu\nu} \right) F_{\mu\nu}^{(1)} + 2 \left( \lambda \right)^{(0)} F_{\mu\nu}^{(1)} \]
\[ - \eta_{\mu\nu} \left( L_{\nu}^{(1)} + L_{\mu}^{(2)} \right) , \]  
(4·4)

where we have used the following notations: \( G_{\mu}^{(2)} \{ \} \) is the second-order part of \( G_{\mu} \{ \} \), and its indices are raised by the same rule as those on \( P_{\nu}^{(2)} \), while \( \{ \rho \}^{(0)} \), \( F_{\mu\nu}^{(0)} \) and \( J_{[\mu][\nu]}^{(0)} \) are the linear approximation of the Christoffel symbol, the Lorentz gauge field strength \( F_{\mu\nu}^{(1)} \), and the tensor \( J_{[\mu][\nu]}^{(1)} \) given by (A·10), respectively.

Next, \( L_{\nu}^{(1)} \) is given by (A·7) with \( F_{\mu\nu} \) replaced by \( F_{\mu\nu}^{(1)} \). Finally, \( L_{\nu}^{(1)} \) and \( L_{\nu}^{(2)} \) are
\[ F_{\mu\nu}^{(1)} = (\alpha + 2a/3) (t_{\mu\nu} - t_{\nu\mu}) + (\beta - 2a/3) (v_{\mu} v_{\nu} - \eta_{\mu\nu} v_{\nu}) \]
\[ - \frac{1}{3} (\gamma + 3a/2) \varepsilon_{\mu\nu} a^{\mu} , \]  
(4·5)

\[ L_{\nu}^{(1)} = (\alpha + 2a/3) (t_{\mu\nu} + t_{\nu\mu}) + (\beta - 2a/3) (v_{\mu} v_{\nu}) + (\gamma + 3a/2) (a_{\nu} a^{\mu}) . \]  
(4·6)

It follows from the second-order part of the identity (2·1) that \( P_{\nu}^{(2)} \) satisfies
\[ \eta_{\mu\nu} P_{\nu}^{(2)} = 0 , \]  
(4·7)

when the linear parts, \( P_{\nu}^{(1)} \) and \( Q_{\nu}^{(1)} \), are both vanishing. Thus, the energy-momentum tensor \( t_{\mu\nu} \) given by (4·3a) is conserved
\[ \eta_{\mu\nu} t_{\mu\nu} = 0 , \]  
(4·8)

owing to the linearized gravitational field equations in vacuum, (4·1) and (4·2).

The total energy-momentum vector of the weak gravitational and torsion fields, defined by
\[ P_{\mu} = \int d^{3}x t_{\mu}^{(0)} , \]  
(4·9)

is conserved and transforms like a four-vector under Lorentz transformations.

It was shown in §§ 4 and 5 of III that the linearized gravitational field equa-
Gravity from Poincaré Gauge Theory of the Fundamental Particles

Equations (4·1) and (4·2) can be reduced to the linearized Einstein equation in vacuum for the massless graviton field $\phi_{\mu\nu}$:

$$\Box \phi_{\mu\nu} - \partial^\rho (\partial_\rho \phi_{\mu\nu} + \partial_\mu \phi_{\nu\rho}) + \gamma_{\mu\rho} \partial^\rho \phi_{\nu\sigma} = 0,$$  (4·10)

and the six free Klein-Gordon equations for the six irreducible torsion fields, $\sigma, B, \bar{\sigma}, \bar{B}, \chi_{\mu}$ and $\bar{t}_{\mu\nu}$, with the following mass and spin parity $J^P$:

- $m_\sigma = \frac{2a (\beta - 2a/3)}{\beta (a_x + 12a_e)}$, $J^P = 0^+$ for $\sigma$, (4·11a)
- $m_B = \frac{2(\gamma + 3a/2)}{3(a_x + a_e)}$, $J^P = 0^-$ for $B$, (4·11b)
- $m_\bar{\sigma} = \frac{9(\alpha + 2a/3) (\beta - 2a/3)}{2(\alpha + \beta)} (a_x + a_e)$, $J^P = 1^-$ for $\bar{\sigma}$, (4·11c)
- $m_{\bar{B}} = \frac{-2(\alpha + 2a/3) (\gamma + 3a/2)}{(a - 4\gamma/9) (a_x + a_e)}$, $J^P = 1^+$ for $\bar{B}$, (4·11d)
- $m_\chi = \frac{-2a (\alpha + 2a/3)}{a (3a_x + 2a_e)}$, $J^P = 2^+ \chi_{\mu}$, (4·11e)
- $m_{\bar{\chi}} = \frac{3(\alpha + 2a/3)}{3a_x + 4a_e}$, $J^P = 2^- \bar{t}_{\mu\nu}$. (4·11f)

The fields $\bar{\sigma}, \bar{B}, \chi_{\mu}$ and $\bar{t}_{\mu\nu}$ are divergenceless and traceless in vacuum. The $\chi_{\mu}$ is symmetric, and the $\bar{t}_{\mu\nu}$ is symmetric in $\lambda$ and $\mu$ and satisfies the cyclic identity, $\bar{t}_{\mu\nu} + \bar{t}_{\nu\rho} + \bar{t}_{\rho\mu} = 0$. The gravitational field $h_{\mu\nu}$ and the three irreducible parts of the torsion field, $\nu_{\mu}, a_\mu$ and $t_{\mu\nu}$, are represented as

$$h_{\mu\nu} = \phi_{\mu\nu} + \frac{3a_0 (3a_x + 2a_e)}{a^2} \chi_{\mu}, \frac{b (a_x + 12a_e)}{a^2} (\eta_{\mu\rho} - \partial_\rho \partial_\nu / m_\sigma) \sigma,$$  (4·12a)

$$\nu_{\mu} = \bar{\nu}_{\mu} + \partial_\mu \sigma / m_\sigma,$$  (4·12b)

$$a_\mu = \bar{a}_\mu + \partial_\mu B / m_B,$$  (4·12c)

$$t_{\mu\nu} = \bar{t}_{\mu\nu} + 2 (\partial_\mu \chi_{\nu}) - \partial_\nu \chi_{\mu} / m_\chi^2 - \frac{-2a/3}{\alpha + 2a/3} \left\{ \frac{1}{3} (\eta_{\mu\nu} \bar{\nu}_{\rho} - \eta_{\nu\rho} \bar{\nu}_{\mu} - (\partial_\mu \bar{\nu}_{\nu} - \partial_\nu \bar{\nu}_{\mu}) / m_\chi^2 \right\}$$

$$- \frac{\gamma + 3a/2}{3 (\alpha + 2a/3) m_\chi^2} (\bar{\chi}_{\nu\rho} \partial_\mu + \bar{\chi}_{\rho\nu} \partial_\lambda) \bar{\nu}_{\lambda} \bar{\nu}_{\sigma},$$  (4·12d)

where $h_{\mu\nu} = h_{\mu\nu} - (1/2) \eta_{\mu\nu} h$ with $h = \bar{\chi}_{\mu\nu} h$. In this section we assume that the parameters, $(\alpha + 2a/3), (\beta - 2a/3)$ and $(\gamma + 3a/2)$, are nonvanishing: Namely, we assume that the six irreducible torsion fields, $\sigma, B, \bar{\sigma}, \bar{B}, \chi_{\mu}$ and $\bar{t}_{\mu\nu}$, are
massive. We will discuss in a separate paper of this series the case that these irreducible torsion fields are massless.

The linearized field equations (4·1) and (4·2) are invariant under the gauge transformation specified by

$$h'_{mn} = h_{mn} + \partial_m A_n + \partial_n A_m,$$

$$T'_{mn} = T_{mn},$$

(4·13)

where $A_m$ are small but otherwise arbitrary functions. According to (4·12a) $\sim$ (4·12d), the massless graviton field $\phi_m$ changes like

$$\phi'_{mn} = \phi_{mn} + \partial_m A_n + \partial_n A_m - \gamma_{mn} \partial_i A_i,$$

(4·14)

under the gauge transformation, whereas the irreducible torsion fields are kept unchanged. The energy-momentum tensor $t_{mn}$ of (4·4) is transformed into another conserved energy-momentum tensor $t'_{mn}$ formed of $h'_{mn}$ and $T_{mn}$: The $t_{mn}$ does not coincide with $t_{mn}$, but the total energy-momentum vector does not depend on the gauge functions $A_m$.$^9$ Consequently, we can calculate the total energy-momentum vector $P^a$ by assuming the gauge condition,

$$\partial^a \phi_{mn} = 0,$$

(4·15)

under which Eq. (4·10) becomes the d’Alembert equation,

$$\Box \phi_{mn} = 0.$$  

(4·16)

From (4·11a) $\sim$ (4·11f) we see that the six irreducible torsion fields are unambiguously labelled with their spin-parity. So we shall henceforth denote by $\phi^{(j)}$ or simply by $\phi^{j}$ the irreducible torsion field with spin-parity $=j^r$: For example, $\phi^{(0)}$ or 2$^r$ shall represent the field $\chi_{mn}$.

Now let us calculate the energy and momentum of the seven classes of the Klein-Gordon fields, namely, of the massless graviton field and the six irreducible torsion fields. Using (4·12a) $\sim$ (4·12d) in (4·9) with (4·4), we first of all notice that $P^a$ is made of four parts:

$$P^a = P^{(0)x} + \sum_i P^{(i)x} + \sum_j Q^{(j)x} + \sum_{ij} Q^{(i)j},$$

(4·17)

where $i$ and $j$ run over the $j^r$-values of the six irreducible torsion fields, 0$^r$, 1$^r$ and 2$^r$. Here $P^{(i)x}$ and $P^{(j)x}$ are the quadratic terms of $\phi_m$ and $\phi^{j}$ respectively, while $Q^{(j)x}$ and $Q^{(i)j}$ are made of mixed terms bilinear in $\phi_m$ and $\phi^{j}$ and in

$^9$ Consider another gauge functions $A'_m$ with $A'_m \to A_m$ as $t \to -\infty$ and $A'_m \to 0$ as $t \to +\infty$, and denote the energy-momentum tensor formed of $h''_{mn} = h_{mn} + \partial_m A'_n + \partial_n A'_m$ and $T'_{mn}$ as $t''_{mn}$. Then, since $t_{mn}$, $t'_{mn}$ and $t''_{mn}$ are all conserved, we have

$$\int d^4x t''_{mn} = \int_{-\infty}^\infty d^4x t''_{mn} = \int_{-\infty}^\infty d^4x t_{mn} = \int_0^\infty d^4x t_{mn}.$$
Gravity from Poincaré Gauge Theory of the Fundamental Particles 2233

\( \Phi_0 \) and \( \Phi_0' \), respectively. Since the massless graviton field and the six irreducible torsion fields obey the free Klein-Gordon equations, it follows from the conservation of \( P^\alpha \) that each of \( P^{(0)}, P^{(1)}, Q^{(0)}, \) and \( Q^{(1)} \) is also conserved. In the momentum representation each term of \( Q^{(0)} \) necessarily involves the time-dependent factor \( \exp \left[ \pm i (\omega_\mu - \omega_\mu') t \right] \) or \( \exp \left[ \pm i (\omega_\mu + \omega_\mu') t \right] \) with \( \omega_\mu = |k| \) and \( \omega_\mu' = \left( k^2 + M_f^2 \right)^{1/2} \) (where \( M_f \) denotes the mass of \( \Phi^{(0)} \)), showing that all the terms in \( Q^{(0)} \) should be cancelled by each other. In the same manner we see that similar compensation should take place also in \( Q^{(1)} \). Thus, we find that

\[
Q^{(0)} = Q^{(1)} = 0.
\] (4.18)

Let us next calculate the energy and momentum, \( P^{(0)}, \) of the massless graviton field by assuming that the irreducible torsion fields are all vanishing. According to (4.12a) \( \sim \) (4.12d), we have

\[
\phi_{\mu\nu} = \bar{h}_{\mu\nu}, \quad T_{\mu\nu} = 0
\] (4.19)
in this case, and so the linearized Lorentz gauge field strength \( F_{\mu\nu}^{(0)} \) reduces to the linearized Riemann-Christoffel curvature tensor \( R_{\mu\nu\rho\sigma}^{(0)} \). The energy-momentum tensor of gravitation (4.4) then becomes

\[
T_{\mu\nu} = - \left[ 2 a G_{\mu\nu}^{(0)} \right] R_{\rho\sigma}^{(0)} J^{[\rho\sigma]}_{\mu\nu} \eta_{\rho\sigma} L_{\rho\sigma}^{(0)} \].
\] (4.20a)

Since the linearized Ricci tensor \( R_{\mu\nu}^{(0)} \) and the linearized scalar curvature \( R^{(0)} \) are vanishing due to (4.10) and (4.19), it follows from (A.10) and (A.7) that \( J^{[\rho\sigma]}_{\mu\nu} \) and \( L_{\rho\sigma}^{(0)} \) are given by

\[
J^{[\rho\sigma]}_{\mu\nu} = \frac{3}{2} a R_{\rho\sigma}^{(0)} (\{ \}), \quad L_{\rho\sigma}^{(0)} = \frac{3}{4} a \zeta R_{\rho\sigma}^{(0)} (\{ \}) R^{(0)}_{\rho\sigma} (\{ \}).
\] (4.20b, 4.20c)

By virtue of the Bach-Lanczos identity for the Riemann-Christoffel curvature tensor, \( R_{\mu\nu\rho\sigma}^{(0)} (\{ \}) \)

\[
eq 2 R_{\mu\nu\rho\sigma}^{(1)} (\{ \}) R^{(0)} (\{ \}) + 2 R_{\mu\nu}^{(0)} (\{ \}) R^{(0)}_{\rho\sigma} (\{ \}) - R^{(0)} (\{ \}) R_{\rho\sigma}^{(0)} (\{ \})
\]

\[
+ \frac{1}{4} \eta_{\rho\sigma} [ R_{\mu\nu\rho\sigma}^{(1)} (\{ \}) R^{(0)}_{\rho\sigma} (\{ \}) - 4 R_{\mu\nu}^{(0)} (\{ \}) R^{(0)}_{\rho\sigma} (\{ \}) + R (\{ \})^2 ]
\] (4.21)

we see that the second and third terms of (4.20a) cancel with each other, giving

\[
T_{\mu\nu} = - 2 a G_{\mu\nu}^{(0)} (\{ \}) \quad \text{for the massless graviton field.}
\] (4.22)

This coincides with the expression for the energy-momentum tensor of the weak
gravitational field in General Relativity, because we have chosen the parameter \( a = 1/2 \kappa = 1/16\pi G \) with \( G \) Newton’s gravitational constant.

As for the energy-momentum vector \( P^{(i)} \) of the irreducible torsion field \( \Phi^{(i)} (i = J^p = 0^+, 1^, 2^+) \), we obtain

\[
P^{(i)} = \int d^p x t^{(i)\text{rel}}
\]

with

\[
t^{(i)\text{rel}} = (1/N^{(i)}) \left[ \delta^{\alpha \beta} \delta^{\gamma \delta} \delta^{(i)} - \frac{1}{2} \eta^{\alpha \beta} (\partial_{\alpha} \Phi^{(i)} \partial_{\beta} \Phi^{(i)} + M_{\alpha \beta} \Phi^{(i)} \Phi^{(i)}) \right],
\]

where the factors \( N^{(i)} \) are given by

\[
N^{(p)} = \frac{4a^3 (\beta - 2a/3)}{3d^2 (a_4 + 12a_2)^2} \quad \text{for } 0^+,
\]

\[
N^{(q)} = \frac{2(\gamma + 3a/2)}{9(a_1 + a_0)} \quad \text{for } 0^-,
\]

\[
N^{(v)} = -\frac{9(\alpha + 2a/3)}{4(\alpha + \beta)^2 (a_4 + a_0)} \quad \text{for } 1^-, \quad (4.24a)
\]

\[
N^{(v)} = \frac{(\alpha + 2a/3)^2}{(\alpha - 4\gamma/9)^2(2a_3 + a_4)} \quad \text{for } 1^+, \quad (4.24b)
\]

\[
N^{(p)} = -\frac{2a^3 (\alpha + 2a/3)}{9a^2 (3a_2 + a_0)^2} \quad \text{for } 2^+,
\]

\[
N^{(v)} = \frac{3}{2(3a_2 + 4a_0)} \quad \text{for } 2^-,
\]

(See Appendix B for derivation.) Here the summation over tensor indices on \( \Phi^{(i)} \) is carried out: For \( i = 2^+ \), for example, \( \partial^{(i)} \Phi^{(i)} \partial^{(i)} \Phi^{(i)} \) represents \( \partial_{\alpha \beta}^{rel} \partial_{\alpha \beta}^{rel} \).

The energy of the massless graviton field is positive-definite, while the energy of the irreducible torsion field \( J^p \) is positive- or negative-definite according as the factor \( N^{(J^p)} \) is positive or negative. The condition of positive mass and positive energy for the irreducible torsion fields is given by

\[
\beta (\beta - 2a/3) > 0 \quad \text{and } a_4 + 12a_2 > 0 \quad \text{for } 0^+,
\]

\[
\gamma + 3a/2 > 0 \quad \text{and } a_4 + a_2 < 0 \quad \text{for } 0^-,
\]

\[
(\alpha + 2a/3) (\alpha + 2a/3) (\beta - 2a/3) < 0 \quad \text{and } a_4 + a_2 < 0 \quad \text{for } 1^-,
\]

\[
(\alpha - 4\gamma/9) (\alpha + 2a/3) (\gamma + 3a/2) < 0 \quad \text{and } 2a_3 + a_4 > 0 \quad \text{for } 1^+,
\]

\[
\alpha (\alpha + 2a/3) < 0 \quad \text{and } 3a_2 + 2a_3 > 0 \quad \text{for } 2^+,
\]

\[
\alpha + 2a/3 < 0 \quad \text{and } 3a_2 + 4a_3 < 0 \quad \text{for } 2^-.
\]
Gravity from Poincaré Gauge Theory of the Fundamental Particles

which consist of two classes of the constraints on the parameters: One is the constraints on the parameters \( a, \beta \) and \( \gamma \), whereas the other gives the constraints on the parameters \( a, a, \ldots \) and \( a \). As for the former class, (4.26e) and (4.26f) are incompatible with each other, indicating that \( 2^+ \) and \( 2^- \) cannot coexist as normal fields.\(^*\) As for the latter class, on the other hand, (4.26c) \( \sim \) (4.26f) cannot be fulfilled at the same time due to the identity,

\[
2\{(2a_3 + a_4) - (a_3 + a_4)\} + \{5(3a_3 + 2a_4) - (3a_3 + 4a_4)\} = 0.
\]

(4.27)

Let us require as the basic postulate that each of the six irreducible torsion fields \((0^+, 1^+, 2^+)\) should either be normal or be frozen at the place of matter with the mass being infinite.\(^**\) Then, the irreducible torsion fields are divided into two groups, namely, the group of normal fields and the group of fields frozen at the place of matter: We shall call these two groups as the normal multiplet and the frozen multiplet, respectively. For any irreducible torsion field belonging to the normal multiplet the condition (4.26) must be satisfied, while the condition of infinite mass should be fulfilled for the field belonging to the frozen multiplet.

In view of the expressions (4.11a) \( \sim \) (4.11f) for the mass, the condition of infinite mass reads

\[
\beta(a_3 + 12a_4) = 0 \quad \text{for } 0^+, \quad (4.28a)
\]
\[
a_3 + a_4 = 0 \quad \text{for } 0^-, \quad (4.28b)
\]
\[
(\alpha + \beta)(a_4 + a_5) = 0 \quad \text{for } 1^+, \quad (4.28c)
\]
\[
(a - 4\gamma/9)(2a_3 + a_4) = 0 \quad \text{for } 1^-, \quad (4.28d)
\]
\[
\alpha(3a_3 + 2a_4) = 0 \quad \text{for } 2^+, \quad (4.28e)
\]
\[
3a_3 + 4a_4 = 0 \quad \text{for } 2^-; \quad (4.28f)
\]

It is possible to choose various combinations of the irreducible torsion fields as the normal multiplet. In order to find out all the possibilities systematically, we classify normal multiplets into the following four classes, I, II, III and IV, according as whether \( 2^+ \) and \( 2^- \) are normal or not.\(^***\)

(I) The class I: \( 2^- \) is normal but \( 2^+ \) is frozen with the condition,

\[
3a_3 + 2a_4 = 0 \quad \text{for } 0^+, \quad (4.29a)
\]
\[
3a_3 + 4a_4 < 0, \quad -2a/3 > \alpha \quad \text{for } 0^-. \quad (4.29b)
\]

\(^*\) A normal field means a field with positive mass and positive energy.

\(^**\) When the mass \( M \) tends to infinity, the field \( \Phi^{(0)} \) satisfying the Klein-Gordon equation is given by \( \Phi^{(0)} = \lim_{M \to \infty} (-j^{(0)} / M^2) = \text{finite} \). This is assured from the expressions for \( j^{(0)} \), denoted by (III·4·6), (III·4·10), (III·4·13), (III·4·17), (III·4·21) and (III·4·25).

\(^***\) We discard the case that both \( 2^+ \) and \( 2^- \) are normal, because we already know that this is impossible. The classes II and IV correspond to the fact that there are two possible ways to make the mass of \( 2^+ \) infinite: one is to put \( \alpha = 0 \) and the other is to choose \( 3a_3 + 2a_4 = 0 \).
The class II: $2^+$ is normal but $2^-$ is frozen with the condition,
\[ 3a_2 + 4a_3 = 0, \quad 3a_2 + 2a_3 > 0, \quad 0 > a > -2a/3. \]

The class III: $2^+$ and $2^-$ are both frozen with the condition,
\[ 3a_2 + 4a_3 = 0, \quad a = 0. \]

The class IV: $2^+$ and $2^-$ are both frozen with the condition,
\[ 3a_2 + 4a_3 = 0, \quad 3a_2 + 2a_3 = 0. \]

It is convenient to divide each of these four classes further into four subclasses according as whether $(\alpha + \beta)$ and $(\alpha - 4\gamma/9)$ are vanishing or not. It is now easy to find all the possible normal multiplets and the conditions of the parameters associated with them. We show in Table I the maximal normal multiplet in each subclass and the conditions of the parameters. Smaller normal multiplets in each subclass can be obtained as the limiting cases where the mass of a member (or members) of the maximal normal multiplet in the subclass becomes infinite.

Table I. (a) The maximal normal multiplets of the classes I and II.

<table>
<thead>
<tr>
<th>Condition of $\alpha + \beta$ and $\alpha - 4\gamma/9$</th>
<th>Class I</th>
<th>Class II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha + \beta) = 0, (\alpha - 4\gamma/9) &gt; 0$</td>
<td>$2^+, 1^+, 0^-$</td>
<td>$2^+, 1^+, 0^-$</td>
</tr>
<tr>
<td>$a_1 + a_3 = 0, a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 = 0, a_1 + a_3 &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>$a_1 + 12a_3 &gt; 0, a_1 + 12a_3 &lt; 0$</td>
<td>$a_1 + 12a_3 &gt; 0, a_1 + 12a_3 &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>$0 &gt; a &gt; -2a/3$</td>
<td>$0 &gt; a &gt; -2a/3$</td>
<td></td>
</tr>
</tbody>
</table>

$^*$ In some subclasses there are two maximal normal multiplets.
Gravity from Poincaré Gauge Theory of the Fundamental Particles 2237

Table I. (b) The maximal normal multiplets of the classes III and IV.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Class III</th>
<th>Class IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha + \beta \neq 0, \neq \alpha - 4\gamma /9$</td>
<td>$(1^+, 1^-, 0^-)$</td>
<td>$(0^*, 0^-)$</td>
</tr>
<tr>
<td>$a_1 + 2a_2 = 0$</td>
<td>$a_1 + a_2 = 0$</td>
<td>$a_1 + a_2 = 0$</td>
</tr>
<tr>
<td>$2a_1 + a_3 &gt; 0$</td>
<td>$2a_1 + a_3 &gt; 0$</td>
<td>$2a_1 + a_3 = 0$</td>
</tr>
<tr>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
</tr>
<tr>
<td>$a_1 + a_2 &lt; 0$</td>
<td>$a_3 + 2a_2 &gt; 0$</td>
<td>$a_1 + 2a_2 &gt; 0$</td>
</tr>
<tr>
<td>$2\alpha /3 &gt; \beta &gt; 0$</td>
<td>$0^* \beta$ or $\beta &gt; 2\alpha /3$</td>
<td>$0^* \beta$ or $\beta &gt; 2\alpha /3$</td>
</tr>
<tr>
<td>$\gamma &gt; 0$</td>
<td>$\gamma &gt; 0$</td>
<td>$\gamma &gt; -3\alpha /2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Condition</th>
<th>Class III</th>
<th>Class IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha - 4\gamma /9 = 0 \neq \alpha + \beta$</td>
<td>$(1^-, 0^+, 0^-)$</td>
<td>$(1^+, 0^+, 0^-)$</td>
</tr>
<tr>
<td>$a_1 + 2a_2 = 0$</td>
<td>$a_1 + a_2 = 0$</td>
<td>$a_1 + a_2 = 0$</td>
</tr>
<tr>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
</tr>
<tr>
<td>$a_1 + a_2 &lt; 0$</td>
<td>$a_3 + 2a_2 &gt; 0$</td>
<td>$a_1 + 2a_2 &gt; 0$</td>
</tr>
<tr>
<td>$2\alpha /3 &gt; \beta &gt; 0$</td>
<td>$0^* \beta$ or $\beta &gt; 2\alpha /3$</td>
<td>$0^* \beta$ or $\beta &gt; 2\alpha /3$</td>
</tr>
<tr>
<td>$\gamma &gt; 0$</td>
<td>$\gamma &gt; 0$</td>
<td>$\gamma &gt; -3\alpha /2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Condition</th>
<th>Class III</th>
<th>Class IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha + \beta = 0 - 4\gamma /9$</td>
<td>$(0^-, 1^-, 2^-)$</td>
<td>$(0^+, 0^+, 0^-)$</td>
</tr>
<tr>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
<td>$a_1 + a_3 &lt; 0$</td>
</tr>
</tbody>
</table>

We can observe the following conclusions from Table I:

1. The maximal number of normal fields is three. There are nine possible normal triplets:
   
   $(0^-, 1^-, 2^-)$, $(0^+, 1^+, 2^-)$ and $(0^-, 0^+, 2^-)$ in the class I,
   
   $(0^-, 1^+, 2^-)$, $(0^-, 1^+, 2^-)$ and $(0^-, 0^+, 2^-)$ in the class II,
   
   $(0^-, 1^-, 1^+)$ in the class III,
   
   $(0^-, 0^+, 1^+)$ in the class IV,
   
   $(0^-, 0^+, 1^+)$ in the classes III and IV.

2. Any pair except for $(2^+, 2^-)$ is allowed as a normal doublet.

3. Any of the six irreducible torsion fields is allowed as a normal singlet.

4. The normal multiplet can be null, in which case all the irreducible torsion fields are frozen at the place of matter.

§ 5. Conclusions

We have investigated the particle spectrum of Poincaré gauge theory, based
on the gravitational action which is linear in the Lorentz gauge field strength and quadratic in the translation and Lorentz gauge field strengths. Besides the usual graviton, the torsion field gives rise to the six massive particles of \(2^+, 2^-, 1^+, 1^-, 0^+\) and \(0^-\), each of which obeys the Klein-Gordon equation. The main purpose of the present paper is to see under what conditions these particles travel in vacuum (through the Klein-Gordon equation) with positive mass and positive energy. In other words, we searched for the conditions of no ghost and no tachyon.

We shall uniquely designate all the fields in question as the well-known symbol of \(J^p\); namely, besides \(2^+\) for the massless graviton field, \(2^+\) for the massive \(Z^+\) field, \(2^-\) for the massive \(\tilde{T}^\pm\) field, \(1^+\) for the massive \(\tilde{\theta}^\pm\) field, \(0^-\) for the massive \(\phi^\pm\) field, and finally, \(0^+\) for the massive \(B\) field.

The result is as follows. Among the six massive particles there occur at most only three particles which are normal. (By ‘normal’ we mean the particle can propagate in vacuum under the following conditions: (i) it satisfies the Klein-Gordon equation, (ii) it has positive mass and (iii) it has also positive energy.) Among the three particles the following nine normal triplets are possible; \((2^+, 1^-, 0^-), (2^+, 1^-, 0^-), (2^+, 0^+, 0^-)\) and their parity conjugates, and \((1^+, 0^-, 0^-)\) and its parity conjugate, and finally, \((1^+, 1^-, 0^-)\).

On the condition that \(2^+\) and \(2^-\) cannot coexist as the normal field, one can reduce the above nine triplets to 14 \((6 \times 2 - 1)\) possible doublets like \((2^+, 1^+)\) and \((0^-, 0^-)\) and so forth and to six possible singlets, \(2^+, 2^-, 1^+, 1^-, 0^-\) and \(0^+\).

Of particular importance is a singlet of \(1^+\), a massive axial-vector particle, whose excitation is generated by the spin 1/2 fundamental particles through their intrinsic spin angular-momentum tensor. This issue will be discussed in a forthcoming paper of this series.

### Appendix A

--- *Alternative Expression for \(L_F\) and \(J_{\{ijmn\}}\)*

We have constructed in I the quadratic Lagrangian density of the Lorentz gauge field, \(L_F\), by using the irreducible decomposition of the Lorentz gauge field strength \(F_{ijmn}\). For practical calculations, it is convenient to represent \(L_F\) explicitly in terms of \(F_{ijmn}, F_{ij} = F_{n\cdot imj}\) and \(F = \eta^{ij} F_{ij}\).

The totally antisymmetric part \(A_{ijmn}\) of (I·3·10) can be rewritten as

\[
A_{ijmn} = -\frac{1}{24} \varepsilon_{ijmn} (\varepsilon_{pqrs} F_{pqrs}), \quad (A\cdot1)
\]

so we have

\[
A_{ijmn} A_{ijmn} = -\frac{1}{24} (\varepsilon_{ijmn} F_{ijmn})^2. \quad (A\cdot2)
\]

The quadratic forms of other irreducible parts are represented as
Gravity from Poincaré Gauge Theory of the Fundamental Particles 2239

\[ B_{ijmn} = \frac{1}{32} (s_{ijmn} F^{ijmn})^2 + \frac{3}{8} (F_{ijmn} F^{ijmn} + F_{ijmn} F^{mnij}) - \frac{3}{4} (F_{ijF^{ij}} + F_{ijF^{ij}}) + \frac{1}{4} F^2, \quad (A\cdot3) \]

\[ C_{ijmn} = \frac{1}{2} (F_{ijmn} F^{ijmn} - F_{ijmn} F^{mnij}) - (F_{ijF^{ij}} - F_{ijF^{ij}}), \quad (A\cdot4) \]

\[ E_{ij} = \frac{1}{2} (F_{ijF^{ij}} - F_{ijF^{ij}}), \quad (A\cdot5) \]

\[ I_{ij} = \frac{1}{2} (F_{ijF^{ij}} + F_{ijF^{ij}}) - \frac{1}{4} F^2. \quad (A\cdot6) \]

Using (A\cdot2) \sim (A\cdot6) in (1\cdot23), we have

\[ L_F = b_1 (F_{ijmn} F^{ijmn}) + b_2 (F_{ijmn} F^{mnij}) + b_3 (F_{ijmn} F^{ijmn}) \]

\[ + b_4 (F_{ijF^{ij}} + b_5 F^2 + b_6 (s_{ijmn} F^{ijmn})^2, \quad (A\cdot7) \]

where the parameters, \( b_1, b_2, \ldots \) and \( b_6 \), are related to the parameters, \( a_1, a_2, \ldots \) \( a_6 \), introduced in I by

\[ b_1 = \frac{1}{8} (3a_2 + 4a_3), \quad b_2 = \frac{1}{8} (3a_2 - 4a_3), \quad b_3 = -\frac{1}{4} (3a_2 + 4a_3 - 2a_4 - 2a_5), \]

\[ b_4 = -\frac{1}{4} (3a_2 - 4a_3 + 2a_4 + 2a_5), \quad b_5 = \frac{1}{4} (a_2 - 3a_3), \quad b_6 = -\frac{1}{96} (4a_1 - 3a_2), \quad (A\cdot8) \]

or conversely

\[ a_1 = b_1 + b_2 - 24b_6, \quad a_2 = \frac{4}{3} (b_1 + b_2), \quad a_3 = b_1 - b_2, \quad a_4 = 2(b_1 - b_2) + b_1 - b_4, \]

\[ a_5 = 2(b_1 + b_2) + b_3 + b_4, \quad a_6 = \frac{1}{6} (b_1 + b_2) + \frac{1}{4} (b_3 + b_4) + b_3. \quad (A\cdot9) \]

In accordance with this alternative expression for \( L_F \), the tensor \( J_{(I\Xi mnp)} \) which appears in the gravitational field equations of (1\cdot24) and (1\cdot30), can be rewritten as

\[ J_{(I\Xi mnp)} = K_{(I\Xi mnp)} \quad (A\cdot10) \]

with \( K_{ijmn} \) defined by

\[ K_{ijmn} = 2 \{ b_1 F_{ijmn} + b_2 F_{ijmn} + \eta_{lm} (b_5 F_{lm} + b_4 F_{lm}) \]

\[ + b_7 \eta_{lm} (b_2 F_{lm} + b_3 F_{lm}) \} \quad (A\cdot11) \]
Appendix B.

---Energy and Momentum of the Irreducible Torsion Field---

In vacuum we can write the irreducible torsion fields \( \Phi^{(i)}(x) \) \((i = J^p = 0, 1^+, 2^+)\) as the Fourier integral

\[
\Phi^{(i)}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3k}{2k^3} [\tilde{\Phi}^{(i)}(k)e^{ikx} + \tilde{\Phi}^{(i)*}(k)e^{-ikx}]
\]

(B.1)

with \( k^2 = (k^2 + M^2)^{\frac{1}{2}} \), where a star means complex conjugation, and \( M_i \) denotes the mass of \( \Phi^{(i)}(x) \). The Fourier transform \( \tilde{\Phi}^{(i)}(k) \) is a tensor with the same symmetry properties as \( \Phi^{(i)}(x) \), and furthermore the contraction of \( \tilde{\Phi}^{(i)}(k) \) with \( k^i \) vanishes for \( i = 1^+, 2^+ \). Since the total energy-momentum vector \( P^{(i)} \) of the field \( \Phi^{(i)} \) is a Lorentz vector, it is written in the momentum representation as

\[
P^{(i)} = \int \frac{d^3k}{2k^3} f^{(i)}(k),
\]

(B.2)

where \( f^{(i)}(k) \) is a vector formed of \( k^i, \tilde{\Phi}^{(i)}(k) \) and \( \tilde{\Phi}^{(i)*}(k) \).

For the uniform torsion field with \( \phi_m = 0 \) and

\[
\partial_m \Phi^{(i)}(x) = 0 \quad (\alpha = 1, 2, 3), \quad \partial_m \partial_m \Phi^{(i)}(x) = -M^2 \Phi^{(i)}(x),
\]

(B.3)

we can show by a straightforward calculation that the energy-momentum tensor \( t_{\alpha\nu} \) of (4.4) is rewritten for \( \nu = 0 \) as

\[
t_{\alpha\nu} = \sum_i (1/2N^0) [\partial_\nu \Phi^{(i)} \partial_\alpha \Phi^{(i)} + M^2 \Phi^{(i)} \Phi^{(i)*}],
\]

(B.4)

with \( i \) running over \( 0^+, 1^+ \) and \( 2^+ \), showing that the vector \( f^{(i)}(k) = 0 \) is given by

\[
f^{(i)}(k = 0) = (1/2N^0) [\tilde{\Phi}^{(i)*}(k = 0) \tilde{\Phi}^{(i)}(k = 0) + \phi^{(i)}(k = 0) \phi^{(i)*}(k = 0)] M_i.
\]

(B.5)

Here the factors \( N^0 \) are defined by (4.25a) \( \sim (4.25f) \) and the summation over tensor indices on \( \Phi^{(i)} \) and \( \tilde{\Phi}^{(i)} \) are carried out in (B.4) and (B.5): In Eq. (B.4), for example, \( \Phi^{(i)} \Phi^{(j)} = \delta_{ij} \delta^{(i)} \) for \( i = 2^+ \). Since \( f^{(i)}(k) \) is a vector, it follows from (B.5) that

\[
f^{(i)}(k) = (1/2N^0) [\tilde{\Phi}^{(i)*}(k) \tilde{\Phi}^{(i)}(k) + \phi^{(i)}(k) \phi^{(i)*}(k)] k^\nu.
\]

(B.6)

Consequently, \( P^{(i)} \) is given by
Gravity from Poincaré Gauge Theory of the Fundamental Particles

\[ P^{(i)} = \left( \frac{1}{2N^{(i)}} \right) \int \frac{d^3 k}{2k} \left[ \bar{\phi}^{(i)\ast}(k) \phi^{(i)}(k) + \bar{\phi}^{(i)}(k) \phi^{(i)\ast}(k) \right] k^0 \]

\[ = \int d^3 x \epsilon^{(i)\mu\nu}, \quad (B-7) \]

where \( \epsilon^{(i)\mu\nu} \) is defined by (4·24).

References

7) K. Hayashi and A. Bregman, Ann. of Phys. 75 (1973), 562.
8) R. Bach, Math. Z. 9 (1921), 110.
10) Y. Ne'eman, preprint TUAP 763-79.

Note added in proof: After completing this work we received a reprint of similar work by E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. 21 (1980), 3269. The condition of positive mass and positive energy, given here by (4·26), agrees with the ghost-free and tachyon-free condition given by (29) in their paper.