GEOPHYSICS, VOL. XXV, NO. 3 (JUNE, 1960), PP. 642–648, 1 FIG.

SCATTERING OF ELASTIC WAVES BY SMALL INHOMOGENEITIES*

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ABSTRACT

Rayleigh scattering theory is extended to determine the perturbation on an arbitrarily prescribed elastic wave field produced by small inhomogeneities in an otherwise homogeneous, isotropic medium. The general result is applied to the specific problems of the scattering of both plane *P*- and *S*-waves. It is found that a change in compressibility acts at a distance as a simple source and a change in density as a dipole, as in the acoustical problem, while a change in shear modulus contributes both simple-source and quadrapole fields.

INTRODUCTION

We consider here the scattering of elastic waves by small inhomogeneities in an isotropic medium (see (1) below for a more precise definition of *small*). The corresponding acoustical (longitudinal waves only) and optical (transverse waves only) problems were solved long ago by Rayleigh (see *Theory of Sound*, §§296 and 375), while the mathematical framework for the solution of the elastic problem goes back to Stokes (1849). Rayleigh (1871) pointed out that, on dimensional grounds, the scattered field at distances large compared with the wave length must be proportional to $V\kappa^2 r^{-1}$, where V denotes the volume of the small inhomogeneity, r the distance to the point of observation, and κ the wave number (inversely proportional to the wave length).

Rayleigh's scattering law applies also to acoustical scattering by a small obstacle of infinite density and stiffness (e.g., a rigid, immobile sphere; see *Theory of Sound*, §334) and to elastic scattering by a rigid obstacle of finite density (Wolf, 1945). Elastic scattering by a rigid obstacle of infinite density appears to constitute a singular exception, however, in that the constraints required to fix the rigid body render the scattered field proportional to $A\kappa r^{-1}$, where A denotes the area of the obstacle (Yamakawa, 1956; see also Knopoff, 1959).

A second, general characteristic of Rayleigh scattering is that a deviation in compressibility acts as a simple source, yielding a spherically symmetric, scattered wave, whereas a deviation in density acts as a dipole, yielding a scattered wave proportional to the cosine of the angle between the direction of incidence (of a plane wave) and the direction of observation. We anticipate that *P*-wave scattering also will partake of these qualities and that, in addition, deviations in shear modulus will lead to a quadrapole radiation (in analogy with radiation from faults).

FORMAL SOLUTION

We consider an isotropic elastic medium that is homogeneous except in a small domain V, where it is slightly inhomogeneous Let λ_0, μ_0 and ρ_0 denote the

* Manuscript received by the Editor November 23, 1959.

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Lamé constants and the density outside of V; then by *slightly inhomogeneous*, we mean that

$$\lambda = \lambda_0 + \lambda_1, \cdots, \quad |\lambda_1/\lambda_0| \ll 1, \cdots \quad . \tag{1}$$

Similarly, letting $\boldsymbol{q}^{(0)}(\boldsymbol{r})$ denote a displacement that satisfies the equations of motion for $\lambda \equiv \lambda_0, \cdots$, we may define the complete displacement as

$$\boldsymbol{q}(\boldsymbol{r}) = \boldsymbol{q}^{(0)}(\boldsymbol{r}) + \boldsymbol{q}^{(1)}(\boldsymbol{r}), \quad |\boldsymbol{q}^{(1)}| \ll |\boldsymbol{q}^{(0)}|, \tag{2}$$

where $\boldsymbol{q}^{(1)}$ denotes the perturbation displacement excited by $\boldsymbol{q}^{(0)}$ in consequence of the inhomogeneity in V. We shall assume that $\boldsymbol{q}^{(0)}$, and therefore $\boldsymbol{q}^{(1)}$ also, exhibits the harmonic time dependence exp $(i\omega t)$ with the usual convention with respect to complex amplitudes. Given $\boldsymbol{q}^{(0)}$, we then seek $\boldsymbol{q}^{(1)}$.

We first observe that the total force per unit volume is given by

$$f_i = \rho\omega^2 q_i + \sigma_{ij,j}, \tag{3}$$

where $\rho\omega^2 q_i$ denotes the inertial force associated with the acceleration $-\omega^2 q_i$ and

$$\sigma_{ij} = \lambda\delta_{ij}q_{k,k} + \mu(q_{i,j} + q_{j,i}), \tag{4}$$

denotes the Cartesian stress tensor. Single subscripts denote components of a vector (e.g., q_i is a Cartesian component of $\boldsymbol{q}$), double subscripts components of the stress tensor and subscripts following a comma, differentiation with respect to the corresponding Cartesian coordinate; repeated subscripts imply summation according to the usual convention, and δ_{ij} is the Kronecker delta. We next observe that, from the viewpoint of an observer outside of V, $\boldsymbol{q}^{(1)}$ may be regarded as having been generated by the perturbation force obtained by replacing ρ, λ and μ in (3) and (4) by ρ_1, λ_1 and μ_1; accordingly, we may determine $\boldsymbol{q}^{(1)}(\boldsymbol{r})$ as a superposition of disturbances generated by the infinitesimal forces $\boldsymbol{f}(\boldsymbol{r}')dV(\boldsymbol{r}')$ integrated through V.

The disturbance produced by a time-dependent force acting at a point in an elastic medium was determined originally by Stokes (1849; see also Love, 1944, §212 and Rayleigh, 1945, §378) and is given by

$$4\pi\rho_0\omega^2\boldsymbol{q} = -\nabla\nabla\cdot\mathbf{G}\left(t - \frac{r}{\alpha}\right) + \nabla\times\nabla\times\mathbf{G}\left(t - \frac{r}{\beta}\right),$$
$$\boldsymbol{G}(t) = \frac{1}{r}\boldsymbol{F}(t), \tag{5a}$$

in vector notation or

$$4\pi\rho_0\omega^2 q_i = \left[G_j\left(t - \frac{r}{\beta}\right) - G_j\left(t - \frac{r}{\alpha}\right)\right]_{,ij} - \left[G_i\left(t - \frac{r}{\beta}\right)\right]_{,jj}, \tag{5b}$$

in tensor notation; here $\boldsymbol{F}$ denotes the force, r the distance from its point of application to the point of observation, and α and β the speeds of dilatational and shear waves—viz.,

$$\alpha^2 = (\lambda_0 + 2\mu_0)/\rho_0, \qquad \beta^2 = \mu_0/\rho_0. \tag{6a, b}$$

It then follows from the principle of superposition that we may calculate the disturbance produced by a distribution of force $\boldsymbol{f}(\boldsymbol{r}, t)$ by evaluating $\boldsymbol{G}_i$ according to

$$\boldsymbol{G} = \iiint f_i\left(\boldsymbol{r}', t - \frac{R}{\alpha}\right)\frac{dV(\boldsymbol{r}')}{R}, \tag{7}$$

where

$$R = \left| \boldsymbol{r} - \boldsymbol{r}' \right| = \left[(x_i - x_i')(x_i - x_i')\right]^{1/2}. \tag{8}$$

Introducing the time dependence exp $(i\omega t)$, suppressing the explicit appearance of t, and abbreviating $dV(\boldsymbol{r}')$ by dV', we may replace (7) by

$$\boldsymbol{G}(\boldsymbol{r}) = \iiint \boldsymbol{f}(\boldsymbol{r}')e^{-i\kappa_\alpha R}\,\frac{dV'}{R}, \tag{9}$$

where

$$\kappa_\alpha = \omega/\alpha. \tag{10}$$

The volume integral in (9), and subsequently, is over V, and we may replace α by β as required.

Substituting (3) in (9) with $\rho = \rho_1$ etc., integrating the term in $\sigma_{ij,j}$ by parts, noting that

$$\frac{\partial}{\partial x_i'}\,\frac{e^{-i\kappa R}}{R} = -\,\frac{\partial}{\partial x_i}\,\frac{e^{-i\kappa R}}{R}, \tag{11}$$

and invoking the fact (from the definition of V) that λ_1 and μ_1 vanish on the surface bounding V, we obtain

$$G_i^{(1)} = \iiint \left\{\rho_1\omega^2 q_i + \left[\lambda_1\delta_{ij}q_{k,k} + \mu_1(q_{i,j} + q_{j,i})\right]\left(\frac{\partial}{\partial x_j}\right)\right\}\frac{e^{-i\kappa R}}{R}\,dV' \tag{12}$$

where κ denotes either κ_α or κ_β, and the argument of q_i is $\boldsymbol{r}'$. We remark that the substitution of (12) in (5b) yields an exact but implicit result, which becomes approximate but explicit if $\boldsymbol{q}$ is approximated by $\boldsymbol{q}^{(0)}$ in the determination of $\boldsymbol{G}$. The superscript (1) on G_i implies that its substitution in (5b) yields only $q_i^{(1)}$.

FAR FIELD APPROXIMATION

We now invoke the approximation

$$\kappa r \gg 1 \gg \kappa r', \tag{13}$$

thereby implying that the dimensions of V are small compared with the wave length and that the distance from V to the point of observation is large compared with the wave length. We then may approximate R by r in (12) and approximate the derivatives with respect to x_i according to (Note that $r_{,i}=x_i/r$ are the direction cosines of $\mathbf{r}$ and that $r_{,i}r_{,i}=1$).

$$\left(\frac{\partial}{\partial x_i}\right)\left(\frac{e^{-i\kappa R}}{R}\right) \doteq -i\kappa r_{,i}\frac{e^{-i\kappa r}}{r} \tag{14}$$

in virtue of $\kappa r \gg 1$, while $\kappa r' \ll 1$ permits the approximation

$$\iiint \rho_1(\mathbf{r}')q_i(\mathbf{r}')dV' \doteq q_i^{(0)} \iiint \rho_1 dV', \tag{15}$$

and similarly for the other terms in (12). These approximations lead to

$$G_i^{(1)} = \frac{e^{-i\kappa r}}{r}\left\{P_1\omega^2 q_i - i\kappa[\lambda_1\delta_{ij}q_{k,k} + \mu_1(q_{i,j}+q_{j,i})]r_{,j}\right\} \tag{16}$$

in place of (12), while substituting (16) in (5b) yields

$$4\pi\rho_0\omega^2 q_i^{(1)} = \kappa_\alpha^2\frac{e^{-i\kappa_\alpha r}}{r}r_{,i}\left\{P_1\omega^2 r_{,j}q_j - i\kappa_\alpha[\Lambda_1 q_{k,k} + M_1(q_{j,k}+q_{k,j})r_{,j}r_{,k}]\right\}$$

$$+ \kappa_\beta^2\frac{e^{-i\kappa_\beta r}}{r}(\delta_{ij} - r_{,i}r_{,j})[P_1\omega^2 q_j - i\kappa_\beta M_i(q_{j,k}+q_{k,j})r_{,k}], \tag{17}$$

where

$$P_1, \Lambda_1, M_1 = \iiint (\rho_1, \lambda_1, \mu_1)dV', \tag{18}$$

and q_i and its derivatives may be replaced by $q_i^{(0)}$ and its derivatives evaluated at the origin. Further investigation suggests that the approximations may be optimized by choosing the origin at the centroid of the distributions ρ_1, λ_1 and μ_1 (assuming that these distributions have a common centroid).

We specifically remark that the approximations (16) and (17) do not assume ρ_1, λ_1 and μ_1 to be uniform throughout V; indeed, any attempt to invoke such an assumption would have rendered the analysis far more difficult in consequence of the implicit discontinuities (infinite gradients) at the boundaries of V. This statement notwithstanding, the final approximations are independent of the details of the distributions, and we may (if convenient) take ρ_1, λ_1 and μ_1 to be uniform in evaluating the integrals of (18).

SCATTERING OF PLANE P-WAVE

We consider the plane P-wave

$$q_i^{(0)} = n_i e^{-i\kappa_\alpha n_j x_j}, \tag{19}$$

to be incident on V; n_i are the components of the wave front normal, and the amplitude of $\boldsymbol{q}^{(0)}$ is unity. Substituting (19) in (17) and noting that

$$q_i^{(0)} = n_i, \qquad q_{k,k}^{(0)} = -\,i\kappa_\alpha, \qquad q_{i,j}^{(0)} + q_{j,i}^{(0)} = -\,2i\kappa_\alpha n_i n_j \text{ at } r = 0, \tag{20}$$

we obtain

$$\begin{aligned} 4\pi\rho_0 \boldsymbol{q}^{(0)} = {}& \kappa_\alpha^2 \frac{e^{-i\kappa_\alpha r}}{r} \boldsymbol{r}_1 \left\{ P_1(\boldsymbol{n}\cdot\boldsymbol{r}_1) - \frac{1}{\alpha^2}\left[\Lambda_1 + 2M_1(\boldsymbol{n}\cdot\boldsymbol{r}_1)^2\right] \right\} \\ & + \kappa_\beta^2 \frac{e^{-i\kappa_\beta r}}{r} \left[\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{r}_1)\boldsymbol{r}_1\right] \left[P_1 - \frac{2M_1}{\alpha\beta}(\boldsymbol{n}\cdot\boldsymbol{r}_1) \right], \end{aligned} \tag{21}$$

where

$$\boldsymbol{n}\cdot\boldsymbol{r}_1 = n_j r_{,j} = \cos\theta, \tag{22}$$

$\boldsymbol{r}_1$ being the unit radial vector and θ the polar angle between $\boldsymbol{r}$ and $\boldsymbol{n}$. If we also introduce the unit vector $\boldsymbol{\theta}_1$ (see Figure 1), we may rewrite (21) as

$$\begin{aligned} 4\pi\rho_0 \boldsymbol{q}^{(1)} = {}& \kappa_\alpha^2 \frac{e^{-i\kappa_\alpha r}}{r} \boldsymbol{r}_1 \left[-\left(\frac{\Lambda_1 + M_1}{\alpha^2}\right) + P_1 \cos\theta - \frac{M_1}{\alpha^2}\cos 2\theta \right] \\ & + \kappa_\beta^2 \frac{e^{-i\kappa_\beta r}}{r} \boldsymbol{\theta}_1 \left(-P_1 \sin\theta + \frac{M_1}{\alpha\beta}\sin 2\theta \right). \end{aligned} \tag{23}$$

Generalizing Rayleigh's observations for the acoustical problem (*Theory of Sound*, §296), we remark that the scattered wave of (23) comprises a simple

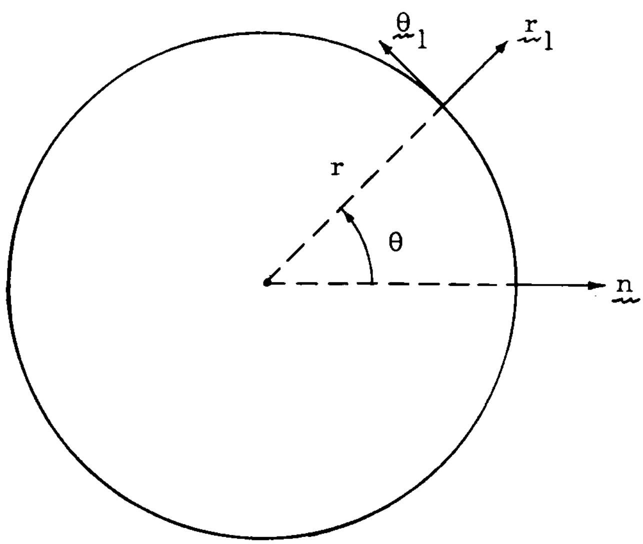

FIG. 1. The Spherical Polar Coordinates for the Scattered Wave of Equation (23).

source of strength proportional to Λ_1+M_1, a dipole of strength proportional to P_1, and a quadrapole of strength proportional to M_1. If $\mu\equiv 0$ we must discard the S-wave terms in (23), which then reduces to Rayleigh's result.

SCATTERING OF A PLANE S-WAVE

We may specify a plane S-wave by

$$\overset{(0)}{q_i} = m_i e^{-i\kappa_\beta n_j x_j}, \qquad m_j n_j = 0, \tag{24}$$

where m_i are the direction cosines of the transverse displacement of unit amplitude. We then obtain

$$\overset{(0)}{q_i} = m_i, \quad \overset{(0)}{q_{k,k}} = 0, \quad \overset{(0)}{q_{i,j}} + \overset{(0)}{q_{j,i}} = -i\kappa_\beta(m_i n_j + m_j n_i) \text{ at } r = 0, \tag{25}$$

and

$$\begin{aligned} 4\pi\rho_0 \boldsymbol{q} = {} & \kappa_\alpha^2 \frac{e^{-i\kappa_\alpha r}}{r} (\boldsymbol{m}\cdot\boldsymbol{r}_1)\boldsymbol{r}_1 \left[P_1 - \frac{M_1}{\alpha\beta}(\boldsymbol{n}\cdot\boldsymbol{r}_1) \right] \\ & + \kappa_\beta^2 \frac{e^{-i\kappa_\beta r}}{r} \left\{ [\boldsymbol{m} - (\boldsymbol{m}\cdot\boldsymbol{r}_1)\boldsymbol{r}_1] \left[P_1 - \frac{M_1}{\beta^2}(\boldsymbol{n}\cdot\boldsymbol{r}_1) \right] \right. \\ & \left. - [\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{r}_1)\boldsymbol{r}_1] \frac{M_1}{\beta^2} (\boldsymbol{m}\cdot\boldsymbol{r}_1) \right\}. \end{aligned} \tag{26}$$

Introducing θ_m and θ_n according to

$$\boldsymbol{m}\cdot\boldsymbol{r}_1 = \cos\theta_m, \qquad \boldsymbol{n}\cdot\boldsymbol{r}_1 = \cos\theta_n, \tag{27}$$

and unit vectors $\boldsymbol{\theta}_m$ and $\boldsymbol{\theta}_n$, with $\boldsymbol{\theta}_n$ directed in an $\boldsymbol{r}_1$, $\boldsymbol{n}$ plane as in Figure 1 and similarly for $\boldsymbol{\theta}_m$ in an $\boldsymbol{r}_1$, $\boldsymbol{m}$ plane, we may transform (26) to

$$\begin{aligned} 4\pi\rho_0 \overset{(1)}{\boldsymbol{q}} = {} & \kappa_\alpha^2 \frac{e^{-i\kappa_\alpha r}}{r} \boldsymbol{r}_1 \left[P_1 \cos\theta_m - \frac{M_1}{\alpha\beta} \cos\theta_m \cos\theta_n \right] \\ & + \kappa_\beta^2 \frac{e^{-i\kappa_\beta r}}{r} \left\{ \boldsymbol{\theta}_m \left[-P_1 \sin\theta_m + \frac{M_1}{\beta^2} \sin\theta_m \cos\theta_n \right] \right. \\ & \left. + \boldsymbol{\theta}_n \frac{M_1}{\beta^2} \cos\theta_m \sin\theta_n \right\}. \end{aligned} \tag{28}$$

We again may interpret the scattered wave in terms of a dipole field proportional to P_1 and a quadrapole field proportional to M_1. There is, however, no simple source field in consequence of the absence of dilation in the incident wave. If the medium is assumed to be incompressible we may discard the P-wave terms in (28), which then reduces to Rayleigh's result (1871) for the optical problem (based on an elastic aether); see also *Theory of Sound* §375 for the special case of density variations only.

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