
DISCUSSION

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The author's extension of the theory of the development of the turbulent free-shear layer is indeed a welcome addition to our understanding of this problem. For some time the writer has been studying this flow in the incompressible case, primarily through reference to experiment. He is thus disappointed that only one correlation (Fig. 5) with experiment is offered and that this is in terms of \( \eta_p \), a factor not easily evaluated from measured velocity profiles. Most of the writer's comments thus center around a plea that the author indicate the predicted variation with downstream distance of width characterizing factors easily evaluated from experiment.

The spread of the turbulent layer with downstream distance is linear in the asymptotic limit where the velocity profile is then universal (similar). The author's theory includes this at very small \( \psi \) values and small \( \eta_p \); however, there is great leeway in deciding just when the asymptotic situation occurs. The several presentations shown in the figures imply that this result is for \( \psi \) values above 1000. The writer's analysis of some layers developing from rather thick turbulent boundary layers suggests that often much smaller distances yield effectively asymptotic behavior in certain gross quantities although the detailed velocity profiles have not yet reached the universal, similar condition. For the “width” of the shear layer, a convenient measure is the lateral distance between the loci of \( \psi = 0.9 \) and 0.1; i.e., \( b = y_{0.9} - y_{0.1} \). This is easily picked off plots of the velocity profiles obtained from experimental measurement. As an integral, and thus more precisely defined width definition, one has the momentum thickness \( \theta = \int (1 - \psi) dy \). When either of these widths is evaluated from turbulent free-shear-layer measurements, one obtains results such as shown in the lower part of Fig. 12 for the experiment of Liepmann and Laufer [18]. In this case the initial boundary layer was laminar (momentum-thickness Reynolds number about 150) and transition to turbulent free-shear flow did not occur for 6 cm. The \( \theta \) value indicated in the figure at transition is based on the growth rate of the laminar free-shear layer. An initial laminar layer is not often found; when the initial boundary layer is turbulent a region of rapid growth in \( \theta \) or \( b \) toward the asymptotic linear behavior appears when the boundary layer is one which developed in an appreciable pressure gradient. In the case of flat-plate boundary layers the writer has found little or no change in \( \theta \) or \( b \) even right near the start of the shear layer.

The observed linear variation in shear-layer width with downstream distance leads naturally to the useful virtual-origin concept, initially propounded by Kirk [11] in 1954. As indicated by the figure, in terms of a virtual origin at \( z_0 = -4.5 \) cm, this concept adequately describes the spread of the Liepmann and Laufer layer for z-distances of 20 cm and greater. It is significant to note that at 20 cm the velocity profile was still appreciably non-similar. If one employs the laminar initial boundary-layer thickness—a dubious procedure here—the corresponding \( \theta \) is 200. The author's theory suggests a considerably greater development distance. Other experiments with initial layers that are turbulent suggest much shorter development distances for the widths.

Several years ago H. F. Johnson, under the writer's supervision, measured the development of the free-shear layer from boundary layers of thickness 0.95, 0.60, and 1.20 in. with the two thicker ones having velocity profiles characteristic of turbulent flat-plate flow. The thickness Reynolds number of the thinnest initial layer suggests that it may have been laminar but measurements at \( z = 4 \) in. (\( \psi \) of about 80) do not suggest behavior such as noted for the Liepmann and Laufer experiment. The \( b \) and \( \theta \) variations with downstream distance from all three experiments follow the virtual-origin concept almost from the beginning. Only slight deviations in the constancy of the \( b/(x-x_0) \) and \( \theta/(x-x_0) \)
ratios appear for the thicker layers at $\psi$-values as low as 3 and 6. Of course, the velocity profiles at these locales were not similar in detail but this seems to have had little effect on the gross behavior. The velocities did seem to be similar at the last measurement station ($x = 16$ in., yielding $\psi$-values of 13 and 27). In a less complete experiment by the writer in 1954, as part of a diffuser flow study, with approach boundary-layer development grossly similar to that of Chapman [8], the shear layer developing from a turbulent boundary layer of shape factor (ratio of displacement to momentum thickness) $H = 2.5$ was traversed up to $\psi = 3.4$. The variation in width $b$ with $\psi$ in this case is again found to be linear from $\psi$ of about 0.3, although the traversing did not extend far enough downstream to reach the similar result.

The rather quick development of an asymptotic behavior of these widths of the turbulent free-shear layer, despite much slower development in details of the flow (as indicated by the velocity profiles), together with the utility of the virtual-origin concept, are features of the developing layer which it would seem useful to look for in the theory. Specifically, the theory should enable one to predict the effect of the initial boundary layer on the distance to the virtual origin. In lieu of this, the writer has found empirically that this distance may be calculated as 25 times the momentum thickness, when the development is from turbulent boundary layers of the flat-plate type. This prediction underestimates the distance for the Liepmann and Laufer experiment and for the turbulent boundary layer which developed in a strongly adverse pressure gradient (high $H$-value).

In view of the power-law velocity-profile formulation serving on $\psi$, a rather quick approximation to observed turbulent velocity profiles, the writer was most disappointed in the author's use of this formulation for $\phi_\psi$. Much more satisfactory profile formulations have appeared in the past 15 years. The range in type of initial turbulent boundary layers considered in the analysis should be widened to include layers typical of separation or near-separation flows. Separation due to flow development, and not sharp geometry change, is characterized by $H$-values of 2.5 to 3.5. In terms of the power-law formulation of the paper, these correspond to $n$-values of 1.4 to 0.8. Calculations such as summarized in Fig. 4 thus should be extended to $n = 0.8$ and to lower $\psi$-values.

As one final point the writer notes appreciable divergence of experimentally found values of the turbulent shear stress from those indicated in Fig. 11. Liepmann and Laufer's measurements at $\psi$ (based on initial laminar $\delta$) values of 300, 540, and 750 yield maximum $J$-values of 0.0053 and $J$-values of about 0.007. Measurements by the writer, in the experiment of Johnson with the thinnest boundary layer, at $\psi = 160$ and 320 yield maximum values of 0.0086 and 0.00847. These magnitudes are about two thirds of those predicted at zero Crocco number.

**Author's Closing**

Professor Robertson's interesting response is appreciated. First, it should be noted that the lack of experimental correlation presented in the paper resulted from an earnest desire of the experimental information directed primarily at developing flows. For example, in all 12 investigations discussed by Maydew and Reed [17], most of the experimental velocity profiles were taken far enough from the origin of mixing so that initial boundary-layer effects were essentially negligible. Consequently, there is a general omission of measurements of initial velocity profiles. Furthermore, in circumstances where these data were taken, it was found that the boundary layers were laminar [18]. Recently, however, there have been a number of experimental studies of developing turbulent layers with free-stream turbulence numbers up to 8. These data are being correlated with the present theory and the results will be presented in a later document.

Professor Robertson has also discussed the question, "Where does the flow become fully developed?" While it is true that the present paper has qualitatively confirmed the often-overlooked fact that profile shape similarity (in terms of the position parameter) is attained much earlier than detailed similarity (for example, velocity along a particular streamline), the present results are somewhat surprising in that the developing layer does not converge absolutely to the asymptotic solution in the relatively short distances indicated by some experiments. This point is illustrated by the variations in Fig. 4 which show a crossing of the position parameter curves for developing and asymptotic solutions. Although not shown in Fig. 4, there is an ultimate convergence of the two solutions for very small $\eta_0$ values (0.001 $< \eta_0 < 0.01$). This behavior of the present solution has been verified by an independent development and computation at NASA-Marshall and is presently attributed to the difference between an approach to the asymptotic limit in the differential equation rather than in the solution to the equation. Further study of this aspect of shear-layer development is in progress.

In his remarks, Professor Robertson has dealt at some length with the virtual origin concept. He has expressed the desire that the present theory be utilized to determine the virtual origin. Unfortunately, the foregoing considerations prohibit such a correlation in general, inasmuch as the virtual origin idea presupposes that there is some relatively small distance beyond which the shear layer is fully developed. However, it should be remembered that the virtual origin is an independent postulation, based largely on experimental observation, and therefore one would not expect it to be related in a general way to a more exact theory. This can be illustrated by considering the variations of the momentum thickness $\theta$, as suggested by Robertson. Now the general expression for $\theta$ in an incompressible flow, using the Nomenclature of references [3 and 12] and normalizing with the initial boundary-layer thickness, is $\theta = \psi (1/\eta_0)$ for the asymptotic theory and $\theta = (1 - l_1)/\eta_0$ for the developing flow, where the subscript R denotes the free-stream edge of the shear layer.

It is observed that the asymptotic theory gives zero momentum thickness at the origin while in an actual flow it is, of course, finite. Following Kirk [11], the displacement of the virtual origin is therefore $X_0 = \theta_0 (\psi_1/l_1)$ which, with $\sigma = 12$ and $l_1 = 0.4$, yields $X_0 = 30 \theta_0$. This compares favorably with Robertson's experimental value of 25 $\theta_0$. After normalization, the virtual origin is $\psi_0 = \pi \theta_0 (\eta_0 + 1)/(\eta_0 + 2)$ for a 1/4 power law in incompressible flow. Shown in Fig. 13 are the variations of normalized momentum thickness with development distance for both asymptotic and developing flows. It is observed that the predicted values of $\theta$ from curves $A$ and $B$ are equivalent at $\psi = 20$ but diverge at larger development distances. In addition, the asymptotic solution with the negative virtual origin (curve C) agrees with the developing solution at small values of $\psi$. On the other hand, to obtain reasonable agreement between the two theories at large distances, the displacement of the virtual origin must assume a rather large positive value (curve D). This latter concept has been previously proposed by Bradshaw [21] as a result of experimental observations.

Turning to the question of initial velocity profiles presented in the paper, the power law distributions were used primarily for simplicity since the mixing zone velocity profiles are rather insensitive to the minor variations between such initial profiles and logarithmic distributions, for example. These latter profiles would be somewhat unattractive for the present formulation because of their dependence on the wall shear stress which would lead to an additional independent variable. Of course, it is desirable that a theory incorporating more accurate upstream
boundary-layer information be developed eventually. The omission from the paper of initial profiles with large shape parameters was deliberate inasmuch as an isobaric mixing theory is considered to be of little use near separation where there are large longitudinal pressure gradients and reverse flow.

The discrepancies between measured and theoretical shear stress values, which Professor Robertson mentioned, are unaccounted for at the present. However, it should be noted that these theoretical values are in agreement with the calculations of Chow and Korst (NASA TN-D-1894, Apr., 1963).